Chapter 5 Analyzing Change: Extrema and Points of Inflection

5.1 Optimization

Your calculator can be very helpful in checking your analytic work when you find optimal points and points of inflection. When you are not required to show work using derivatives or when a very good approximation to the exact answer is all that is required, it is a very simple process to use your calculator to find optimal points and inflection points.

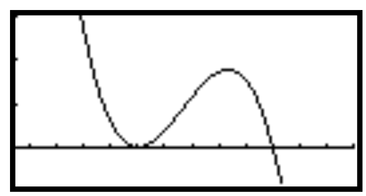
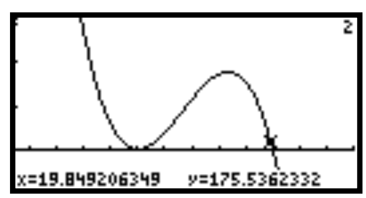
5.1.1 FINDING *X*-INTERCEPTS OF SLOPE GRAPHS

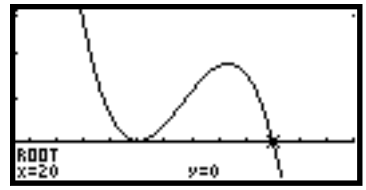
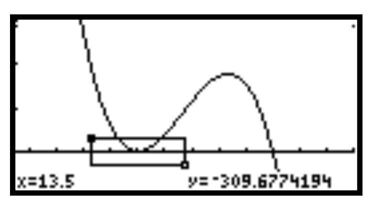
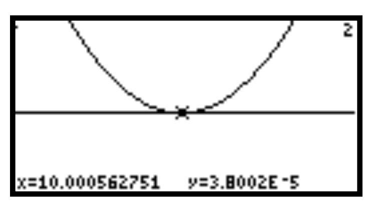
Where the graph of a function has a local maximum or minimum, the slope graph has a horizontal tangent. Where the tangent line is horizontal, the derivative of the function is zero. Thus, finding where the slope graph *crosses* the input axis is the same as finding where a relative maximum or a relative minimum occurs.

Consider, for example, the model for a cable company's revenue for the 26 weeks after it began a sales campaign:

$$R(x) = {}^{-}3x^4 + 160x^3 - 3000x^2 + 24{,}000x \quad \text{dollars}$$

where x is the number of weeks since the cable company began sales.

Enter $R(x)$ in the y1 location of the y(x)= list. Enter either the calculator's derivative *or* your derivative in the y2 location. Turn y1 off (i.e., deselect y1). (If you use your derivative, be sure to first check to see that your derivative and the calculator's derivative are the same!)	
The statement of the problem indicates that x should be graphed between 1 and 26. Graph with ZFIT and then reset the vertical view so that you can better see the high and low points (something like yMin = ⁻800 and yMax = 3000). Draw the slope graph with F5 (GRAPH) .	
TI-85 We find any x-intercepts of the slope graph with GRAPH MORE F1 (MATH) F3 (ROOT) and then use ◁ and/or ▷ to move the cursor near where the graph crosses the x-axis.	

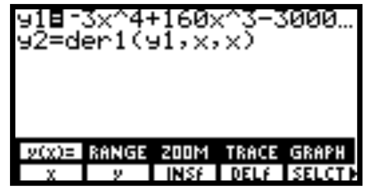
Next, press ENTER . The rightmost x-intercept (root) is at 20. That is, $R'(x) = 0$ at $x = 20$.	
TI-86 We find any x-intercepts of the slope graph by pressing GRAPH MORE F1 (MATH) F1 (ROOT) . Next, press and hold ▷ to move the cursor anywhere to the *left* of where the graph crosses the axis. Press ENTER to mark the location of the *left* bound.	
Use ▷ again to move the cursor to the *right* of the x-intercept, and press ENTER to mark the location of the *right* bound. Notice that the calculator has marked the interval between your two bounds with small triangles at the top of the screen. *The intercept must be between these two marks.*	
You are next asked to provide a guess. Any value in the interval will do. Use ◁ to move near the intercept and press ENTER .	
The location of the x-intercept is displayed. We see that $R'(x) = 0$ at $x = 20$. (The numerical process used to find the root sometimes includes rounding errors. The output should, of course, be 0 but is printed on the screen as $^{-}0.000000005$.)	
Both Now, you must determine if the derivative graph crosses or just touches the x-axis at the location to the left of this intercept. Use ZOOM F1 (BOX) as many times as necessary to magnify that portion of the graph to see what happens there. (See Section 3.3.1 of this *Guide*.)	
After using BOX several times, we see that the graph just touches and does not cross the x-axis near $x = 10$. Using the information gained from $R'(x)$ and the graph of $R(x)$, we find that revenue was greatest 20 weeks after the cable company began sales.	

5.1.2 FINDING OPTIMAL POINTS Once you draw a graph of a function that clearly shows any optimal points, finding the location of those high points and low points is an easy task for your calculator. When a relative maximum or a relative minimum exists at a point, your calculator can find it in a few simple steps. We again use the cable company revenue equation, *R(x)*, from Section 5.1.1.

Enter *R(x)* in the y1 location of the y(x)= list.

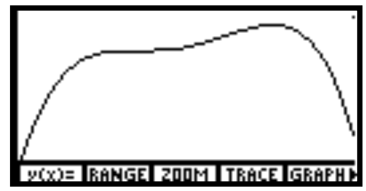
(Any location will do, but if you have other equations in the graphing list, turn them off or clear them.)

Next, draw a graph of the revenue, *R(x)*.

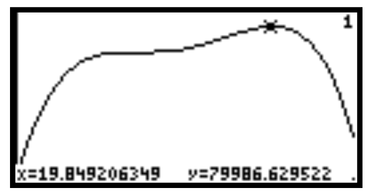


The input *x* should be graphed between 1 and 26.

Set these values and draw a graph with ZFIT. Reset yMax to a slightly larger value, say 85,000, to give a little more room at the top of the screen.

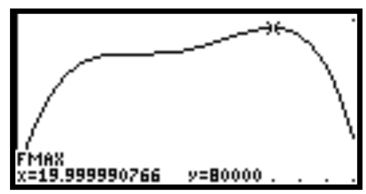


TI-85 Prepare to find the local (relative) maximum by first pressing GRAPH MORE F1 (MATH) MORE F2 (FMAX) . Press and hold ▷ to move the cursor near the highest point on the graph.

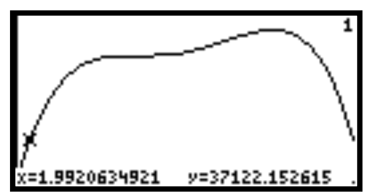


Press ENTER and the maximum value is displayed.

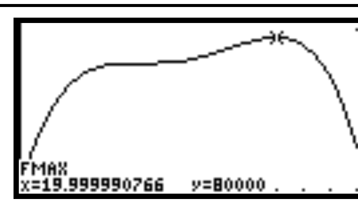
Again, rounding during the calculator's routine can sometimes produce results that are not exact. We see that the revenue is greatest at 20 weeks with a value of *R*(20) = $80,000.



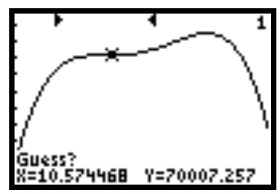
What about the other part of the curve that may contain a peak? Follow the same procedure as indicated above, but this time put the cursor far to the left side of the graph. (We are trying to see if there is a local maximum somewhere around $x = 10$.)



Notice that the calculator returns the maximum at the same value as before. This indicates that there is not another relative maximum value of *R(x)* shown on this screen. (Remember that we found that R′(x) only touched and did not cross the *x*-axis at $x = 10$.)



TI-86 Prepare to find the local (relative) maximum by first pressing GRAPH MORE F1 (MATH) F5 (FMAX). Next, press ▷ to move the blinking cursor that appears to a position to the *left* of the high point. Press ENTER to mark the *left* bound for *x*.

Next, press ▷ to move the blinking cursor to a position to the *right* of the high point. Press ENTER to mark the *right* bound for x.	
Use ◁ to move the cursor near your estimate of the high point and press ENTER . The TI-86 uses your guess to locate the highest point in the region between the two bound marks. The maximum value and the x-value at which it occurs is displayed at the bottom of the screen.	
Revenue is greatest at 20 weeks with a value of $R(20) = \$80{,}000$. Return to the home screen and verify that y1(20) gives the value displayed at the bottom of the graphics screen.	
What about the other part of the curve that may contain a peak? Follow the same procedure as indicated above to investigate the curve on either side of $x = 10$.	
Notice that the calculator returns the maximum at the right end of the interval. This indicates that there is not a relative maximum value in the interval we marked to test. (Recall that we found that $R'(x)$ only touched and did not cross the x-axis at $x = 10$.)	

Both The methods of this section also apply to finding relative or local minimum values of a function. The only difference is that to find the minimum instead of the maximum, you would use the FMIN menu key that is to the left of the FMAX menu key.

5.2 Inflection Points

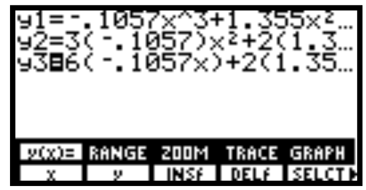
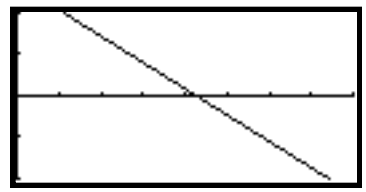
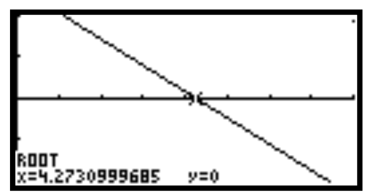
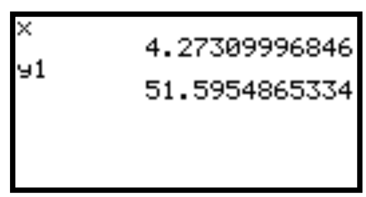
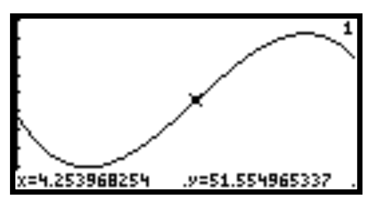
As was the case with optimal points, your calculator can be very helpful in checking your analytic work when you find points of inflection. You can also use the methods illustrated in Section 5.1.2 of this *Guide* to find the location of any maximum or minimum points on the graph of the first derivative to find the location of any inflection points for the function. In fact, your calculator offers three graphical methods for finding inflection points. We investigate these as well as the analytic method in the following discussions.

5.2.1 FINDING *X*-INTERCEPTS OF A SECOND DERIVATIVE GRAPH We first look at using the analytic method of finding inflection points -- finding where the graph of the second derivative of a function *crosses* the input axis.

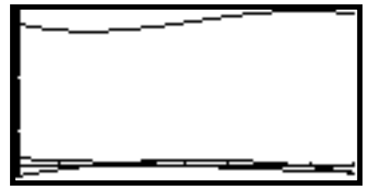
To illustrate, consider a model for the percentage of students graduating from high school in South Carolina from 1982 through 1990 who entered post-secondary institutions:

$$f(x) = {}^{-}0.1057x^3 + 1.355x^2 - 3.672x + 50.792 \text{ percent}$$

where $x = 0$ in 1982.

Enter $f(x)$ in the y1 location of the y(x)= list, your formula for the first derivative in y2 and your second derivative of the function in y3. Turn off y1 and y2.	
The problem says the model is for 1982 through 1990 which corresponds to $0 \le x \le 8$. Thus, you are told the horizontal view. Choose an appropriate vertical view -- possibly y between ⁻4 and 4. Graph $f''(x)$ Remember that you should be able to clearly see any optimal points. Leave room at the bottom of the screen so that trace coordinates will not block your view of any important points on the graph.	 Since the second derivative is a line and we need to find the x-intercept, this is a "good" graph.
Use the methods illustrated in 5.1.1 of this *Guide* to find where the second derivative graph crosses the x-axis. If you are asked to give the inflection *point* of $f(x)$, you should give both an x-value and a y-value.	
Return to the home screen and type X. The x-value you just found is stored in the x-location until you change it by tracing, using a menu item from the CALC menu, etc. Find the y-value by substituting this x-value in the function located in y1.	
Next, look a graph of the function and verify that there does appear to be an inflection point at $x \approx 4.27$. To do this, turn off y3 and turn on y1. Draw the graph of y1 with ZFIT.	

In this problem, it is difficult to find a window that shows a good graph of both the function and its derivatives. However, if you draw a graph of all three, you can roughly see that the location of the inflection point of the function occurs at the location of the maximum of the first derivative and at the location of the x-intercept of the second derivative.

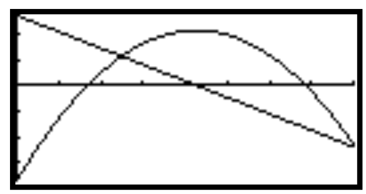


Both the TI-85 and the TI-86 have a second derivative function. The calculator notation for $f''(x)$, the second derivative, is der2 and it is accessed with 2nd ÷ (CALC) F4 (der2) . Use der2 either to check your second derivative formula or enter it in the graphing list instead of your second derivative formula in the process described above.

Enter $f(x)$ in the y1 location of the y(x)= list, der1 in the y2 location, and der2 in the y3 location. Turn off y1.

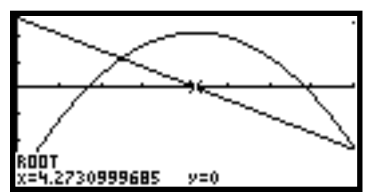
Again note that the only difference in what we are doing here and the above discussion is that we are using the calculator's numerical derivatives instead of the ones that we computed using derivative formulas.

Be sure $0 \le x \le 8$. Draw the graphs of the slope graph, ($f'(x)$ in y2) and the derivative of the slope graph ($f''(x)$ in y3) using ZFIT.



Use the methods of 5.1.1 of this *Guide* to find the x-intercept of the second derivative. (It takes a little longer here than it did before!)

Be sure to use ▽ to move to the graph of the line before tracing to the approximate location of the root.



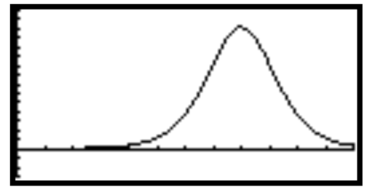
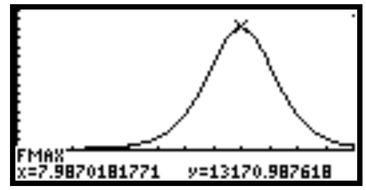
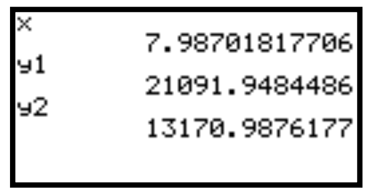
5.2.2 FINDING INFLECTION POINTS WITH YOUR CALCULATOR

Remember that an inflection point is a point of greatest or least slope. Whenever finding the second derivative of a function is tedious or you do not need an exact answer from an analytic solution, you can very easily find an inflection point of a function by finding where the first derivative of the function has a maximum or minimum value.

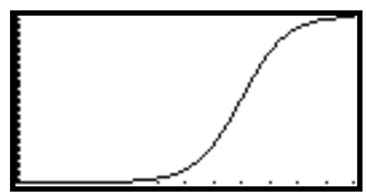
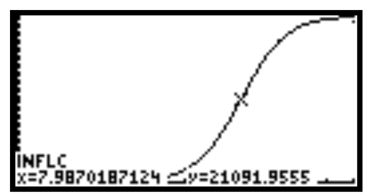
We illustrate this process with the function giving the number of polio cases in 1949: $y = \dfrac{42183.911}{1 + 21484.253e^{-1.248911t}}$ where $t = 1$ on January 31, 1949, $t = 2$ on February 28, 1949, etc.

Enter the function in the y1 location of the y(x)= list and the calculator's numerical derivative in the y2 location.

Turn off y1.

Graph y2 in an appropriate viewing window with *x* between 0 and 12 and something like *y* between $^-3000$ and 15,000 with tick marks every 1000 units. Remember that you should be able to clearly see any optimal points. Leave room at the bottom of the screen so that trace coordinates will not block your view of any important points on the graph.	
Use the methods illustrated in 5.1.2 of this *Guide* to find the maximum of the slope graph. The *x*-value of the maximum of the slope graph is the *x*-value of the inflection point of the function.	
If you are asked to give the inflection point, you should give both an *x*-value and a *y*-value. Find the *y*-value by substituting this *x*-value in the function located in y1. The rate of change at that time is obtained by substituting this *x*-value in y2.	

Our final method is the simplest -- just be certain if you use it that the function does have an inflection point at the location indicated by the calculator.

Draw the graph of the logistic function in y1, *not* its derivative. (That is, turn off y2 and turn on y1.) Have $0 \le x \le 12$ and use ZFIT to set the vertical view.	
TI-85 Press [GRAPH] [MORE] [F1] (MATH) [MORE] [F3] (INFLC) , and move the cursor to the approximate location of where the function changes concavity. Press [ENTER] to find the inflection point.	
TI-86 Press [GRAPH] [MORE] [F1] (MATH) [MORE] [F1] (INFLC) . Next, press [◀] to move the cursor to a position to the *left* of where the function changes concavity. Press [ENTER] to mark the *left* bound.	

Use ▷ to move the cursor to the *right* of where the function changes concavity. Press ENTER to mark the *right* bound. At the Guess? prompt, move the cursor to the approximate location of the inflection point. Press ENTER .	
The inflection point is displayed and marked on the graph.	