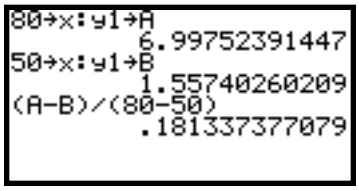
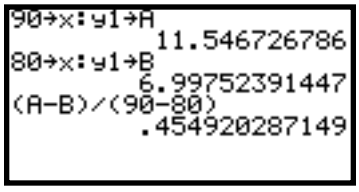
Chapter 3 Describing Change: Rates

3.1 Average Rates of Change

As you calculate average and other rates of change, remember that each numerical answer should be accompanied by units telling how the quantity is measured. You should also be able to interpret each numerical answer. It is only through their interpretations that the results of your calculations will be useful in real-world situations.

3.1.1 FINDING AVERAGE RATES OF CHANGE Finding an average rate of change using a model is just a matter of evaluating the model at two different values of the input variable and dividing by the difference in those input values. Consider this example.

The population density of Nevada from 1950 through 1990 can be approximated by the model $P(t) = 0.1273(1.05136)^t$ people per square mile where t is the number of years since 1900. You are asked to calculate the average rates of change between from 1950 through 1980 and between 1980 and 1990.

Enter the equation in the y1 location of the y(x)= list. (Remember that you must use x as the input variable in the graphing list. You do not have to use the first function location -- any of them will do unless you intend to use program TABLE on the TI-85.) Return to the home screen with 2nd EXIT (QUIT) .	
TI-85 The average rate of change of the population density between 1950 and 1980 is $\frac{P(80) - P(50)}{80 - 50}$. Next, evaluate the function at each of these values, store the results to different names, and then find the value of the quotient.	
Repeat the procedure to find the average rate of change of the population between 1980 and 1990.	
TI-86 The average rate of change of the population density between 1950 and 1980 is $\frac{P(80) - P(50)}{80 - 50} = \frac{y_1(80) - y_1(50)}{80 - 50}$. Enter this quotient, remembering to use parentheses around both the numerator and the denominator.	

To find the average rate of change between 1980 and 1990, recall the last expression with 2nd ENTER (ENTRY) and replace the 50 by 90. Press ENTER .	

- **TI-86 Note:** If you have many average rates of change to calculate, you could put the average rate of change formula in the graphing list: y2 = (y1(A) – y1(B))/(A – B). (You, of course, need to have the model in y1.) Then, on the home screen, store the inputs of the two points in *A* and *B*: 80 → A : 90 → B. All you need do then is type y1 and press enter. Store the next set of inputs into *A* and *B* and use 2nd ENTER to recall y1 to find the average rate of change between the two new points. Try it!

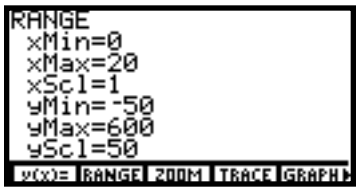
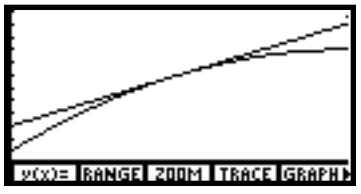
Both Recall that rate of change units are output units per input units. We see that on average, the population density increased by about 0.18 person per square mile per year between 1950 and 1980 and by approximately 0.45 person per square mile per year between 1980 and 1990.

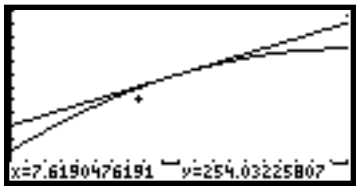
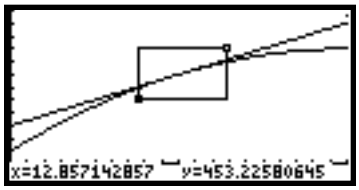
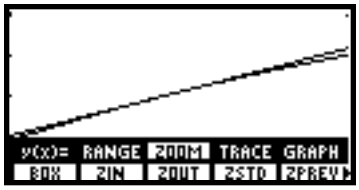
3.3 Tangent Lines

We first examine the principle of local linearity which says that if you are close enough, the tangent line and the curve are indistinguishable. We then use the calculator to draw tangent lines. There are two ways you can have your calculator draw a tangent line at a point on a curve. In this section, we consider one of these. The other method will be discussed in Chapter 4 of this *Guide*.

3.3.1 MAGNIFYING A PORTION OF A GRAPH

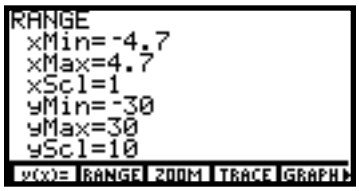
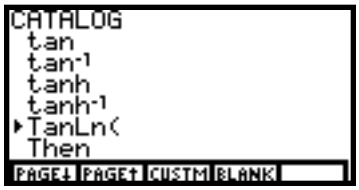
The ZOOM menu of your calculator allows you to magnify any portion of the graph of a function. Suppose we are investigating the graph of $y = -x^2 + 40x + 50$ and the tangent line, $y = 20x + 150$, to the graph of this function at $x = 10$.

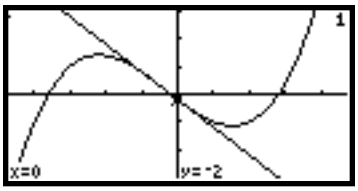
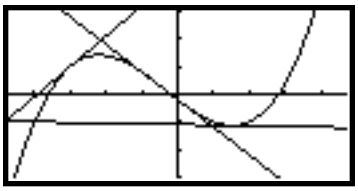
Enter $y = -x^2 + 40x + 50$ in y1 and $y = 20x + 150$ in y2. Set the view shown to the right with RANGE on the TI-85 (or WIND on the TI-86).	
Graph the function and the line tangent to it (y1 and y2) at $x = 10$. (On the TI-86, first be certain that all plots are off.) We now want to "box in" the point of tangency and magnify that portion of the graph.	

Press F3 (ZOOM) F1 (BOX) and use ◁ to move the cursor to the left of and ▽ to move the cursor down from the point of tangency. (You may not have the same values as those shown on the right.) Press ENTER to fix the lower left corner of the box.	
Use ▷ and △ to move the cursor to the opposite corner of your "zoom" box. Press ENTER to magnify the portion of the graph inside the box. Look at the view you now see with RANGE (or WIND). Repeat the above process if necessary.	
It is easy to see that the graph of the function and the graph of the tangent line are almost the same close to the point of tangency.	

- You should verify that the function and its tangent have close output values near the point of tangency by tracing the graphs near the point of tangency. Recall that you jump from one function to the other with △ or ▽ and that the number in the upper right-hand corner of the screen tells you on which function you are tracing.

3.3.2 DRAWING A TANGENT LINE The GRAPH (DRAW) menu of your calculator contains the instruction to draw a tangent line to a curve at a point. To illustrate the process, we draw several tangent lines on the graph of $f(x) = x^3 + x^2 - 10x - 2$. We also investigate what the calculator does when you ask it to draw a tangent line where the line cannot be drawn.

Clear any previously-entered functions in the Y(X)= list, and enter $f(x) = x^3 + x^2 - 10x - 2$ in Y1. Set the view shown to the right with RANGE (or WIND). Press F5 (GRAPH) .	
TI-85 Return to the home screen with 2nd EXIT (QUIT) . Draw the tangent line to the curve at $x = 0$ with 2nd CUSTOM (CATALOG) – (T) , use ▽ to locate TanLn(and then press ENTER . Type Y1 and press , 0) ENTER .	 (See box below.)

TI-86 Return to the home screen with 2nd EXIT (QUIT). Draw the tangent line to the curve at $x = 0$ with 2nd CUSTOM (CATALG-VARS) F1 (CATLG) – (T), use ▽ to locate TanLn(and then press ENTER. Type y1 and press , 0) ENTER.	(See box above.)
Both Notice that the tangent line cuts through the curve at $x = 0$. It appears that $(0, -2)$ is an inflection point.	
Return to the home screen, and recall the last entry with 2nd ENTER (ENTRY). Edit the statement so that you can draw the tangent line at $x = -3$.	
Once again recall the last entry on the home screen, and then draw the tangent line at $x = 1.5$. The tangent line is almost, but not quite, horizontal at $x = 1.5$.	

Let us now look at some special cases:

1. What happens if the tangent line is vertical? We consider the function $f(x) = (x + 1)^{1/3}$ which has a vertical tangent at $x = -1$.
2. How does the calculator respond when the tangent line cannot be drawn at a point? We illustrate what happens with $g(x) = |x| - 1$, a function that has a sharp point at $(0, -1)$.
3. Does the calculator draw the tangent line at the joining point(s) of a piecewise continuous function? We consider two situations:
   a. $h(x)$, a piecewise continuous function that is continuous at all points and
   b. $m(x)$, a piecewise continuous function that is not continuous at $x = 1$.

1. Enter the function $f(x) = (x + 1)^{1/3}$ in the y1 location of the y(x)= list. Remember that anytime there is more than one symbol in an exponent and you are not sure of the calculator's order of operations, enclose the exponent in parentheses.	

Draw the graph of the function with
EXIT F3 (ZOOM) MORE F4 (ZDECM) . Return to the ho
screen and type the instruction TanLn(y1, ⁻1).

Press ENTER . A vertical tangent line at $x = -1$ does not draw (as it should), and the message on the right appears. Press F5 (QUIT) to quit.

THIS IS INCORRECT!

2. Clear y1 and enter the function $g(x) = |x| - 1$. The absolute value symbol is obtained with
2nd X (MATH) F1 (NUM) F5 (ABS) x-VAR .

Draw the graph of the function with
EXIT F3 (ZOOM) MORE F4 (ZDECM) .

Return to the home screen and type the instruction TanLn(y1, 0). Press ENTER .

This error message is correct! There is a sharp point at (0, ⁻1), and the limiting positions of secant lines from the left and the right of that point are different. A tangent line cannot be drawn at (0, ⁻1) because the instantaneous rate of change at that point does not exist.

3a. Clear y1 and enter, as indicated, the function

$$h(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x & \text{when } x > 1 \end{cases}$$

[Recall that the inequality symbols are accessed with 2nd 2 (TEST)].

$h(x)$ is continuous for all values of x.

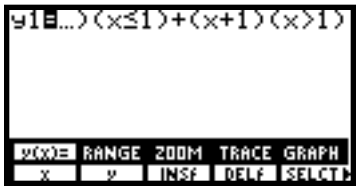
Draw the graph of the function with
EXIT F3 (ZOOM) MORE F4 (ZDECM) .

Return to the home screen and enter TanLn(y1, 1).

The calculator is correct -- the tangent line can not be drawn because secant lines drawn with points on the right and left of $x = 1$ do not approach the same slope.

3b. Edit y1 to enter, as indicated, the function

$$m(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x + 1 & \text{when } x > 1 \end{cases}$$

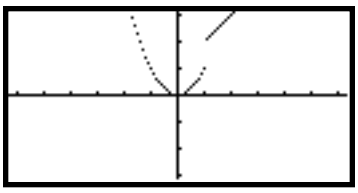


TI-85 Press EXIT MORE F3 (FORMT) and choose DrawDot.

TI-86 Have the cursor on the first line of the function, press F1 (y(x)=) MORE F3 (STYLE), and press F3 five more times to choose the dotted line to the *left* of y1.

Both Draw the graph of the function with EXIT F3 (ZOOM) MORE F4 (ZDECM).

Since $m(x)$ is not continuous at $x = 1$, the instantaneous rate of change does not exist at that point. The tangent line cannot be drawn at (1, 1).

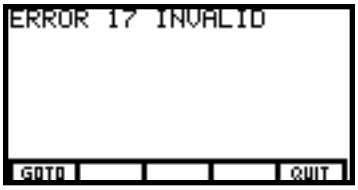


TI-85 Press GRAPH MORE F3 (FORMT) and return your calculator to Connected mode.

TI-86 Press F1 (y(x)=) MORE F3 (STYLE) to return the slanted line to normal graphing or clear the function.

Both Return to the home screen and type the instruction TanLn(y1, 1). Press ENTER.

The calculator is correct; a tangent line cannot be drawn when $x = 1$.



Caution: Be certain that the instantaneous rate of change exists at a point before using your calculator to draw a tangent line at that point. Because of the way your calculator computes instantaneous rates of change, it may draw a tangent line at a point on a curve where the tangent line does not exist. If you receive an error message, be certain you understand why that message is the result of your action.

3.5 Percentage Change and Percentage Rates of Change

The calculations in this section involve no new calculator techniques. When calculating percentage change or percentage rates of change, you have the option of using a program or the home screen.

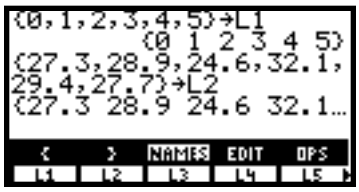
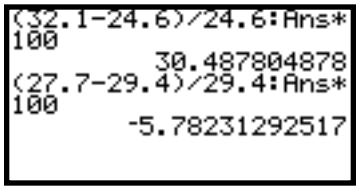
3.5.1 CALCULATING PERCENTAGE CHANGE

Recall that program DIFF stores percentage changes (also called percentage differences) in output data in list L5. Consider the following data giving quarterly earnings for a business:

Quarter ending	Mar 1994	June 1994	Sept 1994	Dec 1994	Mar 1995	June 1995

Earnings (millions) 27.3 28.9 24.6 32.1 29.4 27.7

First, we enter the data in the calculator's lists L1 and L2.

TI-85 Align the input data so that x is the number of quarters since March 1994. Input x in L1 and earnings (in millions) in L2.	
Run program DIFF and view the percentage change in list L5. Notice that the percentage change from the end of September 1994 through December 1994 is about 30.5%. Also, from the end of March 1995 through June 1995, the percentage change is approximately −5.8%.	
TI-86 Align the input data so that x is the number of quarters since March 1994. Input x in L1 and earnings (in millions) in L2.	
Run program DIFF and view the percentage change in list L5. Notice that the percentage change from the end of September 1994 through December 1994 is about 30.5%. Also, from the end of March 1995 through June 1995, the percentage change is approximately −5.8%.	
Both You may find it easier to calculate these using the percentage change formula than have the program do it for you.	

3.5.2 CALCULATING PERCENTAGE RATE OF CHANGE Consider again the quarterly earnings for a business. Suppose you are told or otherwise find that the rate of change at the end of the June 1994 is 1.8 million dollars per quarter. Evaluate the percentage rate of change at the end of June 1994.

Divide the rate of change at the end of June 1994 by the earnings, in millions, at the end of June 1994 and multiply by 100 to obtain the percentage rate of change at that point. The percentage rate of change in earnings at the end of June 1994 was approximately 6.2% per quarter.