Chapter 5 Analyzing Change: Extrema and Points of Inflection

5.1 Optimization

Your calculator can be very helpful in checking your analytic work when you find optimal points and points of inflection. When you are not required to show work using derivatives or when a very good approximation to the exact answer is all that is required, it is a very simple process to use your calculator to find optimal points and inflection points.

5.1.1 FINDING *X*-INTERCEPTS OF SLOPE GRAPHS Where the graph of a function has a local maximum or minimum, the slope graph has a horizontal tangent. Where the tangent line is horizontal, the derivative of the function is zero. Thus, finding where the slope graph *crosses* the input axis is the same as finding where a relative maximum or a relative minimum occurs.

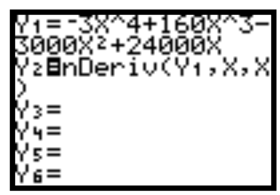
Consider, for example, the model for a cable company's revenue for the 26 weeks after it began a sales campaign:

$$R(x) = {}^{-}3x^4 + 160x^3 - 3000x^2 + 24{,}000x \text{ dollars}$$

where x is the number of weeks since the cable company began sales.

Enter $R(x)$ in the Y1 location of the Y= list. Enter either the calculator's derivative *or* your derivative in the Y2 location. Turn Y1 off.

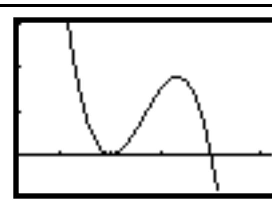
(If you use your derivative, be sure to first check to see that your derivative and the calculator's derivative are the same!)



TI-82 The statement of the problem indicates that x should be graphed between 1 and 26. Experiment and find an appropriate height for the window. Possible values include Ymin = ⁻800 and Ymax = 3000.

Draw the slope graph with GRAPH .

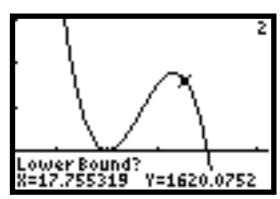
(If you prefer to use program AUTOSCL to set the window, you must switch the location of the functions because the function being graphed must be in Y1.)

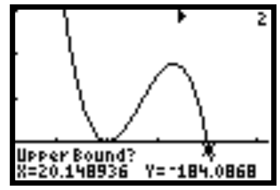


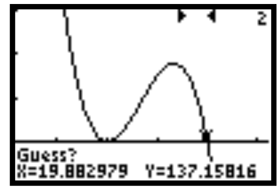
R′(x) is a cubic, so you should look for a cubic shape and have a good view of the *x*-intercepts.

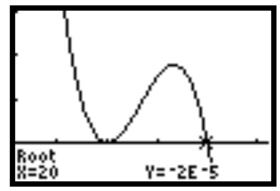
We find any x-intercepts of the slope graph using 2nd TRACE (CALC) 2 (root) .

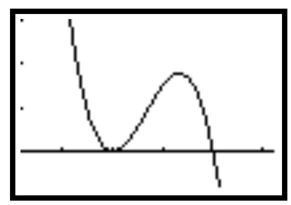
Use ▷ to move the cursor anywhere to the *left* of where the graph crosses the x-axis. Press ENTER to mark the location of the *lower* bound.

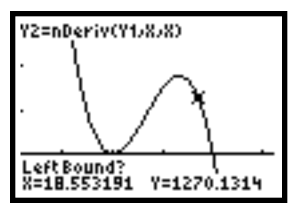
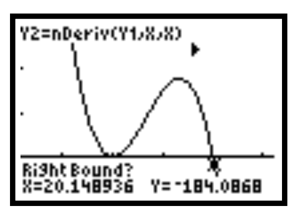


<table>
<tr><td>Use [▷] again to move the cursor to the right of the x-intercept, and press [ENTER] to mark the location of the upper bound.</td><td>
</td></tr>
<tr><td>Notice that the calculator has marked the interval between your two bounds with small triangles at the top of the screen. The intercept must be between these two marks.

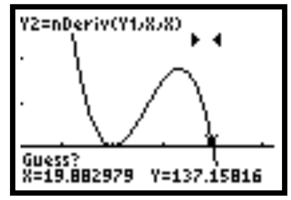
You are next asked to provide a guess. Any value in the interval will do. Use [◁] to move near the intercept and press [ENTER] .</td><td>
</td></tr>
<tr><td>The location of the x-intercept (root) is displayed.
We see that $R'(x) = 0$ at $x = 20$.

(The numerical process used to find the root sometimes includes rounding errors. The output should, of course, be 0 but is printed on the screen as $^{-}0.00002$.)</td><td>
</td></tr>
<tr><td>TI-83 The statement of the problem indicates that x should be graphed between 1 and 26. Set these values in the window, and draw the slope graph with [ZOOM] [△] (0: ZoomFit) [ENTER] .

For a better view, reset the window values to Ymin = $^{-}$800 and Ymax = 3000. Draw the slope graph with [GRAPH] .</td><td></td></tr>
<tr><td>We find any x-intercepts of the slope graph using [2nd] [TRACE] (CALC) [2] (zero) .

Use [▷] to move the cursor anywhere to the left of where the graph crosses the x-axis. Press [ENTER] to mark the location of the left bound.</td><td>
</td></tr>
<tr><td>Use [▷] again to move the cursor to the right of the x-intercept, and press [ENTER] to mark the location of the right bound. Notice that the calculator has marked the interval between your two bounds with small triangles at the top of the screen. The intercept must be between these two marks.</td><td>
</td></tr>
</table>

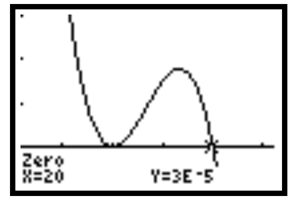
You are next asked to provide a guess. Any value in the interval will do. Use [◁] to move near the intercept and press [ENTER] .



The location of the x-intercept (zero) is displayed.

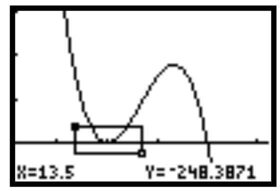
We see that $R'(x) = 0$ at $x = 20$.

(The numerical process used to find the root sometimes includes rounding errors. The output should, of course, be 0 but is printed on the screen as $^{-}0.00003$.)



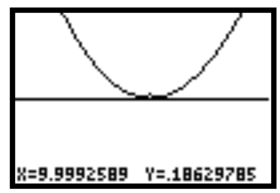
Both Now, you must determine if the derivative graph crosses or just touches the x-axis at the location to the left of this intercept.

Use Zbox as many times as necessary to magnify that portion of the graph to see what happens there. (See Section 3.3.1 of this *Guide*.)



After using Zbox several times, we see that the graph just touches and does not cross the x-axis near $x = 10$.

Using the information gained from $R'(x)$ and the graph of $R(x)$, we find that revenue was greatest 20 weeks after the cable company began sales.



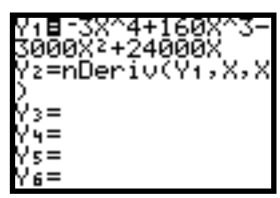
5.1.2 FINDING OPTIMAL POINTS

Once you draw a graph of a function that clearly shows any optimal points, finding the location of those high points and low points is an easy task for your calculator. When a relative maximum or a relative minimum exists at a point, your calculator can find it in a few simple steps. We again use the cable company revenue equation, $R(x)$, from Section 5.1.1.

TI-82 Enter $R(x)$ in the Y1 location of the Y= list.

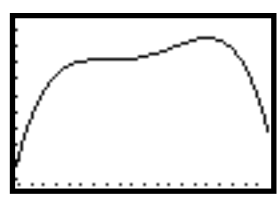
(Any location will do, but if you have other equations in the graphing list, turn them off or clear them.)

Next, draw a graph of the revenue, $R(x)$.



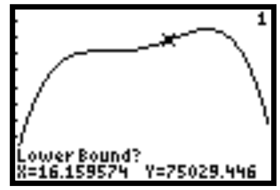
The input x should be graphed between 1 and 26.

You could experiment and find an appropriate height for the window, or you could use program AUTOSCL. (Remember, if you use the program, the function being graphed must be in Y1.)

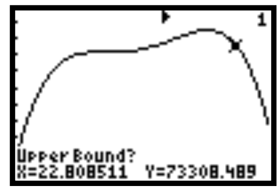


Prepare to find the local (relative) maximum by pressing 2nd TRACE (CALC) 4 (maximum) .

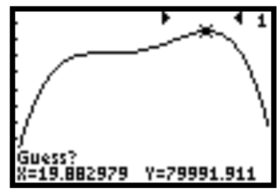
Next, press ▷ to move the blinking cursor that appears to a position to the *left* of the high point. Press ENTER to mark the *lower* bound for *x*.



Next, press ▷ to move the blinking cursor to a position to the *right* of the high point. Press ENTER to mark the *upper* bound for *x*.

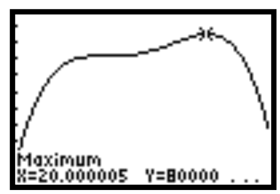


Use ◁ to move the cursor near your estimate of the high point and press ENTER . The TI-82 uses your guess to locate the highest point in the region between the two bound marks. The maximum value and the *x*-value at which it occurs is displayed at the bottom of the screen.



Revenue is greatest at 20 weeks with a value of *R*(20) = $80,000.

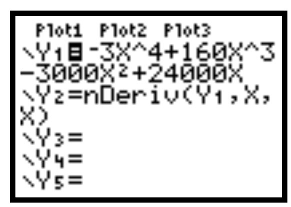
Return to the home screen and verify that Y1(20) gives the value displayed at the bottom of the graphics screen.



TI-83 Enter *R(x)* in the Y1 location of the Y= list.

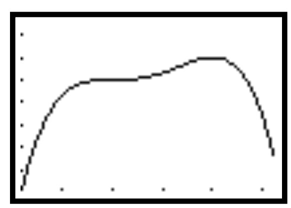
(Any location will do, but if you have other equations in the graphing list, turn them off or clear them.)

Next, draw a graph of the revenue, *R(x)*.



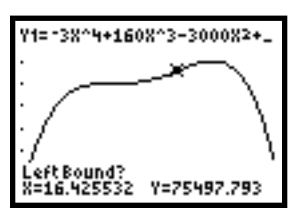
The input (*x*) should be graphed between 1 and 26. You could experiment and find an appropriate height for the window, or you could graph *R(x)* using ZoomFit.

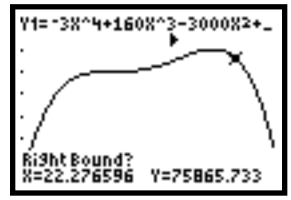
(You might need to reset Ymax to a larger value, say 95,000, so that the high point is not covered up by the equation when you trace the graph.)



Prepare to find the local (relative) maximum by pressing 2nd TRACE (CALC) 4 (maximum) .

Next, press ▷ to move the blinking cursor that appears to a position to the *left* of the high point. Press ENTER to mark the *left* bound for *x*.



<table>
<tr><td>Next, press [▷] to move the blinking cursor to a position to the right of the high point. Press [ENTER] to mark the right bound for x.</td><td>
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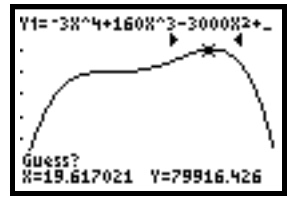
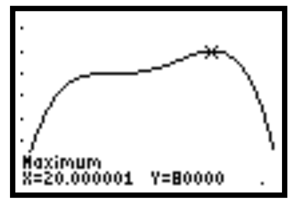
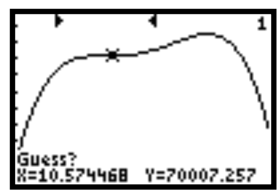
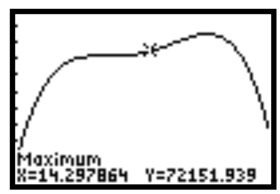
<tr><td>Use [◁] to move the cursor near your estimate of the high point and press [ENTER] . The TI-83 uses your guess to locate the highest point in the region between the two bound marks. The maximum value and the x-value at which it occurs is displayed at the bottom of the screen.</td><td>
</td></tr>
<tr><td>Revenue is greatest at 20 weeks with a value of R(20) = $80,000.

Return to the home screen and verify that Y1(20) gives the value displayed at the bottom of the graphics screen.</td><td>
</td></tr>
<tr><td>Both What about the other part of the curve that may contain a peak? Follow the same procedure as indicated above to investigate the curve on either side of x = 10.</td><td>
</td></tr>
<tr><td>Notice that the calculator returns the maximum at the right end of the interval. This indicates that there is not a relative maximum value in the interval we marked to test.</td><td>
</td></tr>
</table>

- The methods of this section also apply to finding relative or local minimum values of a function. The only difference is that to find the minimum instead of the maximum, initially press [2nd] [TRACE] (CALC) [3] (minimum) .

5.2 Inflection Points

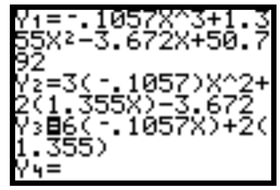
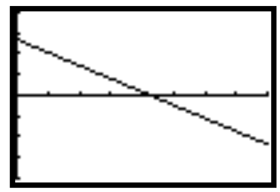
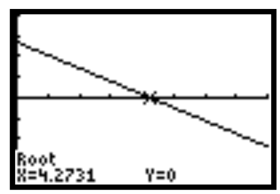
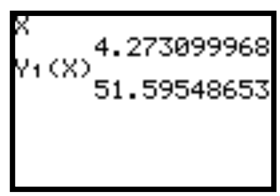
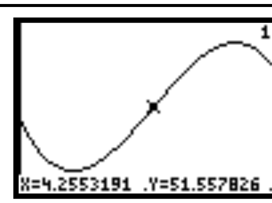
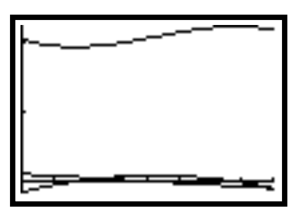
As was the case with optimal points, your calculator can be very helpful in checking your analytic work when you find points of inflection. You can also use the methods illustrated in Section 5.1.2 of this *Guide* to find the location of any maximum or minimum points on the graph of the first derivative to find the location of any inflection points for the function.

5.2.1 FINDING *X*-INTERCEPTS OF A SECOND DERIVATIVE GRAPH We first look at using the analytic method of finding inflection points -- finding where the graph of the second derivative of a function *crosses* the input axis.

To illustrate, consider a model for the percentage of students graduating from high school in South Carolina from 1982 through 1990 who entered post-secondary institutions:

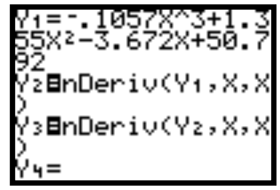
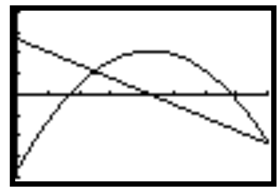
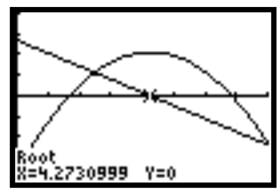
$$f(x) = {}^{-}0.1057x^3 + 1.355x^2 - 3.672x + 50.792 \text{ percent}$$

where $x = 0$ in 1982.

Enter $f(x)$ in the Y1 location of the Y= list, the first derivative in Y2 and the second derivative of the function in Y3. Turn off Y1 and Y2.	
The problem says the model is for 1982 through 1990 which corresponds to $0 \le x \le 8$. Thus, you are told the horizontal view. Choose an appropriate vertical view -- possibly y between $^{-}4$ and 4. Graph $f''(x)$ Remember that you should be able to clearly see any optimal points. Leave room at the bottom of the screen so that trace coordinates will not block your view of any important points on the graph.	 Since the second derivative is a line and we need to find the x-intercept, this is a "good" graph.
Use the methods illustrated in 5.1.1 of this *Guide* to find where the second derivative graph crosses the x-axis. (The TI-82 calls this intercept a "root" while the TI-83 calls it a "zero". Both are acceptable terms.)	
If you are asked to give the inflection *point* of $f(x)$, you should give both an x-value and a y-value. Return to the home screen and type X. The x-value you just found is stored in the x-location until you change it by tracing, using a menu item from the CALC menu, etc. Find the y-value by substituting this x-value in the function located in Y1.	
Next, look a graph of the function and verify that there does appear to be an inflection point at $x \approx 4.27$. To do this, turn off Y3 and turn on Y1. Draw the graph of Y1.	
In this problem, it is difficult to find a window that shows a good graph of both the function and its derivatives. However, if you draw a graph of all three, you can roughly see that the location of the inflection point of the function occurs at the location of the maximum of the first derivative and at the location of the x-intercept of the second derivative.	

Your calculator usually draws an accurate graph of the first derivative of a function when you use nDeriv. However, neither the TI-82 nor the TI-83 has a built-in method to calculate or graph $f''(x)$, the second derivative.

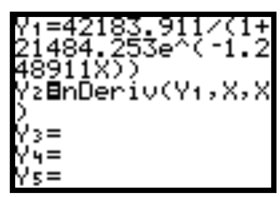
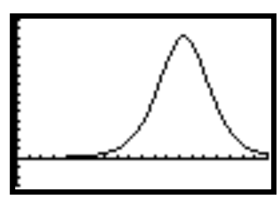
As illustrated below, you can try to use nDeriv($f'(x)$) for $f''(x)$. Be cautioned, however, that nDeriv($f'(x)$) sometimes "breaks down" and gives invalid results. If this should occur, the graph of nDeriv($f'(x)$) appears very jagged and this method should not be used.

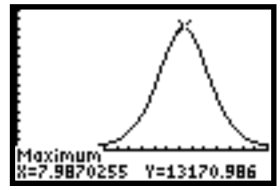
Enter $f(x)$ in the Y1 location of the Y= list, your calculator's numerical derivative in the Y2 location, and the derivative of the calculator's first derivative in the Y3 location. Turn off Y1.	
Draw the graphs in a viewing window that gives a good view of the slope graph, $f'(x)$ in Y2 , and its derivative, $f''(x)$ in Y3. The graph on the right was drawn with $0 \le x \le 8$ and $^-4 \le y \le 4$. (Note that Y3 appears to give good results here.)	
Use the methods of 5.1.1 of this *Guide* to find the x-intercept of the second derivative. (It takes a little longer here than it did before!) Be sure to use  to move to the graph of the line before giving the lower (left) bound for the root (zero).	

5.2.2 FINDING INFLECTION POINTS WITH YOUR CALCULATOR

Remember that an inflection point is a point of greatest or least slope. Whenever finding the second derivative of a function is tedious or you do not need an exact answer from an analytic solution, you can very easily find an inflection point of a function by finding where the first derivative of the function has a maximum or minimum value.

We illustrate this process with the function giving the number of polio cases in 1949: $y = \dfrac{42183.911}{1 + 21484.253e^{-1.248911t}}$ where $t = 1$ on January 31, 1949, $t = 2$ on February 28, 1949, etc.

Enter the function in the Y1 location of the Y= list and the calculator's numerical derivative in the Y2 location. Turn off Y1.	
Graph Y2 in an appropriate viewing window with x between 0 and 12 and something like y between $^-3000$ and 15,000 with tick marks every 1000 units. Remember that you should be able to clearly see any optimal points. Leave room at the bottom of the screen so that trace coordinates will not block your view of any important points on the graph.	

Use the methods illustrated in 5.1.2 of this *Guide* to find the maximum of the slope graph. The x-value of the maximum of the slope graph is the x-value of the inflection point of the function.	
If you are asked to give the inflection point, you should give both an x-value and a y-value. Find the y-value by substituting this x-value in the function located in Y1. The rate of change at that time is obtained by substituting this x-value in Y2.