Chapter 2 Ingredients of Change: Nonlinear Models

2.1 Exponential Functions and Models

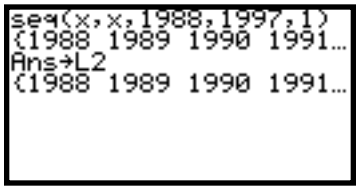
As we consider models that are not linear, it is very important that you be able to use scatter plots, numerical changes in output data, and the underlying shape of the basic functions to be able to identify which model best fits a particular set of data. Finding the model is only a means to an end -- being able to use mathematics to describe the changes that occur in real-world situations.

2.1.1 ENTERING EVENLY-SPACED INPUT VALUES (optional) When an input list consists of many evenly-spaced values, there is a calculator command that will generate the list so that you do not have to type in the values in one by one. The syntax for this sequence command is *seq(formula, variable, first value, last value, increment)*. When entering years that differ by 1, the formula is the same as the variable and the increment is 1. Any letter can be used for the variable -- we choose to use X.

TI-85 Generate the list of years beginning with 1988, ending with 1997, and differing by 1 with:

2nd – (LIST) F5 (OPS) MORE F3 (seq) X-VAR , X-VAR , 1988 , 1997 , 1) ENTER .

Store the values in list L2 with STO L ALPHA 2.



TI-86 Access the data lists with 2nd + (STAT) F2 (EDIT).
Use ▲ to move the cursor over the name L1.

Generate the list of years beginning with 1988, ending with 1997, and differing by 1 with: F5 (OPS) MORE F3 (seq) X-VAR , X-VAR , 1988 , 1997 , 1) ENTER .

2.1.2 FINDING PERCENTAGE CHANGE When the input values are evenly spaced, use program DIFF to compute percentage change in the output values. If the data are perfectly exponential (i.e., every data point falls on an exponential model), the percentage change in the output values is constant. If the percentage change is "close" to constant, this is an indication that an exponential model *may* be appropriate.

Suppose the population of a small town between the years 1988 and 1997 is as follows:

Year	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Population	7290	6707	6170	5677	5223	4805	4420	4067	3741	3442

Clear any old data, and enter the above data in lists L1 (year) and L2 (population). See Section 2.1.1 of this *Guide* for a convenient way to enter the years into L1.

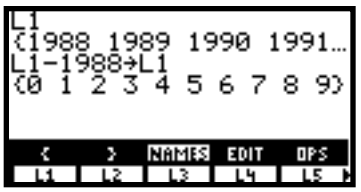
Run program DIFF. At the Choice? prompt, press F3 (%) to generate the list of percentage changes, and observe the percentage change in the output data in list L5. Exit program DIFF with F4 (Quit) .	
You cannot see the entire list, so go to the list editor: **TI-85** 2nd – (LIST) F4 (EDIT) F5 (L5) ENTER Use ▽ and △ to view all of the list.	
TI-86 2nd – (LIST) F4 (EDIT) △ and press ▷ until you are at the position of list L5. Use ▽ and △ to view all of the list.	

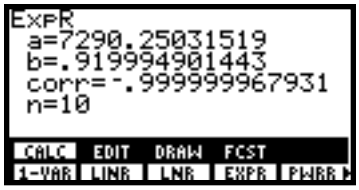
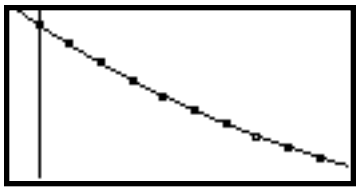
- **Both** The percentage change is very close to constant, so an exponential model may be a good fit.

2.1.3 FINDING AN EXPONENTIAL MODEL Use your calculator to find an exponential model that fits the data. The exponential model is accessed with the statistics menu command EXPR on the TI-85 (ExpR on the TI-86) and is of the form $y = ab^x$. Using the instructions below, construct a scatter plot of the data. Notice that the data curves rather than falling in a straight line pattern. An exponential model certainly seems appropriate!

TI-85 Use program STPLT to construct a scatter plot of the data. **TI-86** Press GRAPH F3 (ZOOM) MORE F5 (ZDATA)	

It is very important to align large numbers (like years) whenever you find an exponential model. The model found by the calculator may not even be correct if you don't!

Other alignments are possible, but we choose to align so that $x = 0$ in 1988. Return to the home screen.		
	TI-85	**TI-86**

TI-85 Press STAT F1 (CALC) , enter L1 as the xlist Name and L2 as the ylist Name. Press ENTER . (If these names already appear, just press ENTER ENTER .)	
Choose the exponential model with F4 (EXPR) . The exponential model is shown on the screen. Copy the model to the *y(x)=* list with MORE F4 (STREG) y1, overdraw the graph on the scatter plot with STPLT, and see that it gives a very good fit.	
TI-86 Press 2nd + (STAT) F1 (CALC) F5 (ExpR) 2nd – (LIST) F3 (NAMES) L1 , L2 , y1 Press ENTER . The model is generated and pasted into the y(x)= list.	
TI-85 Overdraw the model on the scatter plot by running program STPLT. **TI-86** Overdraw the model on the scatter plot by using ▲ to move the cursor over Plot1 in the y(x)= list, press ENTER to turn Plot1 on, and press EXIT F3 (ZOOM) MORE F5 (ZDATA) .	

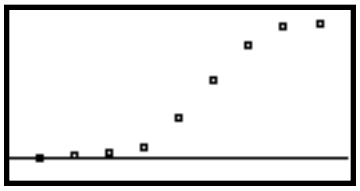
2.1.4 FINDING A LOGISTIC MODEL Use your calculator to find a logistic model of the form $y = \frac{L}{1 + Ae^{-Bx}}$. The logistic model that you obtain may be slightly different from a logistic model found with another calculator. Logistic models in *Calculus Concepts* were found using a TI-83. Refer to the following discussion for the comparable logistic model that best fits the data given in Example 2, Section 2.1 of the text. As with the exponential model $y = ab^x$, large input values must be aligned before fitting a logistic model to data.

Clear any old data, and enter the following in lists L1 and L2:

Aligned end of month	1	2	3	4	5	6	7	8	9
Total number of swimsuits sold	4	12	25	58	230	439	648	748	769

Construct a scatter plot of the data. A logistic model seems appropriate.

TI-85 Program LOGISTIC finds a "best-fit" logistic model rather than being a logistic model with a user-input limiting value L such that no data value is ever greater than L.



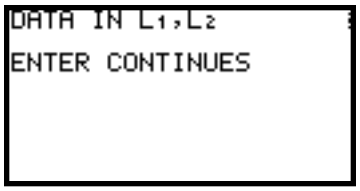
Run program[1] LOGISTIC to fit the logistic model.

Note: To use this program, the input data must be in order, from smallest to largest, in list L1. Have the output data in L2.

Run program LOGISTIC with PRGM followed by the number of the location of the program. Press ENTER .

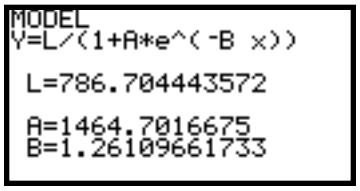
Your program list may not look exactly like this.

The first message you see reminds you that the input data should be in list L1 and the output data in list L2 . If you have not done this, press ON and choose 2 (QUIT) . Enter the data and then rerun the program.



After pressing ENTER to continue, the program displays several messages that you can ignore. (These messages are giving information about some of the advanced calculus techniques used to fit the model. You can see SSE being reduced as the model is fit.)

The program places the equation of the model in the y1 location of the y(x)= list.

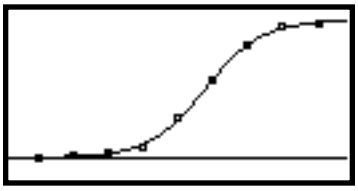


TI-86 This calculator fits a logistic model of the form $y = \frac{a}{1 + b \cdot e^{-cx}} + d$ to data. The form of this logistic model differs from those found by the TI-82, TI-83, and TI-85 because it includes d, a vertical shift from the horizontal axis. When the model is fit to data, the number of data points is displayed, and the model coefficients are in a list called PRegC. (The tolMet = 1 message concerns the tolerance of the TI-86. More information can be found on page 313 of your TI-86 *Guidebook*.)

If you would rather find the same model as the other calculators instead of using the built-in routine, program LOGISTIC can be transferred directly to the TI-86 from a TI-85. If so, read the TI-85 directions above for use of the program.

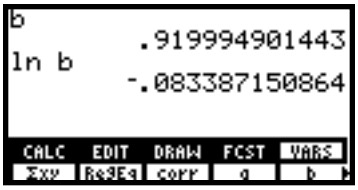
Fit a logistic model to the data and copy the model to the y1 location of the y(x)= list by pressing 2nd + (STAT) F1 (CALC) MORE F3 (LgstR) 2nd – (LIST) F3 (NAMES) L1 , L2 , y1 ENTER .

[1]The authors express their sincere appreciation to Dr. Dan Warner and Robert Simms of the Mathematical Sciences Department at Clemson University for their invaluable help with program LOGISTIC .

This model will take longer to generate than the other models. Notice that the TI-86 uses the variable *a* for the limiting value that your text calls *L*. The model also uses different symbols for the other parameters.	
Both Graph of the model on the scatter plot of the data. (The next section of this *Guide* shows you how to recall the model parameters if they are needed.)	

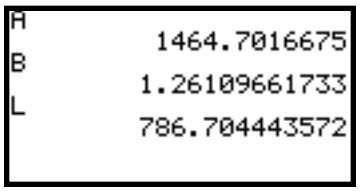
- Provided the input values are evenly spaced, program DIFF might be helpful when you are trying to determine if a logistic model is appropriate for certain data. If the first differences (in list L3 after running program DIFF) *begin small*, *peak in the middle*, and *end small*, this is an indication that a logistic model may provide a good fit to the data. Such is true for this data set because the first differences are 8, 13, 33, 172, 209, 209, 100, and 21.

2.1.5 RECALLING MODEL PARAMETERS Rounding of model parameters can often lead to incorrect or misleading results. You may find that you need to use the full values of model parameters after you have found a model. It would be tedious to copy all these digits into another location of your calculator. You don't have to! The following procedure applies for any model you find using one of the built-in regressions (*i.e.*, from the STAT CALC menu) in your calculator. Of course, once another model is found, previous parameters are no longer stored in the calculator's memory. As an example, we locate the parameter *b* for the exponential model found in Section 2.1.3. However, this same procedure applies to any model you have found using the STAT CALC menu.

TI-85 To recall the value of *b* in the model $y = ab^x$, press [STAT] [F5] (VARS) [MORE] [MORE] [F5] (b) . **TI-86** To recall the value of *b* in the model $y = ab^x$, press [2nd] [+] (STAT) [F5] (VARS) [MORE] [MORE] [F5] (b) .	
Both Press [ENTER] and the full value of *b* is "pasted" where ever you had the cursor before beginning the above keystrokes. Note: Remember that you can only recall the parameters before you use the menu to find a different model. Once a different model is found, the parameters are given for the new model.	

TI-85 The above procedure does not apply to models found with programs entered in the calculator.

If you need to recall the values of A, B, or L for the logistic model, return to the home screen and type ALPHA LOG (A) ENTER , ALPHA SIN (B) ENTER , and then ALPHA 3 (L) ENTER .



2.1.6 RANDOM NUMBERS Imagine all the real numbers between 0 and 1, including the 0 but not the 1, written on identical slips of paper and placed in a hat. Close your eyes and draw one slip of paper from the hat. You have just chosen a number "at random". Your calculator doesn't offer you a choice of all real numbers between 0 and 1, but it allows you to choose, *with an equal chance of obtaining each one*, any of 10^{14} different numbers between 0 and 1 with its random number generator called rand.

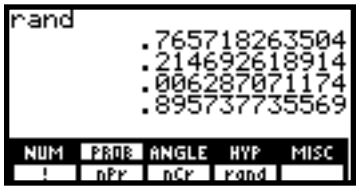
First, "seed" the random number generator. (This is like mixing up all the slips of paper in the hat.)

Pick some number, not the one shown on the right, and store it as the "seed". (Everyone needs to have a different seed, or the choice will not be random.)

The random number generator is accessed with 2nd X (MATH) F2 (PROB) F4 (rand) .

Enter rand again, and press ENTER several times.

Your list of random numbers should be different from the one on the right if you entered a different seed.



If you want to choose, at random, a whole number between 1 and N, enter int(N rand + 1) with 2nd X (MATH) F1 (NUM) F4 (int) (N 2nd F2 (PROB) F4 (rand) + 1) ENTER for a specific value of N.

- Repeatedly press ENTER to choose more random numbers. For instance, the screen to the right shows several values that were chosen at random with N = 10.

2.2 Exponential Models in Finance

You are probably familiar with the compound interest formulas. This section introduces you to some new methods of using your calculator with familiar formulas.

2.2.1 REPLAY OF PREVIOUS ENTRIES TO FIND FORMULA OUTPUTS You can recall expressions previously typed by repeatedly using the calculator's last entry feature. Learn to use this time-saving feature of your calculator.

On the home screen, store 1 in n, press 2nd . (:) to join statements on one line, and type the formula $\left(1+\frac{1}{n}\right)^n$. Press ENTER. The output depends on the value of n. You probably obtained a different output value because you have a different value stored in N. Store 1 in N.	1→N:(1+1/N)^N 2 (Use either upper-case or lower-case n, but be consistent.)
To find the output when $n = 2$, recall the last entry with 2nd ENTER (ENTRY), use ◀ to position the cursor over the 1 in "1→ N" and type 2. Press ENTER. The formula is now evaluated at $n = 2$. Store 3 in n and repeat the procedure.	1→N:(1+1/N)^N 2 2→N:(1+1/N)^N 2.25 3→N:(1+1/N)^N 2.37037037037

- **TI-86 Note**: 2nd ENTER (ENTRY) brings back several previously-entered expressions on this calculator, so the storing and evaluating could be done in two different steps.

Enter y1 = (1 + 1/x)^x Refer to 1.2.1 of this *Guide* to review the information about evaluating outputs using the TABLE.	2nd column for y2,... Press right arrow to scroll through table. [[(1,2) [(2,2.25) [(3,2.37037)]]	
	TI-85	**TI-86**

Because this formula contains only one input variable, you could enter it in the y(x)= list, using x as the input variable, and find the outputs using the TABLE (as indicated above.)

2.2.2 DETERMINING FUTURE VALUE You can save a lot of keystrokes by recalling expressions previously typed by repeatedly using the calculator's last entry feature. When a formula contains more than one input variable, it's easier to recall the last entry on the home screen than to try to use the TABLE. To illustrate, consider the compound interest formula -- one that contains several input variables. Type in the formula for the amount in an account paying $r\%$ interest (compounded n times a year) on an initial deposit of $\$P$ over t years:

$$Amount = P\left(1+\frac{r}{n}\right)^{nt} \text{ dollars}$$

Carefully watch the screen as you type the statements below. The STO key puts the cursor in alphabetic mode, so you must press ALPHA before typing the colon.

The result you obtain when you evaluate the formula with ENTER depends on the values your calculator has stored in P, R, and T. Store 1000 in P, 0.05 in R, 4 in N, and 1 in T and find the amount in the account to be \$1050.95.	P(1+R/N)^(N*T) 4.10989067286 1000→P:.05→R:4→N:1→T: P(1+R/N)^(N*T) 1050.94533691
Recall the last entry with 2nd ENTER (ENTRY), use ◀ to move the cursor, and edit the statements to determine the accumulated amount if \$5000 is invested at 5% interest compounded monthly for 3 years. You should find a result of \$5807.36.	4.10989067286 1000→P:.05→R:4→N:1→T: P(1+R/N)^(N*T) 1050.94533691 5000→P:.05→R:12→N:3→T :P(1+R/N)^(N*T) 5807.36115667

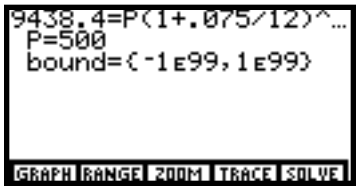
Because the TI-85 will only recall the last-entered expression, you should enter the single line shown above (with colons) so that it can be recalled for editing. However, the TI-86 recalls any number of previously-entered expressions, so you could store each variable separately *or* use the above procedure.

TI-86 Store the values for the various variables, enter the formula, and press ENTER. Then, store new values for the variables and repeatedly press 2nd ENTER (ENTRY) until the formula reappears. Press ENTER to evaluate the formula at the new values.

2.2.3 FINDING PRESENT VALUE The present value of an investment is easily found with the calculator's solver. For instance, suppose you want to solve for the present value P the equation $9438.40 = P\left(1 + \frac{0.075}{12}\right)^{60}$.

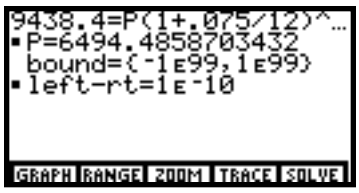
Refer to 1.2.2 of this *Guide* for instructions on using the TI-85's SOLVER. Enter the equation above as shown on the right.

(The TI-86 F2 location is WIND, not RANGE.)



Solve for P to obtain the present value \$6494.49.

(If you prefer, you could find the x-intercept of y1 = 9438.4 – x(1+.075/12)^60 to find the present value. Refer to Section 1.2.3 of this *Guide* for more detailed instructions.)



2.3 Polynomial Functions and Models

You will in this section learn how to fit models to data that have the familiar shape of a parabola or a cubic. Using your calculator to find these models involves basically the same procedure as when using it to find linear and exponential models.

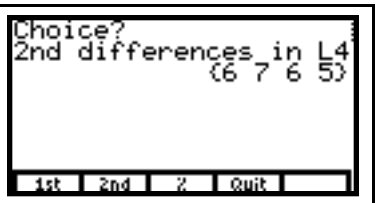
2.3.1 FINDING SECOND DIFFERENCES When the input values are evenly spaced, use program DIFF to compute second differences in the output values. If the data are perfectly quadratic (*i.e.*, every data point falls on a quadratic model), the second differences in the output values are constant. If the second differences are "close" to constant, this is an indication that a quadratic model *may* be appropriate.

Clear any old data, and enter the roofing job data in lists L1 and L2:

Months after January	1	2	3	4	5	6
Number of jobs	12	14	22	37	58	84

Run program DIFF, press F2 at the Choice? prompt, and observe the second differences in list L4.

The second differences are close to constant, so a quadrat- ic model may be a good fit. Construct a scatter plot of the data. A quadratic model seems appropriate!

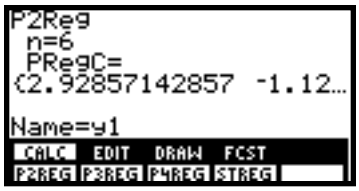


2.3.2 FINDING A QUADRATIC MODEL Use your calculator to obtain a quadratic model that fits the data. The calculator's quadratic model is of the form $y = ax^2 + bx + c$ and is accessed with the command P2REG.

TI-85 Press [STAT] [F1] (CALC) and press the F-keys on the menu to enter L1 as the xlist Name and L2 as the ylist Name. Press [ENTER] after each entry. (If these names already appear, just press [ENTER] [ENTER] .)

Choose the quadratic model with [MORE] [F1] (P2REG) .

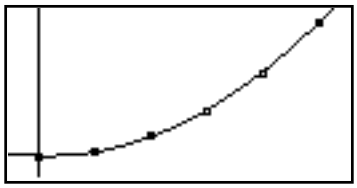
The best fitting quadratic model is displayed. The coefficients of the model $y = ax^2 + bx + c$ are displayed in the list $\{a, b, c\}$ that can be scrolled with [▷] for viewing. Copy the model to the y(x)= list.



TI-86 Press
Error!L1 [,] L2 [,] y1

Press [ENTER] . The model is generated and pasted into the y(x)= list.

Both Overdraw the graph on the scatter plot, and see that this model gives a very good fit to the data.

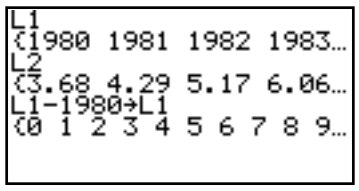


2.3.3 FINDING A CUBIC MODEL Whenever a scatter plot of the data shows a single change in concavity, a cubic or logistic model is appropriate. If a limiting value is apparent, use the logistic model. Otherwise, a cubic model should be considered. When appropriate, use your calculator to obtain the cubic model that best fits data. The calculator's cubic model is of the form $y = ax^3 + bx^2 + cx + d$ and is fit to data with the [P3REG] key.

Clear any old data, and enter the average price in dollars per 1000 cubic feet of natural gas for residential use in the U.S. from 1980 through 1990 in lists L1 and L2:

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Price	3.68	4.29	5.17	6.06	6.12	6.12	5.83	5.54	5.47	5.64	5.77

TI-85 First, clear lists L1 and L2, and then enter the data.
In order to work with smaller coefficients, align the data so that x represents the number of years since 1980.

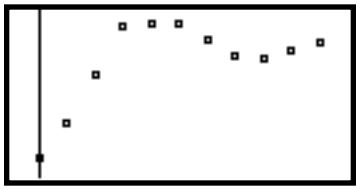


TI-86 First, clear lists L1 and L2, and then enter the data.

In order to work with smaller coefficients, align the data so that x represents the number of years since 1980.

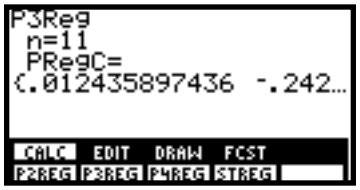
Both Draw a scatter plot of the data.

Notice that a concavity change is evident, but there do not appear to be any limiting values. Thus, a cubic model may fit the data.



TI-85 Press STAT F1 (CALC) and enter L1 as the xlist Name and L2 as the ylist Name. Choose the cubic model with MORE F2 (P3REG).

Copy the model to the y(x)= list using the STREG key.



TI-86 Press
Error!L1 , L2 , y1

Press ENTER. The model is generated and pasted into the y(x)= list.

Both The best fitting cubic model is displayed. The model is displayed, and the coefficients of the model are given in the list $\{a, b, c, d\}$ that can be scrolled with ▷.

Overdraw the model on the scatter plot.