

Chapter 9 Ingredients of Multivariable Change: Models, Graphs, Rates



9.1 Cross-Sectional Models and Multivariable Functions

For a multivariable function with two input variables described by data given in a table with the values of the *first* input variable listed *horizontally* across the top of the table and the values of the *second* input variable listed *vertically* down the left side of the table, obtain cross-sectional models by entering the appropriate data in lists L1 and L2 and then proceeding as indicated in Chapter 1 of this *Guide*.

9.1.1 FINDING CROSS-SECTIONAL MODELS (holding the first input variable constant)

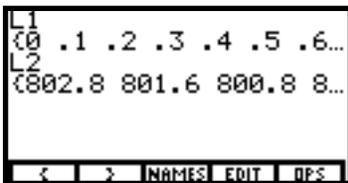
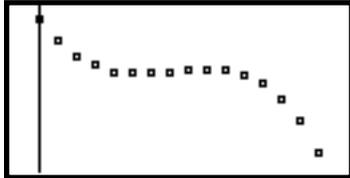
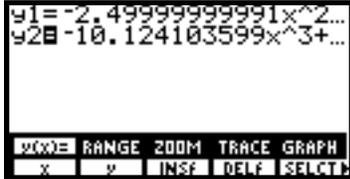
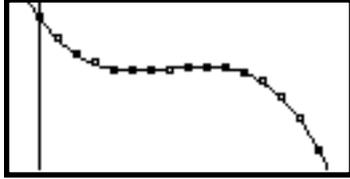
Using the elevation data in Table 9.1 of the text, find the cross-sectional model $E(0.8, n)$ as described below. Remember that “rows” go from left to right horizontally and “columns” go from top to bottom vertically.

When considering $E(0.8, n)$, e is constant at 0.8 and n varies.

<p>So, enter the values for n appearing on the left side of the table (vertically) in L1 and the elevations E obtained from the table in the $e = 0.8$ column in L2.</p>	<p>TI-85</p>	<p>TI-86</p>
<p>Clear previously-entered functions from the $y(x)=$ list. TI-85 Run program STPLT. TI-86 Turn Plot1 on. Both Draw a scatter plot of the $E(0.8, n)$ data.</p>		
<p>The data appear to be quadratic. Fit a quadratic model and copy the model to the $y(x) =$ list. (Refer to Section 2.3.2 of this <i>Guide</i>.)</p>		
<p>Overdraw the graph of the model on the scatter plot. Because we will find several different cross-sectional models, calling different variables by the same names x and y would be very confusing. It is very important that you call the variables by the names they have been assigned in the application. (See below.)</p>	<p>$E(0.8, n) =$ $-2.5n^2 + 2.497n + 799.490$</p>	

- Remember that when finding or graphing a model, your calculator always calls the input variable x and the output variable y . When working with multivariable functions, you must translate the calculator's model $y1 \approx -2.5x^2 + 2.497x + 799.490$ to the symbols used in this problem to obtain $E(0.8, n) = -2.5n^2 + 2.497n + 799.490$.

9.1.2 FINDING CROSS-SECTIONAL MODELS (holding the second input variable constant) Using the elevation data in Table 9.1 of the text, find the cross-sectional model $E(e, 0.6)$ as discussed below. When considering $E(e, 0.6)$, n is constant at 0.6 and e varies.

<p>Thus, enter the values for e appearing across the top of the table (horizontally) in L1 and the elevations E obtained from the table in the $n = 0.6$ row in L2.</p>	 <p style="text-align: center;">TI-85</p>	<p style="text-align: center;">TI-86</p>
<p>You can either turn y_1 "off" by placing the blinking cursor over the equals sign and pressing ENTER or you can delete it.</p> <p>Next, draw a scatter plot of the $E(e, 0.6)$ data.</p>		
<p>The data appears to be cubic. Fit a cubic model and copy it to the y_2 location of the $y(x)=$ list.</p> <p>Be certain that you translate the calculator's model $y_2 \approx -10.124X^3 + 21.347X^2 - 13.972X + 802.809$ to the symbols used in the example. (See the note at the end of Section 9.1.1.)</p>		
<p>Write this cross-sectional model as $E(e, 0.6) = -10.124e^3 + 21.347e^2 - 13.972e + 802.809$</p> <p>Overdraw the graph of the model on the scatter plot.</p>		

9.1.3 EVALUATING OUTPUTS OF MULTIVARIABLE FUNCTIONS Outputs of multivariable functions are found by evaluating the function at the given values of the input variables. One way to find multivariable function outputs is to evaluate them on the home screen. For instance, the compound interest formula tells us that

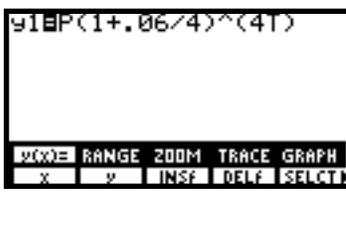
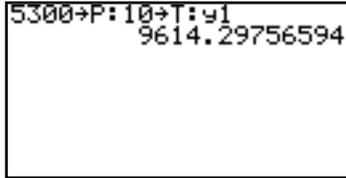
$$A(P, t) = P \left(1 + \frac{0.06}{4} \right)^{4t} \text{ dollars}$$

is the amount to which P dollars will grow over t years in an account paying 6% interest compounded quarterly. Let's review how to find $A(5300, 10)$.

<p>Type, on the home screen, the formula for $A(P, t)$, substituting in $P = 5300$ and $t = 10$. Press ENTER.</p> <p>Again, be warned that you must carefully use the correct placement of parentheses. Check your result with the values in Table 9.4 in the text to see if \$9614.30 is a reasonable amount.</p>	
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Even though it is not necessary in this example, you may encounter activities in this section in which you need to evaluate a multivariable function at several different inputs. Instead of individually typing each one on the home screen, there are easier methods. You will also use the techniques shown below in later sections of this chapter.

When evaluating a multivariable function at several different input values, you may find it more convenient to enter the multivariable function in the graphing list. It is very important to note at this point that while we have previously used x as the input variable when entering functions in the $y(x)=$ list, we do *not* follow this rule when *evaluating* functions with more than one input variable.

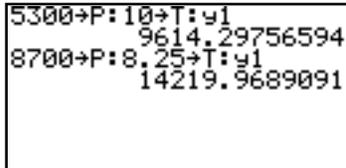
<p>Clear any previously-entered equations. Enter</p> $A(P, t) = P \left(1 + \frac{0.06}{4} \right)^{4t}$ <p>in y1 by pressing ALPHA , (P) (1 + . 0 6 <div style="border: 1px solid black; padding: 2px; display: inline-block;">÷ 4) ^ (4 ALPHA - (T))</div></p>	
<p>Return to the home screen, input the values $P = 5300$ and $t = 10$, and evaluate $A(P, t)$ with the keystrokes</p> <p>5300 P , P 2nd . (: ALPHA 1 0 <div style="border: 1px solid black; padding: 2px; display: inline-block;">STO - (T) and then type y1 and press ENTER .</div></p>	

- **TI-86 Note:** Because the TI-86 can recall many previously-entered expressions with

2nd ENTER

 (ENTRY), this storing process can be done in several different steps.

To find another output for this function, say $A(8700, 8.25)$, repeat the above procedure:

<p>Bring back the last line by pressing <div style="border: 1px solid black; padding: 2px; display: inline-block;">2nd ENTER</div> (ENTRY). Edit the expression to input the values $P = 8700$ and $t = 8.25$. Press <div style="border: 1px solid black; padding: 2px; display: inline-block;">ENTER</div> to evaluate.</p>	
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Warning: Recall that the your calculator recognizes only x as an input variable in the $y(x)=$ list when you *graph* and when you use the *table*. Thus, you should not attempt to graph the current $y1 = A(P, t)$ or use the table because the calculator considers this $y1$ a constant. (Check it out and see that the graph is a straight line at about 14220 and that all values in the table are approximately 14220.)



9.2 Contour Graphs

Because any program that we might use to graph¹ a three-dimensional function would be very involved to use and take a long time to execute, we do not attempt to do such. Instead,

¹A program to graph a multivariable function with two input variables (as well as many other Texas Instruments programs, are available at <http://www.ti.com/calcdocs/80xthing.htm>).

we discover graphical information about three-dimensional graphs using their associated contour curves.

9.2.1 SKETCHING CONTOUR CURVES When you are given a multivariable function model, you can draw contour graphs for functions with two input variables using the three-step process described below. We illustrate by drawing the \$520 constant-contour curve with the function for the monthly payment on a loan of A thousand dollars for a period of t years:

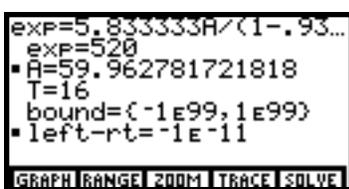
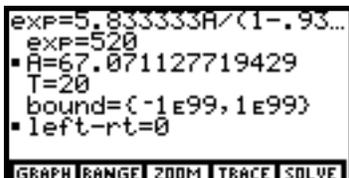
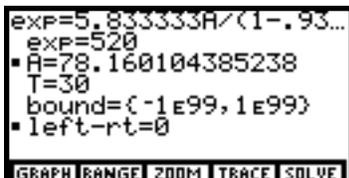
$$M(t, A) = \frac{5.833333A}{1 - 0.932583^t} \text{ dollars}$$

Step 1: Set $M(t, A) = 520$. Since we intend to use the solve routine to find the value of A at various values of t , write the function as

$$M(t, A) - 520 = 0.$$

Step 2: Choose values for t and solve for A to obtain points on the \$520 constant-contour curve. Obtain guesses for the values of t from the table of loan amounts in the table in Figure 9.14 of the text.

You can either enter the $M(t, A)$ formula in y_1 or enter it on the home screen. If you enter the formula in the graphing list, be certain to use the letters A and T and to indicate for which variable you are solving when using the solve routine.

<p>Access the SOLVER with 2nd ENTER (SOLVER), and clear any previously-entered equations with CLEAR.</p> <p>Enter $M(t, A)$ in the eqn location with the keystrokes 5.833333 ALPHA A ÷ (1 - $.932583$ ^ ALPHA T).</p>	 <p>(If necessary, refer to Section 1.2.2 of this Guide to review the instructions for using the solver.)</p>	
<p>Press ENTER. Since $M(t, A) = 520$, enter 520 in the eqn value location. Choose a value for t, say $t = 12$ years and solve for A.</p>		
<p>Choose another value for t, say $t = 16$ years. Solve for A.</p>		
<p>Repeat this process for $t = 20, t = 24, t = 28$, and $t = 30$.</p>		

Make a table of the values of t and A as you find them. You need to find as many points as it takes for you to see the pattern the points are indicating when you plot them on a piece of paper.

Step 3: You need a model to draw a contour graph using your calculator. Even though there are several models that seem to fit the data points obtained in Step 2, their use would be misleading since the real best-fit model can only be determined by substituting the appropriate values in a multivariable function. The focus of this section is to use a graph to study the relationships between input variables, not to find the equation of a model for a contour curve. Thus, we always sketch the contours on paper, not with your calculator.



9.3 Partial Rates of Change

When you hold all but one of the input variables constant, you are actually looking at a function of one input variable. Therefore, all the techniques we previously discussed can be used. In particular, the calculator's numerical derivative `der1` can be used to find partial rates of change at specific values of the varying input variable. Although your calculator does not give formulas for derivatives, you can use it as discussed in Sections 4.4.1 and/or 4.4.2 of this *Guide* to check your algebraic formula for a partial derivative.

9.3.1 NUMERICALLY CHECKING PARTIAL DERIVATIVE FORMULAS

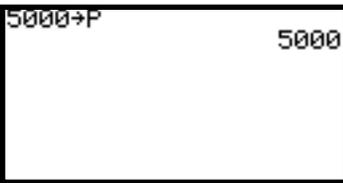
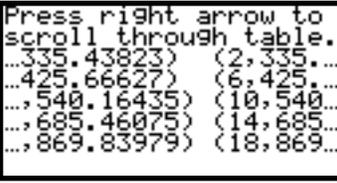
The basic concept behind checking your partial derivative formula is that your formula and the calculator's formula computed with `der1` should have the same outputs when each is evaluated at several different randomly-chosen inputs.

You can use the methods in Section 9.1.3 of this *Guide* to evaluate each formula at several different inputs and determine if the same numerical values are obtained from each formula. If so, your formula is *probably* correct.

You may find it more convenient to use your calculator's TABLE. If so, you must remember that when using the table, your calculator considers x as the variable that is changing. When finding a partial derivative formula, all other variables are held constant except the one that is changing. So, simply call the changing variable x and proceed as in Chapter 4.

Suppose you have a multivariable function $A(P, t) = P(1.061363551)^t$ and find, using derivative formulas, that $\frac{\partial A}{\partial t} = P(\ln 1.061363551)(1.061363551)^t$. We now illustrate how to check to see if you obtained the correct formula for $\frac{\partial A}{\partial t}$.

<p>Have $A(P, t) = P(1.061363551)^t$ in $y1$. Since t is the variable that is changing, replace T with x.</p> <p>Enter $\frac{\partial A}{\partial t}$, as computed by the calculator, in $y2$.</p> <p>(Notice that you must use x as the input variable.)</p>	
<p>Enter your answer for $\frac{\partial A}{\partial t}$ in $y3$.</p> <p>(Remember to use x instead of t.)</p> <p>Since you are checking to see if $y2$ and $y3$ are the same, turn $y1$ off.</p>	

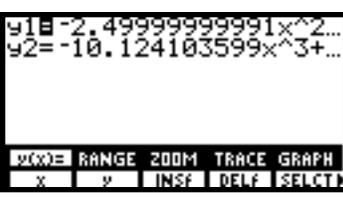
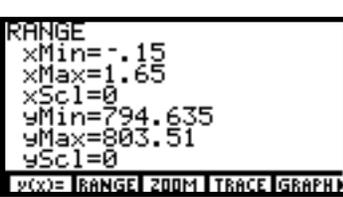
<p>Return to the home screen. Since P is constant, store <i>any</i> reasonable value except 0 in P. (If you prefer, repeat what follows for $P = 1000$, $P = 7500$, and then $P = 190,000$. This, however, is not really necessary.)</p>		
<p>We now use the TABLE. Choose ASK and remember that with ASK, it does not matter what the Tblmin and ΔTbl settings are. Input at least 4 values for $t = x$.</p>	 <p style="text-align: center;">TI-85</p>	<p style="text-align: center;">TI-86</p>

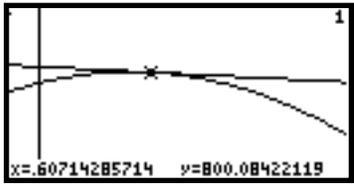
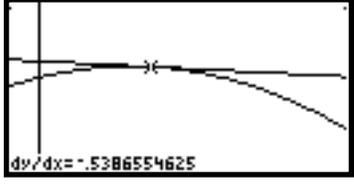
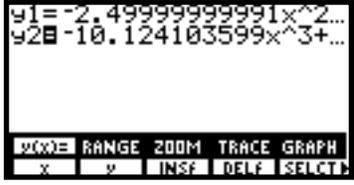
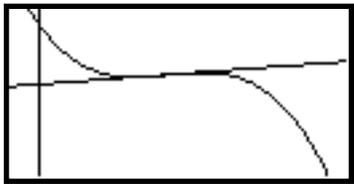
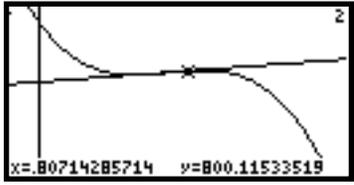
Since the values shown for y_2 and y_3 are approximately the same, our formula for $\frac{\partial A}{\partial t}$ is probably correct.

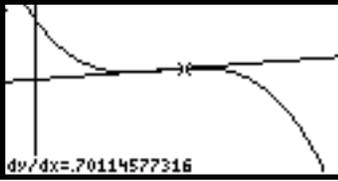
- **TI-85 Note:** Program TABLE will only allow you to ASK for values one at a time. If you prefer, you can choose AUTO with a ΔTbl that will encompass a variety of values for x . This was done to generate the TI-85 table above.
- Note that if you have rounded a model fit by the calculator before computing a partial derivative, then the y_2 and y_3 columns will very likely be *slightly* different.
- To check your formula for $\frac{\partial A}{\partial P}$, repeat the above procedure, but this time replace P by x and store any reasonable value (except 0) in T before going to the table. Remember that the variable that is changing always appears in the denominator of the partial derivative symbol.

9.3.2 VISUALIZING AND ESTIMATING PARTIAL RATES OF CHANGE

A partial rate of change of a multivariable function (evaluated at a specific point) is the slope of the line tangent to a cross-sectional model at a given location. We illustrate this concept with the two elevation cross-sectional models $E(0.8, n)$ and $E(e, 0.6)$.

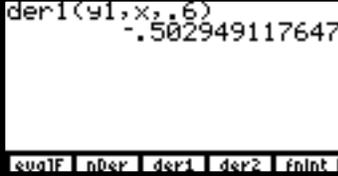
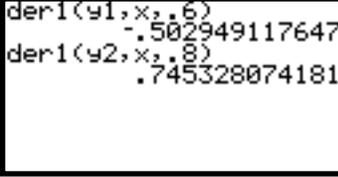
<p>Have $E(0.8, n)$ in y_1 and $E(e, 0.6)$ in y_2. Turn y_1 on and y_2 off using $\boxed{F5}$ (SELECT). (TI-86: Also make sure all Stat Plots are off.)</p>	
<p>Press \boxed{EXIT} $\boxed{F2}$ (RANGE or WIND). The values should be set to those shown to the right. Press $\boxed{2nd}$ \boxed{EXIT} (QUIT) to return to the home screen.</p>	

<p>Draw the tangent line to $E(0.8, n)$ at $n = 0.6$ with $\boxed{2\text{nd}}$ $\boxed{\text{CUSTOM}}$ (CATALOG) $\boxed{-}$ (T), and use $\boxed{\blacktriangledown}$ to select TanLn. Press $\boxed{\text{ENTER}}$ to copy the instruction to the home screen, and press $\boxed{2\text{nd}}$ $\boxed{\text{ALPHA}}$ y $\boxed{1}$ $\boxed{,}$ $\boxed{.6}$ $\boxed{)}$ $\boxed{\text{ENTER}}$.</p>	
<p>Estimate the partial rate of change $\frac{\partial E(0.8, n)}{\partial n}$ at $n = 0.6$ by first pressing $\boxed{\text{MORE}}$ $\boxed{\text{F1}}$ (MATH) $\boxed{\text{F4}}$ (dy/dx) and then pressing $\boxed{\blacktriangleleft}$ to trace to the approximate point of intersection. (Your view will probably be set in such a way that you are not able to locate $n = x = 0.6$ exactly.)</p>	
<p>Next, determine the slope of the tangent line at the point whose coordinates are shown at the bottom of the graphics screen in the previous step with $\boxed{\text{ENTER}}$. Remember that this is only an <i>estimate</i> of the true value of $\frac{\partial E(0.8, n)}{\partial n}$ at $n = 0.6$.</p>	
<p>Now, turn y1 off and y2 on. Return to the home screen.</p>	
<p>Draw the tangent line to $E(e, 0.6)$ at $e = 0.8$ by pressing $\boxed{2\text{nd}}$ $\boxed{\text{ENTER}}$ to return the last command, TanLn(y1, .6), to the home screen and use $\boxed{\blacktriangleleft}$ to edit the instruction to TanLn(y2, .8). Press $\boxed{\text{ENTER}}$.</p>	
<p>Estimate the partial rate of change $\frac{\partial E(e, 0.6)}{\partial e}$ at $e = 0.8$ by first pressing $\boxed{\text{MORE}}$ $\boxed{\text{F1}}$ (MATH) $\boxed{\text{F4}}$ (dy/dx) and then pressing $\boxed{\blacktriangleright}$ to trace to the approximate point of intersection. (Your view will probably be set in such a way that you cannot locate $e = x = 0.8$ exactly.)</p>	

<p>Next, determine the slope of the tangent line at the point whose coordinates are shown at the bottom of the graphics screen in the previous step with ENTER .</p> <p>Remember that this is only an <i>estimate</i> of the true value of $\frac{\partial E(e, 0.6)}{\partial e}$ at $e = 0.8$.</p>	
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9.3.3 FINDING PARTIAL RATES OF CHANGE USING CROSS-SECTIONAL MODELS

The procedure described above gave estimates of the partial rates of change. You will obtain a more accurate answer if you use your calculator's numerical derivative on the home screen to evaluate partial rates of change at specific input values.

<p>Have $E(0.8, n)$ in $y1$ and $E(e, 0.6)$ in $y2$.</p>	
<p>On the home screen, enter the instruction shown to the right to find $\frac{\partial E(0.8, n)}{\partial n}$ evaluated at $e = 0.8, n = 0.6$.</p> <p>(We are using X as the input variable because the cross-sectional models are in the $y=$ list in terms of x.)</p>	
<p>On the home screen, enter the second instruction shown to the right to find $\frac{\partial E(e, 0.6)}{\partial e}$ evaluated at $e = 0.8, n = 0.6$.</p>	

9.4 Compensating for Change

As you have just seen, your calculator finds numerical values of partial derivatives using its `der1` function. This technique can also be very beneficial and help you eliminate many potential calculation mistakes when you find the rate of change of one input variable with respect to another input variable (that is, the slope of the tangent line) at a point on a contour curve.

9.4.1 EVALUATING PARTIAL DERIVATIVES OF MULTIVARIABLE FUNCTIONS

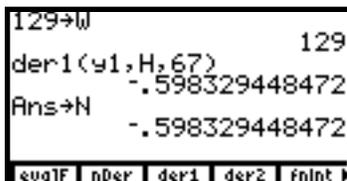
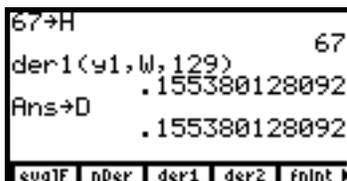
The previous section of this *Guide* indicated how to estimate and evaluate partial derivatives using cross-sectional models. Your calculator can be very helpful when evaluating partial derivatives calculated directly from multivariable function formulas. The most important thing to remember is that you must supply the name of the input variable that is changing and the values at which the partial derivative is evaluated.

Consider, for instance, the formula for a person's body-mass index:

$$B(h, w) = \frac{0.45w}{0.00064516h^2}$$

where h is the person's height in inches and w is the person's weight in pounds. We first find the partial derivatives $\frac{\partial B}{\partial h}$ and $\frac{\partial B}{\partial w}$ at a specific height and weight and then use those values in the next section of this *Guide* to find the value of the derivative $\frac{dw}{dh}$ at that particular height and weight.

Suppose the person in this example is 5 feet 7 inches tall and weighs 129 pounds.

<p>In the previous section, we had to use x as the input variable because we were finding and graphing cross-sectional models. However, here we are given the multivariable formula.</p> <p>Enter $B(h, w)$ in $y1$ using the same letters that appear in the formula above.</p>	
<p>Let's first find $\frac{\partial B}{\partial h}$ at $(h, w) = (67, 129)$. Because h is changing, w is constant. Return to the home screen and store the value of the input that is held constant.</p> <p>Find the value of $\frac{\partial B}{\partial h}$ at $h = 67$ and $w = 129$ by evaluating the expression shown to the right.</p> <p>(Store this value for later use as N.)</p>	 <p>H is the variable, and W is constant.</p>
<p>We now find $\frac{\partial B}{\partial w}$ at $(h, w) = (67, 129)$. Because w is changing, h is constant. Return to the home screen and store the value of the input that is held constant.</p> <p>Find the value of $\frac{\partial B}{\partial w}$ at $h = 67$ and $w = 129$ by evaluating the expression shown to the right.</p> <p>(Store this value for later use as D.)</p>	 <p>W is the variable, and H is constant.</p>

9.4.2 COMPENSATING FOR CHANGE When one input of a two-variable multivariable function changes by a small amount, the value of the function is no longer the same as it was before the change. The methods illustrated below show how to determine the amount by which the other input must change so that the output of the function remains at the value it was before any changes were made.

We use the body-mass index formula of the previous section of this *Guide* to illustrate the procedure for calculating the compensating amount.

We first determine $\frac{dw}{dh}$ at the point $(67, 129)$ on the contour curve corresponding to the person's current body-mass index. The formula is $\frac{dw}{dh} = \frac{-\partial B}{\partial h} \div \frac{\partial B}{\partial w}$. An easy way to remember this formula is that whatever variable is in the numerator of the derivative (in this case, w) is the same variable that appears in the denominator of the partial derivative in the denominator of the quotient. This is why we stored $\frac{\partial B}{\partial w}$ as D (for denominator). The partial derivative with respect to the other variable (here, h) appears in the numerator of the quotient. Don't forget to put a minus sign in front of the numerator.

<p>In the previous section, we stored $\frac{\partial B}{\partial h}$ as N and $\frac{\partial B}{\partial w}$ as D. So, $\frac{dw}{dh} = \frac{-\partial B}{\partial h} \div \frac{\partial B}{\partial w} = -N \div D$.</p> <p>The rate of change is about 3.85 pounds per inch.</p>	
<p>To estimate the weight change needed to compensate for growths of 0.5 inch, 1 inch, and 2 inches if the person's body-mass index is to remain constant, you need to find</p> $\Delta w \approx \frac{dw}{dh} (\Delta h)$ <p>at the various values of Δh.</p>	