Chapter 6 Accumulating Change: Limits of Sums and the Definite Integral

6.1 Results of Change

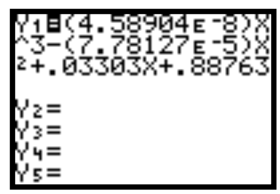
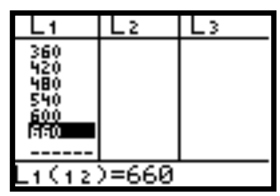
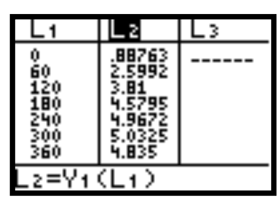
We have thus far seen how to use the calculator to work with rates of change. In this chapter we consider the results of change. Your calculator has many useful features that will assist you in your study of the accumulation of change.

6.1.1 APPROXIMATIONS WITH LEFT RECTANGLES The calculator's lists can be used to perform the calculations needed to approximate, using left rectangles, the area between the horizontal axis, a (non-negative) rate of change function, and two input values.

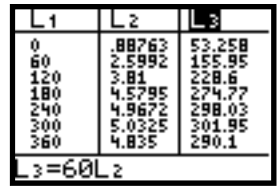
Consider, for example, a model for the number of customers per minute who came to a Saturday sale at a large department store between 9 a.m. and 9 p.m.:

$$c(m) = (4.58904 * 10^{-8})\, m^3 - (7.78127 * 10^{-5})\, m^2 + 0.03303\, m + 0.88763$$

customers per minute where m is the number of minutes after 9 a.m.

Enter this model in the Y[redacted]of the Y= list[redacted] (You must use x as the in[redacted]variable. If you hav[redacted] other equations in the gr[redacted]ist, clear them[redacted] (Remember that "10 to a p[redacted]r" is denoted by [redacted] the calculator. Access [redacted] [,]. Use this symbol or the 10s with powers when entering Y1.)	
Suppose we want to estimate the total number of customers who came to the sale between $x = 0$ and $x = 660$ (12 hours) with 12 rectangles and $\Delta x = 60$. Enter these x-values in list L1. Recall that a quick way to do this is to have L1 darkened and then type **Error!**	 Notice that 9 p.m. is 720 minutes after 9 a.m. When using *left*-rectangle areas, the *rightmost* data point is <u>not</u> included.
Enter the $c(x)$ values calculated from the model in list L2 by pressing [▲] (to darken the name of this list) and typing Y1(L1) with the keystrokes: **TI-82** [2nd] [VARS] (Y-VARS) [1] (Function) [1] (Y1) [( ] [2nd] [1] (L1) [ )] [ENTER] **TI-83** [VARS] (Y-VARS) [▶] [1] (Function) [1] (Y1) [( ] [2nd] [1] (L1) [ )] [ENTER]	 List L2 contains the *heights* of the 12 rectangles.

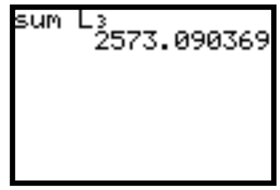
Both Since the *width* of each rectangle is 60, the area of each rectangle is 60**height*. Enter the *areas* of the 12 rectangles in list L3 by using [▲] to darken the name of list L3 and then typing 60L2 with [6] [0] [2nd] [2] (L2) [ENTER] .



Return to the home screen. Find the sum of the areas of the rectangles with

TI-82 [2nd] [STAT] (LIST) [▶] (MATH) [5] (SUM) [2nd] [3] (L3) [ENTER] .

TI-83 [2nd] [STAT] (LIST) [▶] [▶] (MATH) [5] (SUM) [2nd] [3] (L3) [ENTER] .



We estimate, using 12 left rectangles, that 2,573 customers came to the Saturday sale.

Note: The values in Table 6.2 in your text and the final result differ slightly than those in your lists. This is because the unrounded model found with the data was used for computations in the text. If you have the unrounded model available, you should use it instead of the rounded model *c(m)* that is given above.

6.1.2 APPROXIMATIONS WITH RIGHT RECTANGLES

When using left rectangles to approximate the results of change, the rightmost data point is not the height of a rectangle and is not used in the computation of the left-rectangle area. Similarly, when using right rectangles to approximate the results of change, the *leftmost* data point is not the height of a rectangle and is not used in the computation of the right-rectangle area.

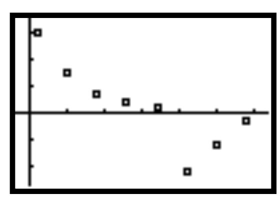
The following data shows the rate of change of the concentration of a drug in the blood stream in terms of the number of days since the drug was administered:

Day	1	5	9	13	17	21	25	29
Concentration ROC (μg/mL/day)	1.5	0.75	0.33	0.20	0.10	⁻1.1	⁻0.60	⁻0.15

First, we fit a piecewise model to the data.

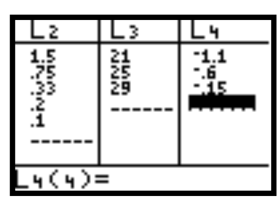
Clear all lists. Enter the days in L1 and the rate of change of concentration in L2.

Draw a scatter plot of the data. It is obvious that a piecewise model should be used with $x = 20$ as the "break" point.



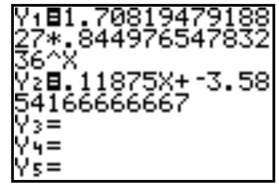
Delete the last three data points from lists L1 and L2 with [DEL] . Enter {21, 25, 29} in L3 and in L4 enter the corresponding outputs: {⁻1.1, ⁻0.60, ⁻0.15}.

Fit an exponential model to the data in L1 and L2 and put that model in Y1.



TI-82 Fit a linear model to the data in L3 and L4 with the keystrokes STAT ▶ (CALC) 5 (LinReg) 2nd 3 (L3) , 2nd 4 (L4) ENTER .

Copy the model to Y2 .

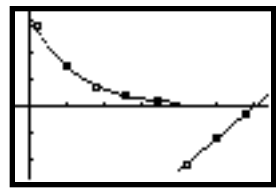


TI-83 Fit a linear model to the data in L3 and L4 and copy it to Y2 with the keystrokes STAT ▶ (CALC) 4 (LinReg) 2nd 3 (L3) , 2nd 4 (L4) , VARS ▶ (Y-VARS) 1 (Function) 1 (Y2) ENTER .

Both (optional) If you want to graph this model on the original scatter plot, first re-enter the data with all the inputs in list L1, all the outputs in list L2.

Return to the graphing list and turn *off* Y1 and Y2.

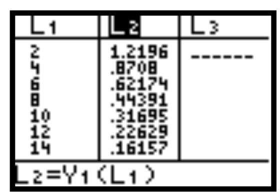
Enter (Y1)(X≤20) + (Y2)(X>20) in Y3. (Use the Y-VARS menu to type Y1 and Y2.) Choose Dot mode, draw the scatter plot and Y3 will graph over it.


Y3 = (Y1)(X≤20)+(Y2)(X>20)

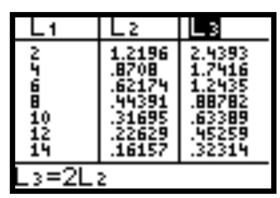
Now, we determine the right-rectangle area for $0 \le x \le 20$:

Clear all lists. Because x starts at 0 and $\Delta x = 2$, enter in L1 the values 0, 2, 4, 6, …, 20 or use seq(X, X, 0, 20, 2) to generate the list. Because we are using right rectangles, *delete* the leftmost input value (0).

Use either the model in Y1 or Y3 to generate the outputs.

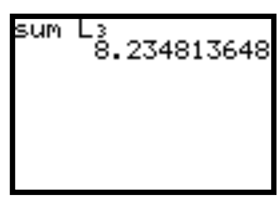


Each rectangle has width 2. The heights of the rectangles are in list L2. Enter the rectangle areas in list L3 as width*height = 2*L2.



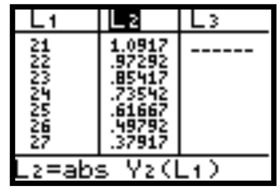
Return to the home screen and enter sum L3 with the keystrokes 2nd STAT (LIST) ▶ (MATH) 5 (SUM) 2nd 3 (L3) ENTER .

We estimate, using 10 right rectangles, that the change in concentration was approximately 8.24 µg/mL.



To estimate, using right rectangles, the change in drug concentration for $20 \le x \le 29$ days with $\Delta x = 1$, follow the same procedure as above. The values in L1 begin with 21 because we must eliminate the leftmost value (20) when using right rectangles. You should use the absolute value of the model in Y2 (or Y3) to generate L2.

Since the width of each rectangle is 1, L2 contains the areas as well as the heights of the 9 right rectangles.



sum L2 = 5.55 µg/mL

6.2 Trapezoid and Midpoint Rectangle Approximations

You can compute areas of trapezoids on the home screen of your calculator or use the fact that the trapezoid approximation is the average of the left- and right-rectangle approximations. Areas of midpoint rectangles are found in the same manner as left and right rectangle areas except that the midpoint of the base of each rectangle is in L1 and no data values are deleted. However, such procedures can become tedious when the number, n, of subintervals is large.

6.2.1 SIMPLIFYING AREA APPROXIMATIONS

When you have a model $y = f(x)$ in Y1 in the Y= list, you will find program NUMINTGL very helpful in determining left-rectangle, right-rectangle, midpoint-rectangle, and trapezoidal numerical approximations for accumulated change. Program NUMINTGL is listed in the TI-82/TI-83 Appendix.

WARNING: This program will not work properly if you have any functions in your Y= list that have letters other than X in them. Delete any such functions (like the logistic equation that might be in Y4) before using program NUMINTGL. If you receive an error while running the program, you may have a picture or program stored to a single-letter name. For instance, if program NUMINTGL is trying to store a number in T and you have a program called T, the calculator stops. Delete or rename any programs or pictures that you have called by a single-letter name before continuing.

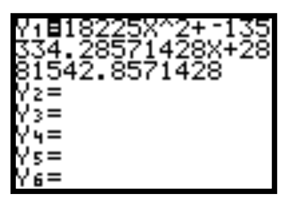
We illustrate using this program with a model for the Carson River flow rates:

$$f(h) = 18{,}225h^2 - 135{,}334.3h + 2{,}881{,}542.9 \text{ cubic feet per hour}$$

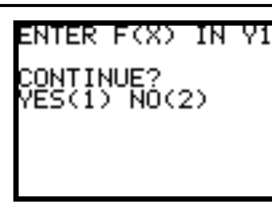
h hours after 11:45 a.m. Wednesday. (The complete model found from the data in your text is used for all the following calculations.)

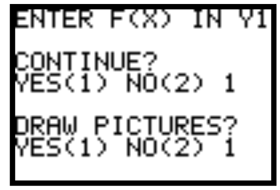
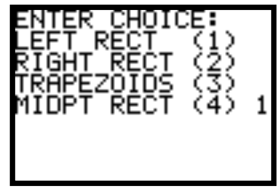
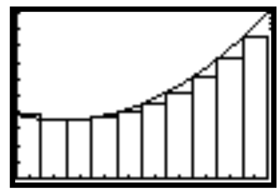
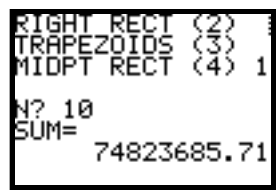
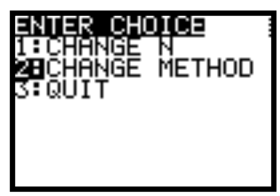
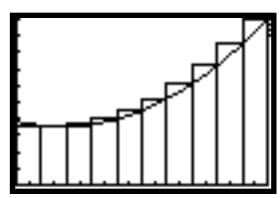
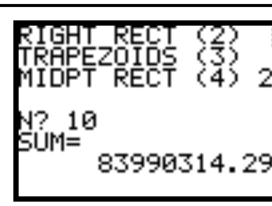
Have $f(h)$ in Y1. Your function *must* be in Y1 for program NUMINTGL to operate.

If you wish to view the approximating rectangles or trapezoids, the program will automatically draw a graph of the function when enter the input interval.



Start program NUMINTGL by pressing PRGM followed by the number next to the location of the program. Press ENTER . (At this point, if you did not enter the function in Y1, enter 2 to exit the program. Enter Y1 and re-run the program.) If your function is in Y1, enter 1 to continue.



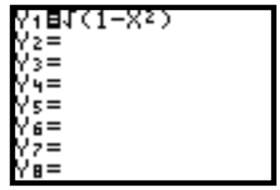
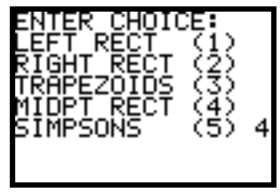
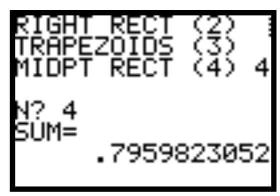
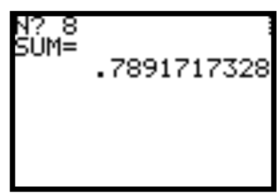
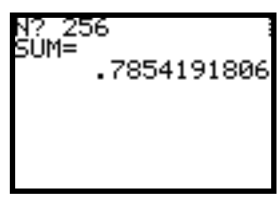
At the next prompt, press [1] [ENTER] to draw the approximating figures or press [2] [ENTER] to obtain only the numerical approximations to the area between *f(h)* and the horizontal axis between 0 and 20. We choose to see some pictures.	
At the LEFT ENDPOINT? prompt, type [0] [ENTER], and at the RIGHT ENDPOINT? prompt, [20] [ENTER] to tell the calculator the input interval. You are next shown a menu of choices. Press [1] [ENTER] to find a left-rectangle approximation.	
Input 10 at the N? prompt and press [ENTER]. The ten left rectangles are shown.	
The left-rectangle area approximation is displayed as SUM when you press [ENTER].	
Press [ENTER] once more and more choices are displayed. Suppose we now want to find the approximating area and see the figure using 10 right rectangles. Press [2] to change the method, and then press [2] [ENTER] to choose right rectangles.	
Enter 10 at the N? prompt and view the right rectangles. Again press [ENTER] and the right-rectangle area approximation is displayed.	
Continue on in this manner and find the trapezoid and midpoint rectangle approximations to the area. When you finish, press [ENTER] and choose 3 to QUIT.	

6.3 The Definite Integral as a Limit of Sums

This section introduces you to a very important and useful concept of calculus -- the definite integral. Your calculator can be very helpful as you study definite integrals and how they relate to the accumulation of change.

6.3.1 LIMITS OF SUMS

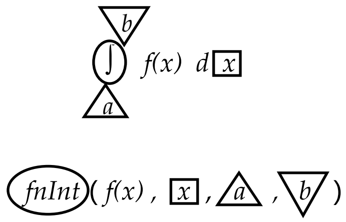
When you are looking for a trend in midpoint-rectangle approximations to the area between a non-negative function and the horizontal axis between two values of the input variable, program NUMINTGL is extremely useful! However, for this use of the program, it is not advisable to draw pictures when n, the number of subintervals, is large.

To construct a chart of midpoint approximations for the area between $f(x) = \sqrt{1 - x^2}$ and the x-axis from x=0 and x=1, first enter $f(x)$ in Y1. (Don't forget to enclose $1 - x^2$ in parentheses.)	
Start program NUMINTGL. Since many subintervals are used with the idea of a limit of sums, do *not* choose to draw the pictures unless you have time to sit and wait for all the rectangles to draw! After entering 0 and 1 as the respective lower and upper limits, choose option 4 for midpoint rectangles.	
Input some number of subintervals, say N = 4. Record on paper the midpoint area approximation 0.7959823052. (If you want an answer accurate to the thousandths position, record at least 4 decimal places.)	
Press ENTER and choose option 1: CHANGE N. Double the number of subintervals to N = 8. Record the midpoint approximation 0.7891717328 (again, to at least 4 decimal places).	
Continue on in this manner, each time choosing option 1: CHANGE N and doubling N until a trend is evident. (Finding a trend means that you can tell what value the approximations are getting closer and closer to, within a specified accuracy, without having to run the program ad infinitum!) Choose QUIT (3) to stop.	

- Remember that the trend indicated by the limit of sums can be interpreted as the area of the region between the function and the input axis *only* when that region lies above the input axis. When the region lies below the input axis, the trend is the negative of the area of the region.

6.5 The Fundamental Theorem

Recall that nDeriv is the calculator's numerical derivative and provides, in most cases, a good approximation to the instantaneous rate of change of a function when that rate of change exists. Your calculator can also give you a numerical approximation for a definite integral of a function. This numerical integrator is called fnInt, and the correspondence between the mathematical definite integral notation $\int_a^b f(x)\,dx$ and the calculator's notation fnInt(f (X), X, a, b) is as shown below:



⬭ indicates we are taking an integral.

□ indicates the letter corresponding to the name of the input variable.

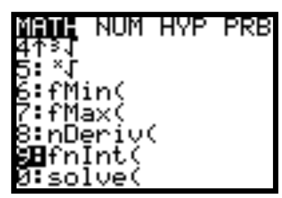
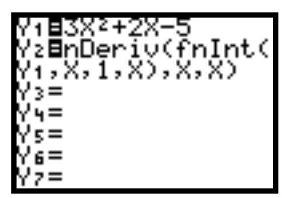
△ indicates the left endpoint of the interval

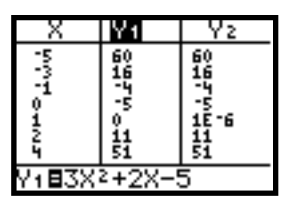
▽ indicates the right endpoint of the interval

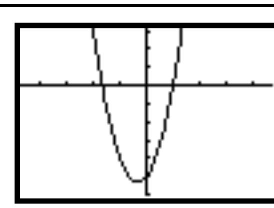
You will find that in most cases, your calculator's numerical integrator gives a very close decimal approximation of the exact value of a definite integral. This use is examined in Section 6.6 of this *Guide*. In this section, we illustrate the Fundamental Theorem of Calculus and see how to draw the graph of a general antiderivative of a function.

6.5.1 THE FUNDAMENTAL THEOREM OF CALCULUS This theorem tells us that the derivative of an antiderivative of a function is the function itself. Let us view this theorem both numerically and graphically.

<table>
<tr><td>Your calculator's numerical integrator is accessed with the keystrokes MATH 9 (fnInt().

As indicated above, it needs to be followed by the function, the variable, a lower limit, and an upper limit (in that order).</td><td></td></tr>
<tr><td>Consider $F'(x) = \frac{d}{dx}\left(\int_1^x 3t^2 + 2t - 5\ dt\right)$. The FTC tells us that $F'(x)$ should equal $3x^2 + 2x - 5$. Enter the functions shown to the right.</td><td></td></tr>
<tr><td>Press 2nd GRAPH (TABLE) and input some different values of x.

Other than a small bit of roundoff error due to the numerical nature of the calculator, Y1 and Y2 are identical!</td><td></td></tr>
<tr><td>Find a suitable viewing window such as x between $^-4.7$ and 4.7 and y between $^-6$ and 3.1. Draw the graphs of Y1 and Y2 separately in this same window and then draw them together. Only one graph appears!

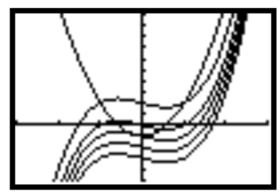
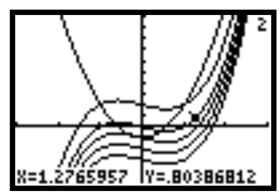
(The graph of Y2 will take a while to draw.)</td><td></td></tr>
</table>

Enter several other functions in Y1 and perform the same explorations as above. Confirm your results with derivative and integral formulas. Are you convinced?

6.5.2 DRAWING ANTIDERIVATIVE GRAPHS All antiderivatives of a specific function differ only by a constant. We explore this idea using the function $f(x) = 3x^2 - 1$ and its antiderivative $F(x) = x^3 - x + C$.

The correct syntax for the calculator's numerical integrator is fnInt(f(X), X, a, b) where $f(x)$ is the function you are integrating, x is the variable of integration, a is the lower limit on the integral, and b is the upper limit on the integral. You do *not* have to use x as the variable unless you are graphing the integral or evaluating it using the calculator's table.

Note that you must supply both a lower limit and an upper limit for fnInt. We can use x for the upper limit, but not both the upper and lower limits. Since we are working with a general antiderivative in this illustration, we do not have the starting point for the accumulation. We therefore just choose some value, say 0, to use as the lower limit to illustrate drawing antiderivative graphs. If you choose a different lower limit, your results will differ from those shown below by a constant.

Enter $f(x)$ in Y1, fnInt(Y1, X, 0, X) in Y2, and $F(x)$ in Y3, Y4, Y5, Y6, and Y7 (using a different value of C in each location) You can try different values of C than those shown on the right.	
Find a suitable viewing window and graph all the functions. (Try x between $^-3$ and 3 and y between $^-5$ and 10.) It seems that the only difference in the graphs (other than the graph of Y1) is that the y-intercept is different. But, isn't C the y-intercept?	
Trace the graphs and then jump between them with ▽. It appears that Y2 and Y3 are the same.	

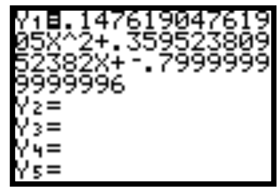
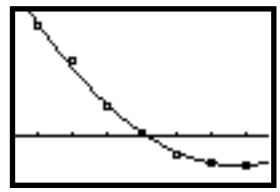
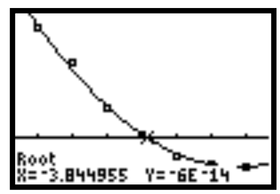
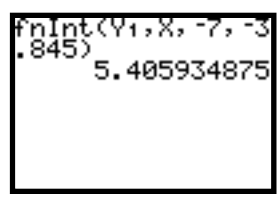
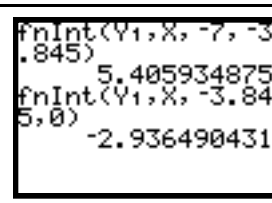
Warning: The methods of Sections 4.4.1 and/or 4.4.2 that we used to check antiderivative formulas are not valid with general antiderivatives. Why? The answer is because in this situation you must choose arbitrarily choose a value for the constant of integration and you must arbitrarily choose a value for the lower limit in order to use the calculator's numerical integrator. However, for most rate-of-change functions where $f(0) = 0$, the calculator's numerical integrator values and your antiderivative formula values should differ by the same constant at every input value where they are defined.

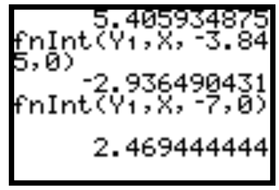
6.6 The Definite Integral

When using the numerical integrator on the home screen, enter fnInt($f(x)$, x, a, b) for a specific function $f(x)$ with input x and specific values of a and b. (Remember that the input variable does not have to be x on the home screen.) If you prefer, $f(x)$ can be in the Y= list and referred to as Y1 (or whatever location you have it in) when using fnInt.

6.6.1 EVALUATING A DEFINITE INTEGRAL ON THE HOME SCREEN

We illustrate the use of your calculator's numerical integrator with the model for the rate of change of the average sea level in meters per year during the last 7000 years. A model for these data is $r(t) = 0.14762t^2 + 0.35952t - 0.8$ meters per year t thousand years from the present. (Note that t is negative since we are talking about the past.)

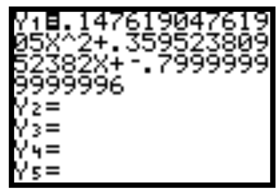
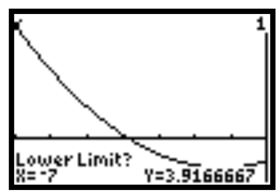
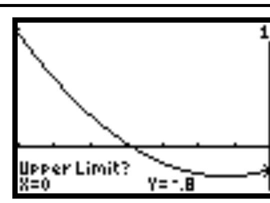
As we have previously mentioned, you should always use the full model (not rounded) for any model you find from data. Find a quadratic to fit the data shown in the *Changing Sea Levels* example in your text, and enter it in some location of the the Y= list, say Y1.	
The model provides a good fit to the data. Because we are asked for *area* between the input axis and the function, we must find where the function becomes negative.	
Use the ideas of Section 1.2.3 of this *Guide* to find the x-intercept (root or zero) of Y1 between 0 and 7 to be $x \approx {}^{-}3.845$.	
TI-82 Return to the home screen and type the expression shown to the right with the keystrokes **Error!** **TI-83** Return to the home screen and type the expression shown to the right with the keystrokes **Error!**	 $\int_{-7}^{-3.845} r(t)\,dt \approx 5.4$ meters
Both The area of the region above the x-axis is about 5.4 meters. Now, find the area of the region below the x-axis. This area equals the negative of the definite integral of the function over that region.	
Evaluate $\int_{-3.845}^{0} r(t)\,dt$ as shown on the right. The area of the region below the x-axis is about 2.9 meters.	

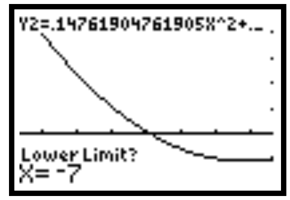
Find $\int_{-7}^{0} r(t)\,dt$. Note that this value is *not* the sum of the two areas. It is their difference.	

- If you evaluate a definite integral using antiderivative formulas and check your answer with the calculator using fnInt, you may sometimes find a slight difference in the last few decimal places. Remember, the TI-82 and the TI-83 are evaluating the definite integral using an approximation technique.

6.6.2 EVALUATING A DEFINITE INTEGRAL FROM THE GRAPHICS SCREEN

Provided that a and b are possible x-values when you trace the graph, you can find the value of the definite integral $\int_a^b f(x)\,dx$ from the graphics screen. (For "nice" trace numbers, you can often find the exact x-value you need if you graph in the ZDecimal screen or set the viewing window so that Xmax – Xmin equals a multiple of 9.4.)

Turn the STAT PLOTS off, and have the model for the average sea level in Y1 (see Section 6.6.1). Suppose we want to find $\int_{-7}^{0} r(t)\,dt$. Graph Y1.	
Because ⁻7 is not a possible trace value in the window set by the stat plot, press WINDOW and change Xmin to ⁻7 and Xmax to 0. Press 2nd TRACE (CALC) 7 (∫f(x)dx) . When the calculator asks for a lower bound, press and hold ◁ until you reach ⁻7.	
Press ENTER to set the lower limit and have the calculator ask for the upper limit. Press and hold ▷ until you reach X = 0.	
Press ENTER to calculate the numerical value of the definite integral. Notice that the two regions whose areas were calculated in the first part of Example 1 in the text are shaded.	

TI-83 This calculator allows you to input any value of x between Xmin and Xmax when finding the value of a definite integral from the graphics screen. You therefore can either follow the above instructions for both calculators or you can do as indicated below. Set the window shown to the right -- the scatter plot window for the average sea level data with Xmax reset to 0.	WINDOW Xmin=⁻7.6 Xmax=0 Xscl=1 Ymin=⁻1.816 Ymax=4.616 Yscl=1 Xres=1
Draw the graph of *r(t)* by pressing GRAPH . Press 2nd TRACE (CALC) 7 (∫f(x)dx) . When the calculator asks for a lower limit, simply type ⁻7 and then press ENTER .	
When the calculator asks for an upper limit, type 0 and then press ENTER . The value of the integral is displayed and the two regions are shaded. Note that if you had not previously changed Xmax from ⁻0.4 to 0, you would have gotten an ERR: INVALID message at this point because 0 would not have been a value included between Xmin and Xmax.