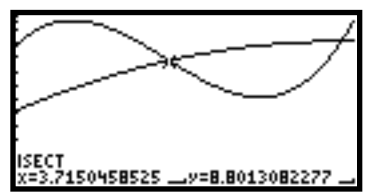
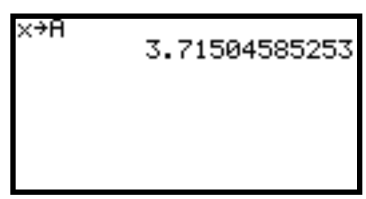
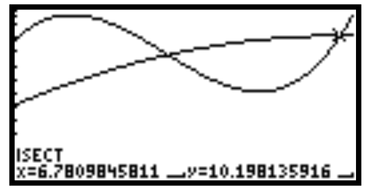
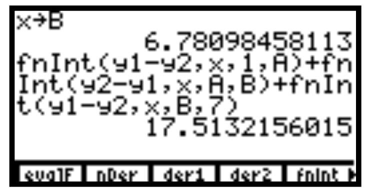
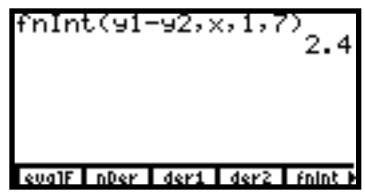
Chapter 7 Analyzing Accumulated Change: More Applications of Integrals

7.1 Differences of Accumulated Changes

This chapter helps you effectively use your calculator's numerical integrator with various applications pertaining to the accumulation of change. In this first section, we focus on the differences of accumulated changes.

7.1.1 FINDING THE AREA BETWEEN TWO CURVES Finding the area of the region enclosed by two functions uses many of the techniques presented in preceding sections. Suppose we want to find the difference between the accumulated change of $f(x)$ from a to b and the accumulated change of $g(x)$ from a to b where $f(x) = 0.3x^3 - 3.3x^2 + 9.6x + 3.3$ and $g(x) = {}^-0.15x^2 + 2.03x + 3.33$. The input of the leftmost point of intersection of the two curves is a and the input of the rightmost point of intersection is b.

Enter $f(x)$ in y1 and $g(x)$ in y2 in the y(x)= list. From Figure 7.9 in the text, we see that xMin = 1, xMax = 7, yMin = 0, and yMax = 12. Set these (RANGE or WIND) values and then graph the two functions. Next, find the two intersection points. (The x-values of these points of intersection will be the limits on the integrals we use to find the area.)	
TI-85 Press [MORE] [F1] (MATH) [MORE] [F5] (ISECT) **TI-86** Press [MORE] [F1] (MATH) [MORE] [F3] (ISECT) **Both** Use [◁] and/or [▷] to move near the point of intersection in the middle of the screen. Press [ENTER] each time the calculator pauses to tell it the first curve, the second curve, and your guess for the point of intersection.	
The leftmost visible point of intersection is displayed. To avoid making a mistak[redacted]opying the x-value and to eliminate as much round[redacted]rror as possible, return to the home screen and st[redacted]his value in A with the keystrokes [x-VAR] [STO▶] A	

Press GRAPH , and repeat the process to find the right-most intersection point. Do not forget to move the cursor near the intersection point on the right after pressing the ISECT key. Store the x-value in B.	
The combined area of the three regions is $\int_1^A (f-g)\,dx + \int_A^B (g-f)\,dx + \int_B^7 (f-g)\,dx.$ (If you are not sure which curve is on top in each of the various regions, trace the curves.)	
Use either the home screen or the graphics screen to find the value of the definite integral $\int_1^7 (f-g)\,dx = \int_1^7 \big(f(x)-g(x)\big)\,dx.$ Note that the value of the integral is *not* the same as the area between the two curves.	

7.3 Streams in Business and Biology

You will find your calculator very helpful when dealing with streams that are accumulated over finite intervals. However, because your calculator's numerical integrator evaluates only definite integrals, you cannot use it to find the value of an improper integral.

7.3.1 FUTURE VALUE OF A DISCRETE INCOME STREAM We use the sequence command to find the future value of a discrete income stream. The change in the future value at the end of T years that occurs because of a deposit of $\$A$ at time t where interest is earned at an annual rate of $100r\%$ compounded n times a year is

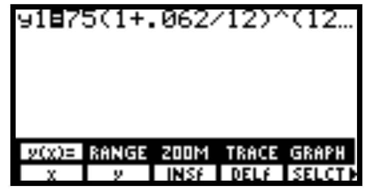
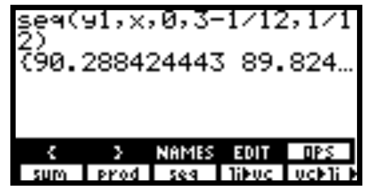
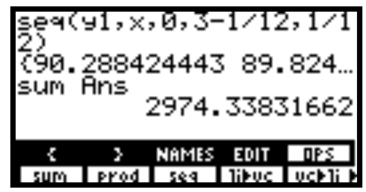
$$f(t) = A\left(1+\frac{r}{n}\right)^{n(T-t)} \text{ dollars per compounding period}$$

where t is the number of years since the first deposit was made. We assume the initial deposit is made at time $t = 0$ and the last deposit is made at time $T - \frac{1}{n}$. The increment for a discrete stream involving n compounding periods and deposits of $\$A$ at the beginning of each compounding period is $\frac{1}{n}$. Thus, the sequence command for finding the future value of the discrete income stream is *seq(f(x), x, 0, T–1/n, 1/n).*

Suppose that you invest \$75 each month in a savings account yielding 6.2% APR compounded monthly. What is the value of your savings in 3 years? To answer this question, note that the change in the future value that occurs due to the deposit at time t is

$$f(t) = 75\left(1+\frac{0.062}{12}\right)^{12(3-t)} \approx 75\left(1.005166667^{12}\right)^{(3-t)} \approx 75(1.06379)^{(3-t)}$$

Now, find the future value of this stream with $A = 75$, $r = 0.062$, $T = 3$, and $n = 12$.

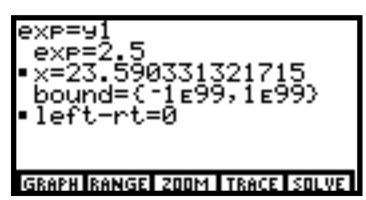
Enter *f(t)*, using x as the input variable, in location y1 of the y(x)= list. Note that if you want an exact answer you should enter the following: y1 = 75(1+ 0.062/12)^(12(3–x)) (You must carefully use parentheses with this form of the equation.)	
Return to the home screen. Enter seq (y1, x, 0, 3 –1/12, 1/12) . (Note that since you start counting at 0, the ending value will be one increment less than the number of years the money accumulates in the account.)	
The sequence is generated. (You can scroll through the list with ▷.) This list contains the heights of the 36 left rectangles in dollars per month. Because the input is in years, dividing by 12 and then multiplying by 12 converts the list to the areas of the 36 rectangles in dollars. The future value we seek is the sum of the heights of the rectangles.	

- When using definite integrals to approximate either the future value or the present value of a discrete income stream or to find the future value or the present value of a continuous stream, use the fnInt command to find the area between the appropriate continuous rate of change function and the *t*-axis from 0 to T.

7.4 Integrals in Economics

Consumers' and producers' surplus, being defined by definite integrals, are easy to find using the calculator. You should always draw graphs of the demand and supply functions and think of the surpluses in terms of area to better understand the questions being asked.

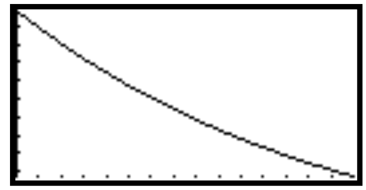
7.4.1 CONSUMERS' SURPLUS Suppose that the demand for mini-vans in the United States can be modeled by $D(p) = 14.12(0.933)^p - 0.25$ million mini-vans when the market price is p thousand dollars per mini-van.

At what price will consumers purchase 2.5 million mini-vans? We solve $D(p) = 2.5$ to find the price. Enter $14.12(0.933)^p - 0.25$ in y1. (You could use the solver on the home screen, but we intend to graph *D(p)*. Don't forget to use x as the input variable.)	
Return to the home screen and use the solver to find that $p \approx \$23.6$ thousand.	

You should always know how many solutions to expect. Because this is an exponential equation, it can have no more than one solution. Refer to Section 1.2.2 of this *Guide* for instructions on using the solver.

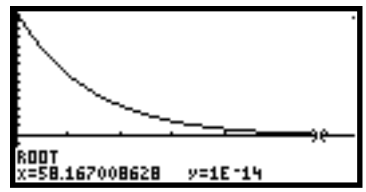
Now, let's find if the model indicates a possible price above which consumers will purchase no mini-vans. If so, we will find that price.

You should have the demand function in y1. Graph y1 in a suitable viewing window. Since the output is mini-vans, we set xMin = 0 and consider the graph for different values of xMax. First, set xMax = 15 (or some other value) and use ZFIT to draw the graph.



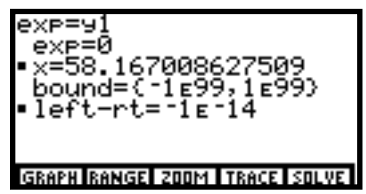
Because we are trying to see if the demand function crosses or touches the input axis, reset yMin to ⁻5 and xMax to a larger value, say 65. Use F3 (GRAPH) to graph the demand function.

It is difficult to tell if there is an intercept. So, use MATH (ROOT) to find, should it exist, the root of the demand function.



Instead of drawing the graph, you could have used the solver to find the location of the *x*-intercept. Notice that here we are looking for where y1 = 0.

Consumers will not pay more than about $58,200 per mini-van.



What is the consumers' surplus when 2.5 million mini-vans are purchased? We earlier found that the market price for this quantity is 23.59033 thousand dollars.

TI-85 The consumers' surplus is *approximately* the area of the region shown to the right (the trace value on the calculator can not set exactly to the market price).

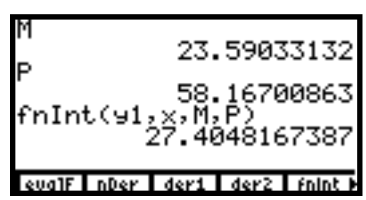
TI-86 You can type in the values for the upper and lower limits and obtain a much better approximation.

(The TI-85 does not shade the region when it finds the value of the integral.)

Find the consumers' surplus, which is the area of the shaded region, by finding the value of

$$\int_M^P D(p)\,dp = \int_{23.59033132}^{58.16700863} \left(14.12(0.933)^p\right) dp.$$

(Remember that you can avoid retyping the long decimal numbers by storing them to various memory locations as you find them.)

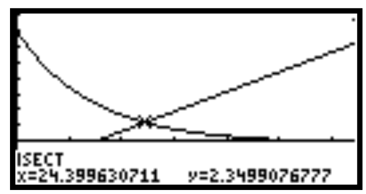
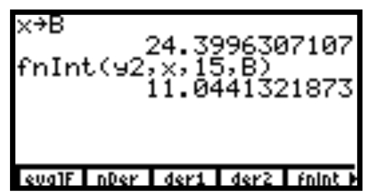


- Carefully watch the units involved in your computations. Refer to the statement of the problem and note that the height is measured in thousand dollars per mini-van and the width in million mini-vans. Thus, the area should be written in units that make sense in the context of this problem:

$$\text{height} * \text{width} \approx 27.4\left(\frac{\text{thousand dollars}}{\text{minivan}}\right)(\text{million minivans}) = \$\,27.4 \text{ thousand million}$$

$$= \$27.4\ (1{,}000)(1{,}000{,}000) = \$27.4(10^9) = \$27.4 \text{ billion dollars}$$

7.4.2 PRODUCERS' SURPLUS When dealing with supply functions, use definite integrals in a manner similar to that for consumers' quantities to find producers' revenue, producers' surplus, and so forth. To illustrate, suppose the demand for mini-vans in the United States can be modeled by $D(p) = 14.12(0.933)^p - 0.25$ million mini-vans and the supply curve is $S(p) = 0$ million mini-vans when $p < 15$ and $S(p) = 0.25p - 3.75$ million mini-vans when $p \geq 15$ where the market price is p thousand dollars per mini-van.

Enter the demand curve in y1 and the supply curve in y2. (Remember to use x as the input variable.) Draw a graph of the demand and supply functions in an appropriate window, say $0 \leq x \leq 65$ and $^-5 \leq y \leq 16$.	
Market equilibrium occurs when $D(p) = S(p)$. Use the methods of Section 7.1.1 of this *Guide* to find the intersection of the two curves. Store the x-value of the intersection as B.	
Producers' surplus is found by evaluating $\int_{15}^{B} S(p)\,dp$	

7.5 Average Value and Average Rates of Change

You need to carefully read any question involving average value in order to determine which quantity is involved. Considering the units of measure in the situation can be of tremendous help when trying to determine which function to integrate when finding average value.

7.5.1 AVERAGE VALUE OF A FUNCTION Suppose that the hourly temperatures shown below were recorded from 7 a.m. to 7 p.m. one day in September.

Time	7am	8	9	10	11	noon	1pm	2	3	4	5	6	7
Temp. (°F)	49	54	58	66	72	76	79	80	80	78	74	69	62

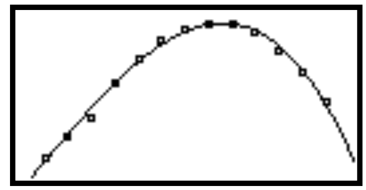
Enter the input data in L1 as the number of hours after midnight: 7 am is 7 and 1 pm is 13, etc.

First, we fit a cubic model

$t(h) = {}^{-}0.03526h^3 + 0.71816h^2 + 1.584h + 13.689$ °F

where h is the number of hours after midnight.

Graph this model on the scatter plot of the data and see that it provides a good fit.



Next, you are asked to calculate the average temperature. Because temperature is measured in this example in degrees Fahrenheit, the units on your result should be °F. When evaluating integrals, it helps to think of the units of the integration result as (height)(width) where the height units are the output units and the width units are the input units of the function that you are integrating. That is,

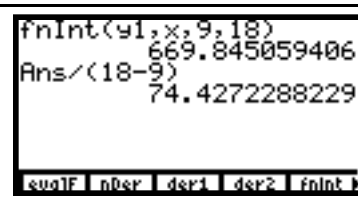
$$\int_{9 \text{ hours}}^{18 \text{ hours}} t(h) \text{ degrees } dh \quad \text{has units of (degrees)(hours).}$$

When we find the average value, we divide the integral by (upper limit – lower limit). So,

$$\text{average value} = \frac{\int_{9 \text{ hours}}^{18 \text{ hours}} t(h) \text{ degrees } dh}{18 \text{ hours} - 9 \text{ hours}} = \frac{\big(T(18) - T(9)\big) \text{ degrees} \cdot \text{hours}}{9 \text{ hours}}$$

where $T(h)$ is an antiderivative of $t(h)$. Because the "hours" cancel, the result is in degrees as is desired. Remember, when finding average value, *the units of the average value are always the same as the output units of the quantity you are integrating*.

Find the average value of the temperature between 9 a.m. and 6 p.m. to be approximately 74.4 °F.



The third part of this example asks you to find the average rate of change of temperature from 9 a.m. to 6 p.m. Again, let the units be your guide. Because temperature is measured in °F and input is measured in hours, the average rate of change is measured in output units per input units = °F per hour. Thus, we find the average rate of change to be

$$\text{average rate of change of temperature} = \frac{\big(t(18) - t(9)\big) \text{ °F}}{(18 - 9) \text{ hours}} = 0.98 \text{ °F per hour}$$

7.5.2 GEOMETRIC INTERPRETATION OF AVERAGE VALUE

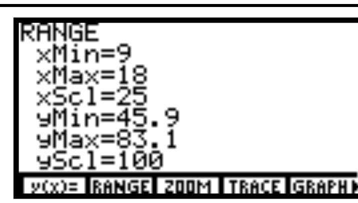
What does the average value of a function mean in terms of the graph of the function? Consider the model we found above for the temperature one day in September between 7 a.m. and 7 p.m.:

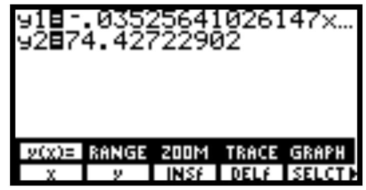
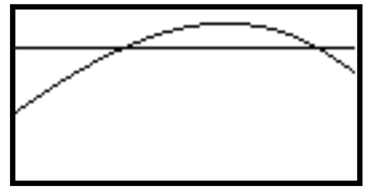
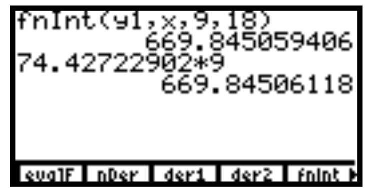
$t(h) = {}^{-}0.03526h^3 + 0.71816h^2 + 1.584h + 13.689$ degrees t hours after 7 a.m.

You should already have the unrounded model for $t(h)$ in y1. If not enter it now.

Reset the viewing window to that given on the right. Notice the input is between 9 a.m. and 6 p.m.

TI-86 Note: Turn off all scatter plots.



Enter the average value in location y2 of the y(x)= list.	
Graph both functions. Notice that the area of the rectangle whose height is the average temperature is (74.42722902)(18 – 9) ≈ 669.8.	
The area of the rectangle equals the area of the region below the temperature function *t(h)* and above the *t*-axis between 9 a.m. and 6 p.m. (In this application, the area does not have a meaningful interpretation because its units are (degrees)(hours).)	

7.6 Probability Distributions and Density Functions

Most of the applications of probability distributions and density functions use technology techniques that have already been discussed. Probabilities are areas whose values can be found by integrating the appropriate density function. A cumulative density function is an accumulation function of a probability density function.

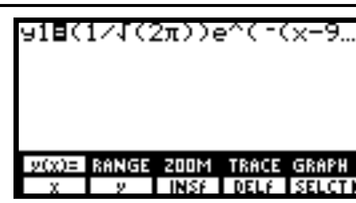
You will find your calculator's numerical integrator especially useful when finding means and standard deviations for some probability distributions because those integrals often contain expressions for which we have not developed an algebraic technique for finding an antiderivative.

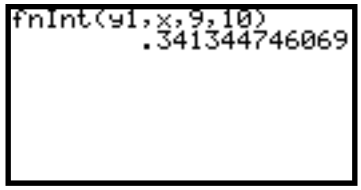
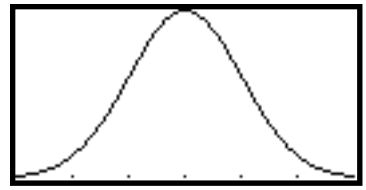
7.6.1 NORMAL PROBABILITIES

The normal density function is the most well known and widely used probability distribution. If you are told that a random variable x has a normal distribution $N(x)$ with mean μ and standard deviation σ, the probability that x is between a and b is

$$\int_a^b N(x)\,dx = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx.$$

Suppose a light-bulb manufacturer advertises that the average life of the company's soft-light bulb is 900 hours with a standard deviation of 100 hours. Suppose also that we know the distribution of the life of these bulbs, with the life span measured in hundreds of hours, is modeled by a normal density function. Find the probability that any one of these light bulbs last between 900 and 1000 hours.

We are told that $\mu = 9$ and $\sigma = 1$. (Note that x is measured in hundreds of hours, so we must convert to these units.) Enter $N(x) = (1/\sqrt{(2\pi)})e^{\wedge}(^{-}(x-9)^2/2)$ in y1 using $\mu = 9$ and $\sigma = 1$. (Very carefully watch your use of parentheses.)	

Find $P(9 < x < 10)$. Graph *N(x)*, remembering that nearly all the area is between $\mu - 3\sigma$ and $\mu + 3\sigma$ = [6, 12]. Use ZFIT to graph.		
With the graph on the screen, use MORE F1 (MATH) **Error!**		