

Chapter 6 Accumulating Change: Limits of Sums and the Definite Integral



6.1 Results of Change

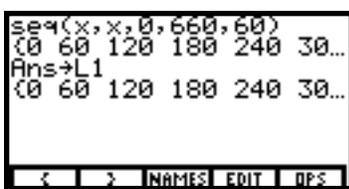
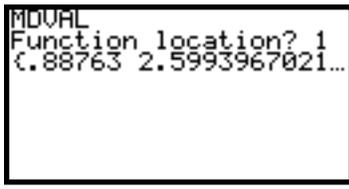
We have thus far seen how to use the calculator to work with rates of change. In this chapter we consider the results of change. Your calculator has many useful features that will assist you in your study of the accumulation of change.

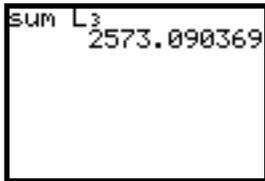
6.1.1 APPROXIMATIONS WITH LEFT RECTANGLES The calculator's lists can be used to perform the calculations needed to approximate, using left rectangles, the area between the horizontal axis, a (non-negative) rate of change function, and two input values.

Consider, for example, a model for the number of customers per minute who came to a Saturday sale at a large department store between 9 a.m. and 9 p.m.:

$$c(m) = (4.58904 \cdot 10^{-8})m^3 - (7.78127 \cdot 10^{-5})m^2 + 0.03303m + 0.88763$$

customers per minute where m is the number of minutes after 9 a.m.

<p>Enter this model in the $y(x)=$ list. (If you have other equations in the graphing list, clear them.)</p> <p>(Remember that "10 to a power" is notated by 10^{\square} on the calculator. Access</p>	
<p>Suppose we want to estimate the total number of customers who came to the sale between $x = 0$ and $x = 660$ (12 hours) with 12 rectangles and $\Delta x = 60$.</p> <p>Enter these x-values in list L1. Recall that a quick way to do this is with \square (2nd) \square (LIST) \square (F5) (OPS) \square (MORE) \square (F3) (seq) \square (x-VAR) \square (,) \square (x-VAR) \square (,) \square (0) \square (660) \square (,) \square (60) \square (ENTER) .</p> <p>Store the values in list L1 with \square (STO) \square (L) \square (ALPHA) 1. (See Section 2.1.1 of this <i>Guide</i>.)</p>	 <p>Notice that 9 p.m. is 720 minutes after 9 a.m. When using <i>left</i>-rectangle areas, the <i>rightmost</i> data point is <u>not</u> included.</p>
<p>TI-85 Enter y-values calculated from the model in list L2 using program MDVAL (model values). This program is in the TI-85/86 Appendix.</p> <p>You can have the model entered in any of the first four locations of the $y(x)=$ list. Tell the program which y location at the Function location? prompt. L2 now contains the <i>heights</i> of the 12 rectangles.</p>	

<p>Since the <i>width</i> of each rectangle is 60, the area of each rectangle is $60 \cdot \text{height}$. Enter the <i>areas</i> of the 12 rectangles in list L3.</p> <p>Press and hold down  to view all the values in L3 or use the LIST EDIT menu to view them.</p>	
<p>TI-86 Enter $c(x)$-values calculated from the model in list L2 with the keystrokes   (STAT)  (EDIT), press  (to darken the name L2), and type $y_1(L1)$ with y_1   L1  .</p> <p>List L2 contains the <i>heights</i> of the 12 rectangles.</p>	
<p>Since the width of each rectangle is 60, the area of each rectangle is $60 \cdot \text{height}$. Enter the <i>areas</i> of the 12 rectangles in list L3 by using  to darken the name of list L3 and then typing $60L2$ with 6 0  L2 .</p>	
<p>Both On the home screen, find the sum of the areas of the rectangles with   (LIST)  (OPS)   (SUM)   (NAMES)  (L3) .</p> <p>We estimate, using 12 left rectangles, that about 2,573 customers came to the Saturday sale.</p>	

Note: The values in Table 6.2 in your text and the final result differ slightly than those in your lists. This is because the unrounded model found with the data was used for computations in the text. If you have the unrounded model available, you should use it instead of the rounded model $c(m)$ that is given above.

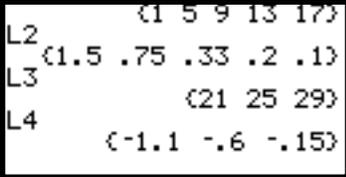
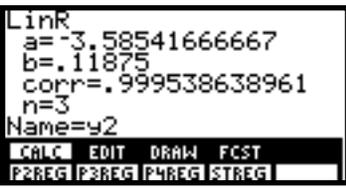
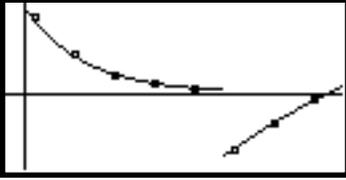
6.1.2 APPROXIMATIONS WITH RIGHT RECTANGLES When using left rectangles to approximate the results of change, the rightmost data point is not the height of a rectangle and is not used in the computation of the left-rectangle area. Similarly, when using right rectangles to approximate the results of change, the *leftmost* data point is not the height of a rectangle and is not used in the computation of the right-rectangle area.

The following data shows the rate of change of the concentration of a drug in the blood stream in terms of the number of days since the drug was administered:

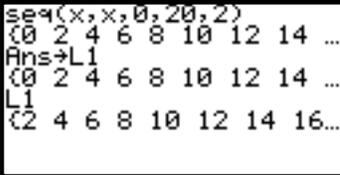
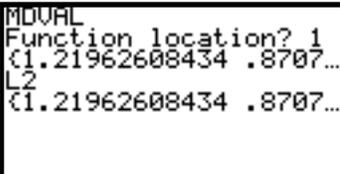
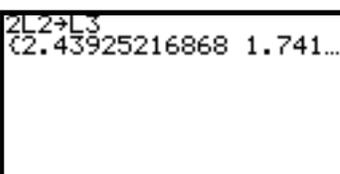
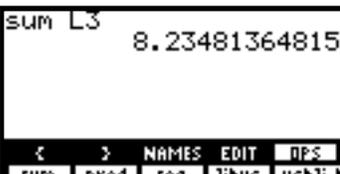
Day	1	5	9	13	17	21	25	29
Concentration	1.5	0.75	0.33	0.20	0.10	-1.1	-0.60	-0.15
ROC ($\mu\text{g}/\text{mL}/\text{day}$)								

First, we fit a piecewise model to the data.

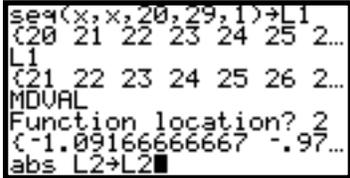
<p>Clear all lists. Enter the days in L1 and the rate of change of concentration in L2.</p> <p>Draw a scatter plot of the data. It is obvious that a piecewise model should be used with $x = 20$ as the "break" point.</p>	
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<p>TI-85 Go to the list editor and delete the last three data points from lists L1 and L2 with F2 (DELI) . Enter the list {21, 25, 29} in L3 and in L4 enter the corresponding outputs: {-1.1, -0.60, -0.15}.</p> <p>Fit an exponential model to the data in L1 and L2 and put that model in y1.</p>	
<p>Fit a linear model to the data in L3 and L4. When you go to the STAT (CALC) menu, you must tell the calculator that you are using different lists by entering L3 as the xlist Name and L4 as the ylist Name.</p>	
<p>Then, fit a linear model with ENTER F2 (LINR) .</p> <p>Copy the linear model to location y2.</p>	
<p>TI-86 Go to the lists with 2nd + (STAT) F2 (EDIT) . Delete the last three data points from lists L1 and L2 by going to the location of those values and pressing DEL . Enter {21, 25, 29} in L3 and in L4 enter the corresponding outputs: {-1.1, -0.60, -0.15}.</p> <p>Fit an exponential model to the data in L1 and L2 and put that model in y1.</p>	
<p>Fit a linear model to the data in lists L3 and in L4 and copy the model to location y2 with 2nd + (STAT) F1 (CALC) F3 (LinR) L3 , L4 , y2</p>	
<p>Both (optional) If you want to graph this model on the original scatter plot, first re-enter the data with all the inputs in list L1, all the outputs in list L2. Return to the graphing list and turn off y1 and y2. Enter (y1)(x ≤ 20) + (y2)(x > 20) in y3. Choose DrawDot format, draw the scatter plot and y3 will graph over it.</p>	

Now, we determine the right-rectangle area for $0 \leq x \leq 20$:

<p>Clear all lists. Because x starts at 0 and $\Delta x = 2$, enter in L1 the values 0, 2, 4, 6, ..., 20 or use <code>seq(x, x, 0, 20, 2)</code> to generate the list.</p> <p>Then, because we are using right rectangles, go to the list editor and <i>delete</i> the leftmost input value (0).</p>	
<p>TI-85 Enter y-values calculated from the model in y_1 into list L2 using program MDVAL.</p> <p>L2 now contains the <i>heights</i> of the 12 rectangles.</p>	
<p>Each rectangle has width 2. The heights of the rectangles are in list L2. Enter the rectangle areas in list L3 as $\text{width} \times \text{height} = 2 \times L_2$.</p>	
<p>TI-86 Use the model in y_1 to generate the outputs as $L_2 = y_1(L_1)$. Remember that the cursor must be on top of the list (with the name L2 darkened) in order to generate the entire list of output values.</p> <p>(Did you remember to use a lower-case y?)</p>	
<p>Each rectangle has width 2. The heights of the rectangles are in list L2. Enter the rectangle areas in list L3 as $\text{width} \times \text{height} = 2 \times L_2$.</p>	
<p>Both On the home screen, find the sum of the areas of the rectangles with <code>2nd</code> <code>[]</code> (LIST) <code>F5</code> (OPS) <code>MORE</code> <code>F1</code> (SUM) <code>EXIT</code> <code>F3</code> (NAMES) <code>F3</code> (L3) <code>ENTER</code>.</p> <p>We estimate, using 10 right rectangles, that the change in concentration was approximately $8.24 \mu\text{g/mL}$.</p>	

To estimate, using right rectangles, the change in drug concentration for $20 \leq x \leq 29$ days with $\Delta x = 1$, follow the same procedure as above. The values in L1 begin with 21 because we must eliminate the leftmost value (20) when using right rectangles. You should use the absolute value of the model in y_2 to generate L2.

<p>Since the width of each rectangle is 1, L2 contains the areas as well as the heights of the 9 right rectangles.</p> <p>sum L2 = 5.55 µg/mL</p>	 <p style="text-align: center;">TI-85</p>	<p style="text-align: center;">TI-86</p>
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6.2 Trapezoid and Midpoint Rectangle Approximations

You can compute areas of trapezoids on the home screen of your calculator or use the fact that the trapezoid approximation is the average of the left- and right-rectangle approximations. Areas of midpoint rectangles are found in the same manner as left and right rectangle areas except that the midpoint of the base of each rectangle is in L1 and no data values are deleted. However, such procedures can become tedious when the number, n , of subintervals is large.

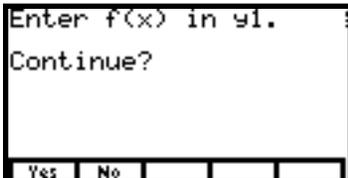
6.2.1 SIMPLIFYING AREA APPROXIMATIONS When you have a model $y = f(x)$ in location y1 in the y(x)= list, you will find program NUMINTGL very helpful in determining left-rectangle, right-rectangle, midpoint-rectangle, and trapezoidal numerical approximations for accumulated change. Program NUMINTGL is listed in the TI-85/TI-86 Appendix.

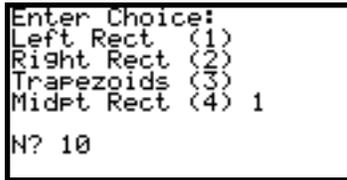
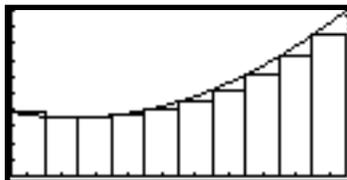
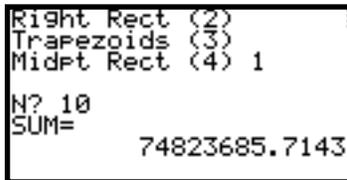
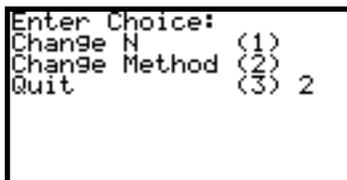
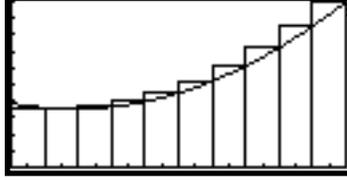
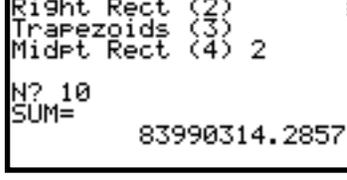
WARNING: This program will not work properly if you have any functions in your y(x)= list that have letters other than x in them. Delete any such functions (like the logistic equation) before using program NUMINTGL. If you receive an error while running the program, you may have a picture or program stored to a single-letter name. For instance, if program NUMINTGL is trying to store a number in T and you have a program called T, the calculator stops. Delete or rename any programs or pictures that you have called by a single-letter name before continuing.

We illustrate using this program with a model for the Carson River flow rates:

$$f(h) = 18,225h^2 - 135,334.3h + 2,881,542.9 \text{ cubic feet per hour}$$

h hours after 11:45 a.m. Wednesday. (The complete model found from the data in your text is used for all the following calculations.)

<p>Have $f(h)$ in y1. Your function <i>must</i> be in y1 for program NUMINTGL to operate.</p> <p>If you wish to view the approximating rectangles or trapezoids, the program will automatically draw a graph of the function when enter the input interval.</p>	
<p>Run program NUMINTGL by pressing PRGM (NAMES) and the appropriate F-key. At this point, if you did not enter the function in y1, press F2 (No) to exit the program. Enter y1 and re-run the program. If your function is in y1, press F1 (Yes) to continue.</p>	

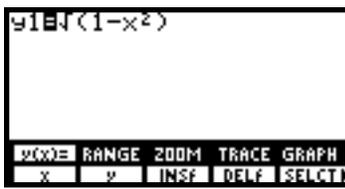
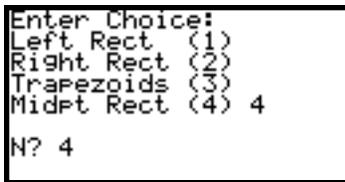
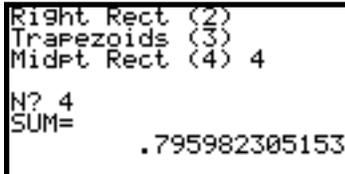
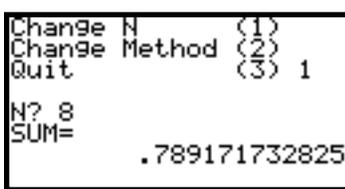
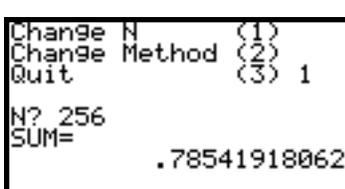
<p>At the next prompt, press F1 (Yes) to draw the approximating figures or press F2 (No) to obtain only the numerical approximations to the area between $f(h)$ and the horizontal axis. We choose to see some pictures and press F1 (Yes).</p>	
<p>At the Lower Limit? prompt, type 0 ENTER, and at the Upper Limit? prompt, 20 ENTER. You are next shown a menu of choices. Press 1 ENTER to find a left-rectangle approximation.</p>	
<p>Input 10 at the N? prompt and press ENTER. The ten left rectangles are shown.</p>	
<p>The left-rectangle area approximation is displayed as "sum" when you press ENTER.</p>	
<p>Press ENTER once more and more choices are displayed. Suppose we now want to find the approximating area and see the figure using 10 right rectangles. Press 2 to change the method, and then press 2 ENTER to choose right rectangles.</p>	
<p>Enter 10 at the N? prompt and view the right rectangles. Again press ENTER and the right-rectangle area approximation is displayed.</p>	
<p>Continue on in this manner and find the trapezoid and midpoint rectangle approximations to the area. When you finish, press ENTER and enter 3 to QUIT.</p>	



6.3 The Definite Integral as a Limit of Sums

This section introduces you to a very important and useful concept of calculus -- the definite integral. Your calculator can be very helpful as you study definite integrals and how they relate to the accumulation of change.

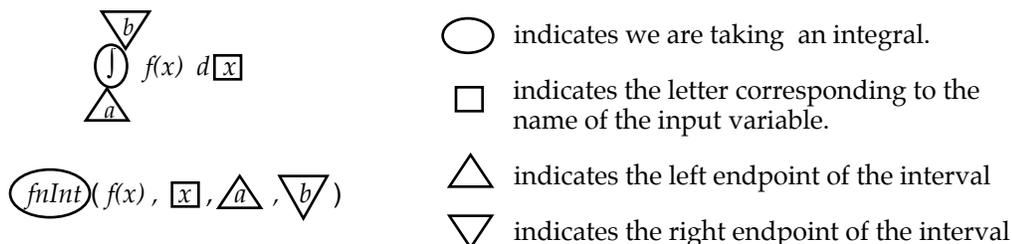
6.3.1 LIMITS OF SUMS When you are looking for a trend in midpoint-rectangle approximations to the area between a non-negative function and the horizontal axis between two values of the input variable, program NUMINTGL is extremely useful! However, for this use of the program, it is not advisable to draw pictures when n , the number of subintervals, is large.

<p>To construct a chart of midpoint approximations for the area between $f(x) = \sqrt{1-x^2}$ and the x-axis from $x=0$ and $x=1$, first enter $f(x)$ in $y1$.</p> <p>(Don't forget to enclose $1-x^2$ in parentheses.)</p>	
<p>Start program NUMINTGL. Since many subintervals are used with the idea of a limit of sums, do <i>not</i> choose to draw the pictures unless you have time to sit and wait for all the rectangles to draw!</p> <p>After entering 0 and 1 as the respective lower and upper limits, choose option 4 for midpoint rectangles.</p>	
<p>Input some number of subintervals, say $N = 4$.</p> <p>Record on paper the midpoint area approximation 0.7959823052. (If you want an answer accurate to the thousandths position, record at least 4 decimal places.)</p>	
<p>Press ENTER and choose option 1: CHANGE N. Double the number of subintervals to $N = 8$.</p> <p>Record the midpoint approximation 0.7891717328 (again, to at least 4 decimal places).</p>	
<p>Continue on in this manner, each time choosing option 1: CHANGE N and doubling N until a trend is evident.</p> <p>(Finding a trend means that you can tell what value the approximations are getting closer and closer to, within a specified accuracy, without having to run the program ad infinitum!) Choose QUIT (3) to stop.</p>	

- Remember that the trend indicated by the limit of sums can be interpreted as the area of the region between the function and the input axis *only* when that region lies above the input axis. When the region lies below the input axis, the trend is the negative of the area of the region.

 **6.5 The Fundamental Theorem**

Recall that **der1** is the calculator's numerical derivative and provides, in most cases, a good value for the instantaneous rate of change of a function when that rate of change exists. Your calculator can also give you a numerical approximation for a definite integral of a function. This numerical integrator is called **fnInt**, and the correspondence between the mathematical definite integral notation $\int_a^b f(x) dx$ and the calculator's notation **fnInt**(*f*(*X*), *X*, *a*, *b*) is as shown below:



You will find that in most cases, your calculator's numerical integrator gives a very close decimal approximation of the exact value of a definite integral. This use is examined in Section 6.6 of this *Guide*. In this section, we illustrate the Fundamental Theorem of Calculus and see how to draw the graph of a general antiderivative of a function.

6.5.1 THE FUNDAMENTAL THEOREM OF CALCULUS This theorem tells us that the derivative of an antiderivative of a function is the function itself. Let us view this theorem both numerically and graphically. Your calculator's numerical integrator is accessed with the keystrokes **Error!**

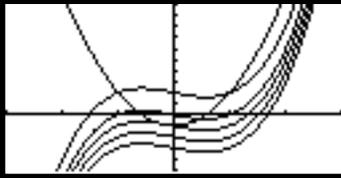
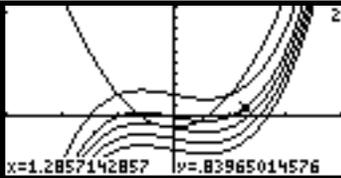
<p>Consider $F'(x) = \frac{d}{dx} \left(\int_1^x 3t^2 + 2t - 5 dt \right)$. The FTC tells us that $F'(x)$ should equal $3x^2 + 2x - 5$. Enter the functions shown to the right.</p>	<p>$y1 = 3x^2 + 2x - 5$ $y2 = nDer (fnInt (y1, x, 1, x), x, x)$</p> <p><i>Note:</i> der1 cannot be used here, so use nDer for dy/dx.</p>
<p>Use the TABLE and input some different values of x. Other than occasionally having a little roundoff error, $y1$ and $y2$ are the same!</p>	<div style="border: 1px solid black; padding: 5px; text-align: center;"> <pre>[[(-5,60) (-5,60)] [(-3,16) (-3,16)] [(-1,-4) (-1,-4)] [(1,0) (1,0)] [(3,28) (3,28)] [(5,80) (5,80)] [(7,156) (7,156)]]</pre> </div> <p style="text-align: center;">TI-85 TI-86</p>
<p>Find a suitable viewing window such as x between -4.7 and 4.7 and y between -6 and 3.1. Draw the graphs of $y1$ and $y2$ separately in this same window and then draw them together. They give the same graph! (The graph of $y2$ will take a while to draw.)</p>	

Enter several other functions in $y1$ and perform the same explorations as above. Confirm your results with derivative and integral formulas. Are you convinced?

6.5.2 DRAWING ANTIDERIVATIVE GRAPHS All antiderivatives of a specific function differ only by a constant. We explore this idea using the function $f(x) = 3x^2 - 1$ and its antiderivative $F(x) = x^3 - x + C$.

The correct syntax for the calculator's numerical integrator is $\text{fnInt}(f(X), X, a, b)$ where $f(x)$ is the function you are integrating, x is the variable of integration, a is the lower limit on the integral, and b is the upper limit on the integral. You do *not* have to use x as the variable unless you are graphing the integral or evaluating it using the calculator's table.

Note that you must supply both a lower limit and an upper limit for fnInt . We can use x for the upper limit, but not both the upper and lower limits. Since we are working with a general antiderivative in this illustration, we do not have the starting point for the accumulation. We therefore just choose some value, say 0, to use as the lower limit to illustrate drawing antiderivative graphs. If you choose a different lower limit, your results will differ from those shown below by a constant.

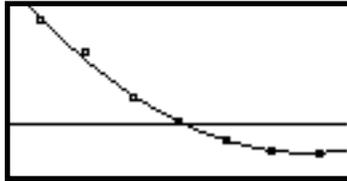
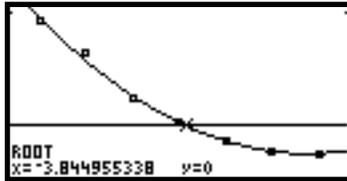
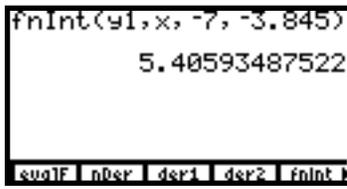
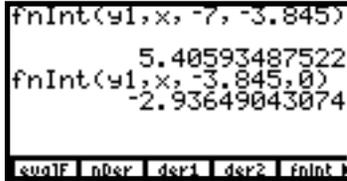
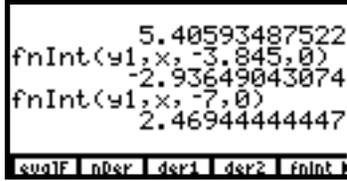
<p>Enter $f(x)$ in $y1$, $\text{fnInt}(y1, x, 0, x)$ in $y2$, and $F(x)$ in $y3$, $y4$, $y5$, $y6$, and $y7$ (using a different value of C in each location)</p> <p>(You can try different values of C than those shown on the right.)</p>	
<p>Find a suitable viewing window and graph all the functions. (Try x between -3 and 3 and y between -5 and 10.)</p> <p>It seems that the only difference in the graphs (other than the graph of $y1$) is that the y-intercept is different. But, isn't C the y-intercept?</p>	
<p>Trace the graphs and then jump between them with . It appears that $y2$ and $y3$ are the same.</p>	

Warning: The methods of Sections 4.4.1 and/or 4.4.2 that we used to check antiderivative formulas are not valid with general antiderivatives. Why? The answer is because in this situation you must choose arbitrarily choose a value for the constant of integration and you must arbitrarily choose a value for the lower limit in order to use the calculator's numerical integrator. However, for most rate-of-change functions where $f(0) = 0$, the calculator's numerical integrator values and your antiderivative formula values should differ by the same constant at every input value where they are defined.

 **6.6 The Definite Integral**

When using the numerical integrator on the home screen, enter $\text{fnInt}(f(x), x, a, b)$ for a specific function $f(x)$ with input x and specific values of a and b . (Remember that the input variable does not have to be x on the home screen.) If you prefer, $f(x)$ can be in the $y(x)=$ list and referred to as $y1$ (or whatever location you have it in) when using fnInt .

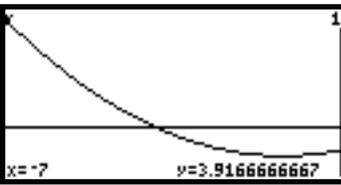
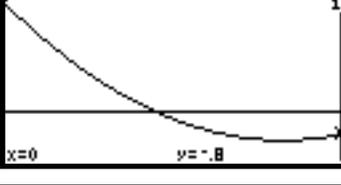
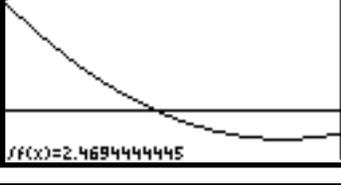
6.6.1 EVALUATING A DEFINITE INTEGRAL ON THE HOME SCREEN We illustrate the use of your calculator's numerical integrator with the model for the rate of change of the average sea level in meters per year during the last 7000 years. A model for these data is $r(t) = 0.14762t^2 + 0.35952t - 0.8$ meters per year t thousand years from the present. (Note that t is negative since we are talking about the past.)

<p>As we have previously mentioned, you should always use the full model (not rounded) for any model you find from data.</p> <p>Find a quadratic to fit the data shown in the <i>Changing Sea Levels</i> example in your text, and enter it in some location of the $y(x)=$ list, say $y1$.</p>	
<p>The model provides a good fit to the data.</p> <p>Because we are asked for <i>area</i> between the input axis and the function, we must find where the function becomes negative.</p>	
<p>Use the ideas of Section 1.2.3 of this <i>Guide</i> to find the x-intercept (root) of $y1$ between 0 and 7 to be $x \approx -3.845$.</p>	
<p>Return to the home screen and type the expression shown to the right. Remember that fnInt is accessed with the keystrokes $\boxed{2\text{nd}} \boxed{+}$ (CALC) $\boxed{F5}$ (fnInt).</p> $\int_{-7}^{-3.845} r(t) dt \approx 5.4 \text{ meters}$	
<p>The area of the region above the x-axis is about 5.4 meters. Now, find the area of the region below the x-axis. This area equals the negative of the definite integral of the function over that region.</p>	
<p>Evaluate $\int_{-3.845}^0 r(t) dt$ as shown on the right. The area of the region below the x-axis is about 2.9 meters.</p>	
<p>Find $\int_{-7}^0 r(t) dt$.</p> <p>Note that this value is <i>not</i> the sum of the two areas. It is their difference.</p>	

- If you evaluate a definite integral using antiderivative formulas and check your answer with the calculator using fnInt, you may sometimes find a slight difference in the last few decimal places. Remember, the TI-85 and the TI-86 are evaluating the definite integral using an approximation technique.

6.6.2 EVALUATING A DEFINITE INTEGRAL FROM THE GRAPHICS SCREEN

Provided that a and b are possible x -values when you trace the graph, you can find the value of the definite integral $\int_a^b f(x) dx$ from the graphics screen. (For “nice” trace numbers, you can often find the exact x -value you need if you graph in the ZDECM screen or set the viewing window so that $xMax - xMin$ equals a multiple of 12.6.)

<p>Have the model for the average sea level in y1 (see Section 6.6.1).</p> <p>Suppose we want to find $\int_{-7}^0 r(t) dt$.</p>	
<p>TI-85 Because -7 is not a possible trace value in the window set by the stat plot, change $xMin$ to -7 and $xMax$ to 0.</p> <p>Graph y1. Press MORE F1 (MATH) F5 ($\int f(x) dx$). The blinking cursor means the TI-85 is waiting for you to give it the lower limit on the integral. Press and hold ◀ until you reach -7.</p>	
<p>Press ENTER to set the lower limit. The calculator is now waiting for you to give it the upper limit on the integral.</p> <p>Press and hold ▶ until you reach $X = 0$.</p>	
<p>Press ENTER to calculate the numerical value of the definite integral.</p>	
<p>TI-86 This calculator allows you to input any value of x between $xMin$ and $xMax$ when finding the value of a definite integral from the graphics screen.</p> <p>Have all STAT PLOTS turned off. Set the window shown on the right -- the scatter plot window for the average sea level data with $xMax$ reset to 0.</p>	

<p>Draw the graph of $r(t)$. Press MORE F1 (MATH) F3 ($\int f(x)dx$). When the TI-86 asks for a lower limit, simply type -7 and then press ENTER.</p>	
<p>When the calculator asks for an upper limit, type 0 and then press ENTER.</p> <p>Note that if you had not previously changed $xMax$ from -0.4 to 0, you would have gotten an ERR: INVALID message at this point because 0 would not have been a value included between $xMin$ and $xMax$.</p>	
<p>The value of the integral $\int_{-7}^0 r(t) dt$ is displayed and the two regions are shaded.</p>	