Chapter 3 Describing Change: Rates

3.1 Average Rates of Change

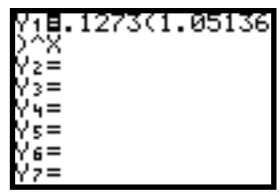
As you calculate average and other rates of change, remember that each numerical answer should be accompanied by units telling how the quantity is measured. You should also be able to interpret each numerical answer. It is only through their interpretations that the results of your calculations will be useful in real-world situations.

3.1.1 FINDING AVERAGE RATES OF CHANGE Finding an average rate of change using a model is just a matter of evaluating the model at two different values of the input variable and dividing by the difference in those input values. Consider this example.

The population density of Nevada from 1950 through 1990 can be approximated by the model $P(t) = 0.1273(1.05136)^t$ people per square mile where t is the number of years since 1900. You are asked to calculate the average rates of change between from 1950 through 1980 and between 1980 and 1990.

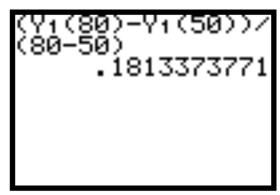
Enter the equation in the Y1 location of the Y= list.

(Remember that you must use x as the input variable in the graphing list. You do not have to use the first function location -- any of them will do.)

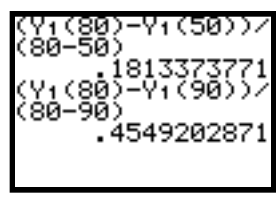


Return to the home screen with [2nd] [MODE] (QUIT) .

The average rate of change of the population density between 1950 and 1980 is $\frac{P(80) - P(50)}{80 - 50} = \frac{\text{Y1}(80) - \text{Y1}(50)}{80 - 50}$. Enter this quotient, remembering to use parentheses around both the numerator and the denominator.



To find the average rate of change between 1980 and 1990, simply recall the last expression with [2nd] [ENTER] (ENTRY) and replace the 50 by 90. Press [ENTER] .



Recall that rate of change units are output units per input units. We see that on average, the population density increased by about 0.18 person per square mile per year between 1950 and 1980 and by approximately 0.45 person per square mile per year between 1980 and 1990.

- If you have many average rates of change to calculate, you could put the average rate of change formula in the graphing list: Y2 = (Y1(A) – Y1(B))/(A – B). (You, of course, need to have a model in Y1.) Then, on the home screen, store the inputs of the two points in A and B: 80 → A : 90 → B. All you need do then is type Y1 and press enter. Store the next set of inputs into A and B and use [2nd] [ENTER] to recall Y1 to find the average rate of change between the two new points. Try it!

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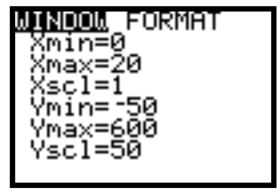
3.3 Tangent Lines

We first examine the principle of local linearity which says that if you are close enough, the tangent line and the curve are indistinguishable. We then use the calculator to draw tangent lines. There are two ways you can have your calculator draw a tangent line at a point on a curve. In this section, we consider one of these. The other method will be discussed in Chapter 4 of this *Guide*.

3.3.1 MAGNIFYING A PORTION OF A GRAPH The ZOOM menu of your calculator allows you to magnify any portion of the graph of a function. Suppose we are investigating the graph of $y = ^{-}x^2 + 40x + 50$ and the tangent line, $y = 20x + 150$, to the graph of this function at $x = 10$.

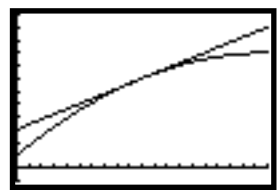
Enter $y = ^{-}x^2 + 40x + 50$ in Y1 and $y = 20x + 150$ in Y2.

Set the view shown to the right with WINDOW.



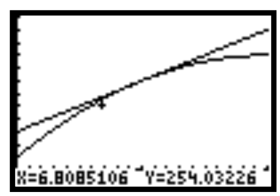
Remember to turn off any stat plot that is on with [2nd] [Y=] (STAT PLOT) [4] (Plots Off) [ENTER], and graph the function and the line tangent to it at $x = 10$ with [GRAPH].

We now want to "box in" the point of tangency and magnify that portion of the graph.



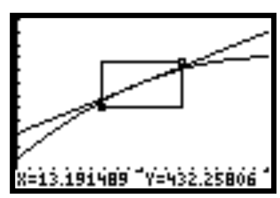
Press [ZOOM] [1] (ZBox), use [◁] to move the cursor to the left of the point of tangency, and use [▽] to move the cursor down from the point of tangency. (You may not have the same values as those shown on the right.)

Press [ENTER] to fix the lower left corner of the box.



Use [▷] and [△] to move the cursor to the opposite corner of your "zoom" box.

The point of tangency should be close to the center of your box.



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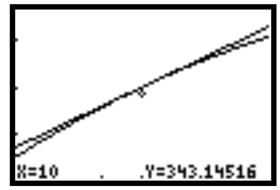
Press [ENTER] to magnify the portion of the graph inside the box. Look at the view you now see with WINDOW. Repeat the above process if necessary.

It is easy to see that the graph of the function and the graph of the tangent line are almost the same very close to the point of tangency.



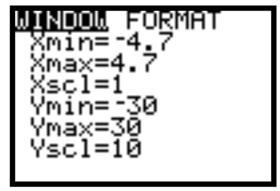
3.3.2 DRAWING A TANGENT LINE

The DRAW menu of your calculator contains the instruction to draw a tangent line to a curve at a point. To illustrate the process, we draw several tangent lines on the graph of $f(x) = x^3 + x^2 - 10x - 2$. We also investigate what the calculator does when you ask it to draw a tangent line where the line cannot be drawn.

Enter $f(x) = x^3 + x^2 - 10x - 2$ in Y1.

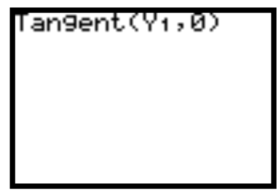
Set the view shown to the right with WINDOW. (The TI-83 window does not have "format" at the top.)

Press [GRAPH] .



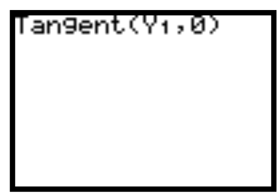
TI-82 Return to the home screen with [2nd] [MODE] (QUIT).

Draw the tangent line to the curve at $x = 0$ with [2nd] **Error!**



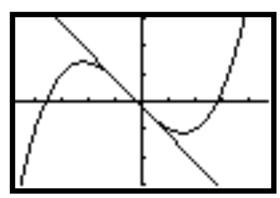
TI-83 Return to the home screen with [2nd] [MODE] (QUIT).

Draw the tangent line to the curve at $x = 0$ with [2nd] **Error!**

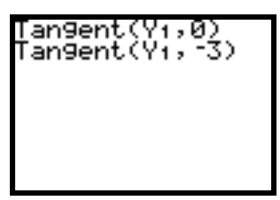


Both Notice that the tangent line cuts through the curve at $x = 0$.

It appears that $(0, ^-2)$ is an inflection point.



Return to the home screen, and recall the last entry with [2nd] [ENTER] (ENTRY) . Edit the statement so that you can draw the tangent line at $x = ^-3$.



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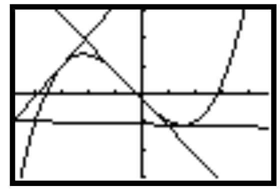
Once again recall the last entry on the home screen, and then draw the tangent line at $x = 1.5$. The tangent line is almost, but not quite, horizontal at $x = 1.5$.	

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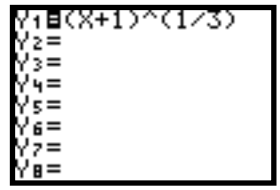
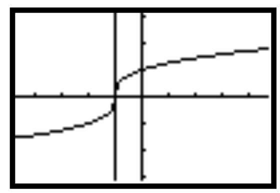
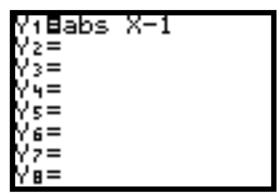
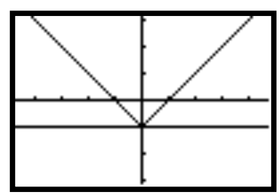
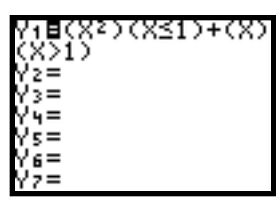
Let us now look at some special cases:

1. What happens if the tangent line is vertical? We consider the function $f(x) = (x+1)^{1/3}$ which has a vertical tangent at $x = {}^{-}1$.
2. How does the calculator respond when the tangent line cannot be drawn at a point? We illustrate what happens with $g(x) = |x| - 1$, a function that has a sharp point at $(0, {}^{-}1)$.
3. Does the calculator draw the tangent line at the joining point(s) of a piecewise continuous function? We consider two situations:
   a. $h(x)$, a piecewise continuous function that is continuous at all points and
   b. $m(x)$, a piecewise continuous function that is not continuous at $x = 1$.

1. Enter the function $f(x) = (x+1)^{1/3}$ in the Y1 location of theY= list. Remember that anytime there is more than one symbol in an exponent and you are not sure of the calculator's order of operations, enclose the power in parentheses.	
Draw the graph of the function with ZOOM 4 (ZDecimal). Return to the home screen and type the instruction Tangent(Y1, $^{-}1$). Press ENTER . The vertical tangent line is correctly drawn.	
2. Clear Y1 and enter the function $g(x) = \|x\| - 1$. The absolute value symbol is obtained with 2nd x^{-1} (ABS) X-T-θ .	
Draw the graph of the function with ZOOM 4 (ZDecimal). Return to the home screen and type the instruction Tangent(Y1, 0). Press ENTER .	
THIS IS INCORRECT! There is a sharp point at $(0, {}^{-}1)$, and the limiting positions of secant lines from the left and the right of that point are different. A tangent line cannot be drawn at $(0, {}^{-}1)$ because the instantaneous rate of change at that point does not exist.	The tangent line above should not be drawn according to *our* definition of instantaneous rate of change. Your calculator's definition is entirely different, and this is why the line is drawn. (See Section 4.3.1 of this *Guide*.)
3a. Clear Y1 and enter, as indicated, the function $h(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x & \text{when } x > 1 \end{cases}$ [Recall that the inequality symbols are accessed with 2nd MATH (TEST)].	 *h(x)* is continuous for all values of *x*.

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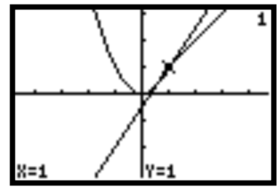
Draw the graph of the function with [ZOOM] [4] (ZDecimal) .

Return to the home screen and enter Tangent(Y1, 1).

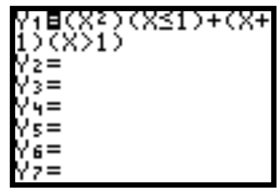
THE GRAPH YOU SEE IS INCORRECT because secant lines drawn with points on the right and left of $x = 1$ do not approach the same slope.



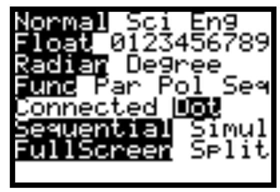
The tangent line above should *not* be drawn.

3b. Edit Y1 to enter, as indicated, the function

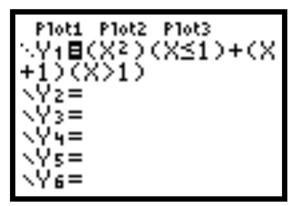
$$m(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x + 1 & \text{when } x > 1 \end{cases}$$



TI-82 Press [MODE] and choose Dot.

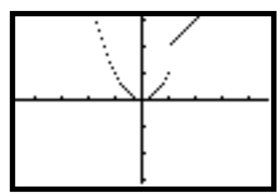


TI-83 Move the cursor to the left of Y1 and press [ENTER] six times to change the slanted line to a dotted line.



Both Draw the graph of the function with [ZOOM] [4] (ZDecimal) .

Since *m(x)* is not continuous at $x = 1$, the instantaneous rate of change does not exist at that point. The tangent line cannot be drawn at (1, 1).

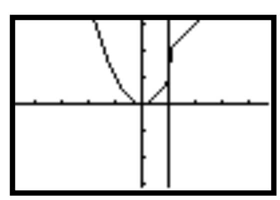


TI-82 Go to the MODE screen and return your calculator to Connected mode.

TI-83 Press [Y=] , use [◄] to move to the left of Y1, and press [ENTER] once to return your calculator to connected mode.

Both On the home screen, type the instruction Tangent(Y1, 1). Press [ENTER] .

THE GRAPH YOU SEE IS INCORRECT. A vertical tangent is drawn, but the tangent line certainly does not exist when $x = 1$.



Caution: Be certain that the instantaneous rate of change exists at a point before using your calculator to draw a tangent line at that point. Because of the way your calcu-

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lator computes instantaneous rates of change, it may draw a tangent line at a point on a curve where the tangent line does not exist.

3.5 Percentage Change and Percentage Rates of Change

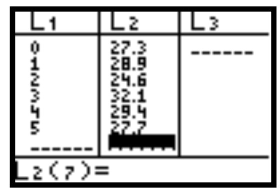
The calculations in this section involve no new calculator techniques. When calculating percentage change or percentage rates of change, you have the option of using a program or the home screen.

3.5.1 CALCULATING PERCENTAGE CHANGE Recall that program DIFF stores percentage changes (also called percentage differences) in output data in list L5. Consider the following data giving quarterly earnings for a business:

Quarter ending	Mar 1994	June 1994	Sept 1994	Dec 1994	Mar 1995	June 1995
Earnings (millions)	27.3	28.9	24.6	32.1	29.4	27.7

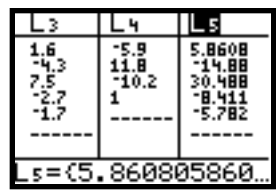
First, we enter the data in the calculator's lists L1 and L2.

Align the input data so that x is the number of quarters since March 1994. Input x in L1 and earnings (in millions) in L2.

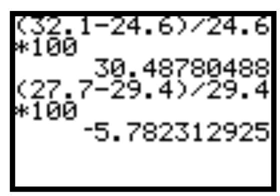


Run program DIFF and view the percentage change in list L5.

Notice that the percentage change from the end of September 1994 through December 1994 is about 30.5%. Also, from the end of March 1995 through June 1995, the percentage change is approximately −5.8%.



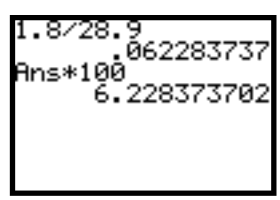
You may find it easier to calculate these values using the percentage change formula than have program DIFF do it for you.



3.5.2 CALCULATING PERCENTAGE RATE OF CHANGE Consider again the quarterly earnings for a business. Suppose you are told or otherwise find that the rate of change at the end of the June 1994 is 1.8 million dollars per quarter. Evaluate the percentage rate of change at the end of June 1994.

Divide the rate of change at the end of June 1994 by the earnings, in millions, at the end of June 1994 and multiply by 100 to obtain the percentage rate of change at that point.

The percentage rate of change in earnings at the end of June 1994 was approximately 6.2% per quarter.



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