Chapter 2 Ingredients of Change: Nonlinear Models

2.1 Exponential Functions and Models

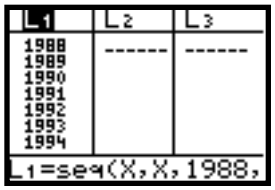
As we consider models that are not linear, it is very important that you be able to use scatter plots, numerical changes in output data, and the underlying shape of the basic functions to be able to identify which model best fits a particular set of data. Finding the model is only a means to an end -- being able to use mathematics to describe the changes that occur in real-world situations.

2.1.1 ENTERING EVENLY-SPACED INPUT VALUES (optional) When an input list consists of many evenly-spaced values, there is a calculator command that will generate the list so that you do not have to type the values in one by one. The syntax for this sequence command is *seq(formula, variable, first value, last value, increment)*. When entering years that differ by 1, the formula is the same as the variable and the increment is 1. Any letter can be used for the variable -- we choose to use *X*.

TI-82 Clear any old data from lists L1 and L2.

Position the cursor at the top of the first list so that the L1 is darkened. Generate the list of years beginning with 1988, ending with 1997, and differing by 1 with the keystrokes:

Error!

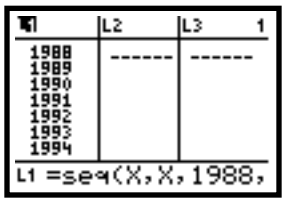


TI-83 Clear any old data from lists L1 and L2.

Position the cursor at the top of the first list so that the L1 is darkened. Generate the list of years beginning with 1988, ending with 1997, and differing by 1 with the keystrokes:

2nd STAT ▶ (OPS)

Error!



2.1.2 FINDING PERCENTAGE CHANGE When the input values are evenly spaced, use program DIFF to compute percentage change in the output values. If the data are perfectly exponential (i.e., every data point falls on an exponential model), the percentage change in the output values is constant. If the percentage change is "close" to constant, this is an indication that an exponential model *may* be appropriate.

Suppose the population of a small town between the years 1988 and 1997 is as follows:

Year	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
Population	7290	6707	6170	5677	5223	4805	4420	4067	3741	3442

Clear any old data, and enter the above data in lists L1 (year) and L2 (population). See Section 2.1.1 of this *Guide* for a convenient way to enter the years into L1.

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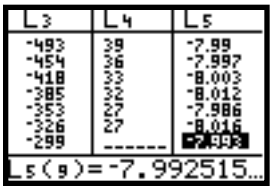
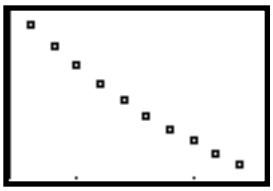
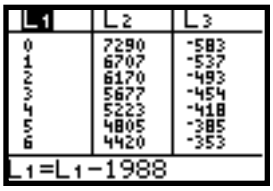
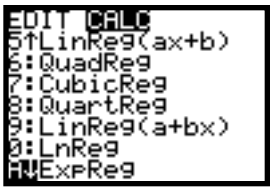
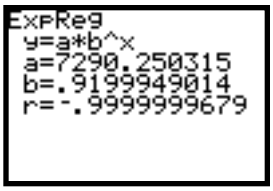
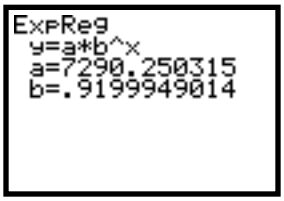
Run program DIFF and press STAT (1: EDIT) ENTER . Observe the first differences in L3, the second differences in L4, and the percentage changes in list L5. (Use ▷ to see L4 and L5.) The percentage change is very close to constant, so an exponential model may be a good fit.	 Use ▽ and △ to view all of the list.

2.1.3 FINDING AN EXPONENTIAL MODEL Use your calculator to find an exponential model that fits the data. The exponential model we use is of the form $y = ab^x$.

Construct a scatter plot of the data. Notice that the data curves rather than falling in a straight line pattern. An exponential model certainly seems appropriate!	
It is very important that you align large numbers (like years) whenever you find an exponential model. The model found by the calculator may not even be correct if you don't! Other alignments are possible, but we choose to align so that $x = 0$ in 1988.	
TI-82 Fit an exponential model to the data by pressing STAT ▷ (CALC) ALPHA A (ExpReg) ENTER .	
The best-fitting exponential model is displayed on the home screen. Copy the model to the Y= list (use VARS 5 ▷ ▷ 7 as explained in Section 1.4.8 of this *Guide*), overdraw the graph on the scatter plot with ZOOM 9 (ZoomStat) , and see that it gives a very good fit to the data.	
TI-83 Fit an exponential model to the data and copy the model to the Y1 location of the Y= list by pressing STAT ▷ (CALC) 0 (ExpReg) 2nd 1 (L1) , 2nd 2 (L2) , VARS ▷ (Y-VARS) 1 (Function) 1 (Y1) . Press ENTER .	

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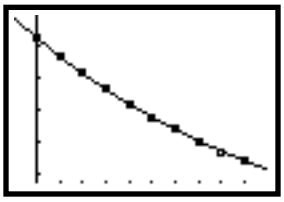
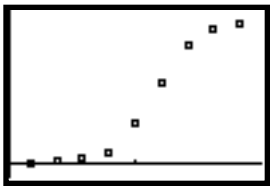
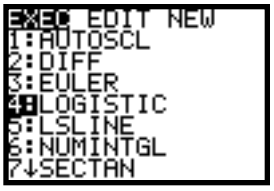
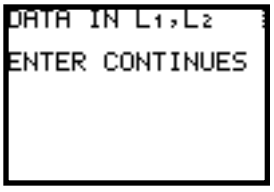
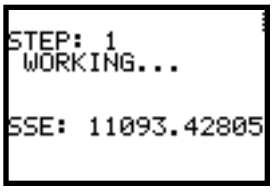
Both Press GRAPH to overdraw the graph of the model on the scatter plot. The model gives a very good fit to the data.	

2.1.4 FINDING A LOGISTIC MODEL Use your calculator to find a logistic model of the form $y = \frac{L}{1 + Ae^{-Bx}}$. The logistic model that you obtain may be slightly different from a logistic model found with another calculator. Logistic models in *Calculus Concepts* were found using a TI-83. Refer to the following TI-82 discussion for the comparable logistic model that best fits the data given in Example 2, Section 2.1 of the text. As with the exponential model $y = ab^x$, large input values must be aligned before fitting a logistic model to data.

(Program LOGISTIC finds a "best-fit" logistic model rather than a logistic model with a user-input limiting value L such that no data value is ever greater than L.)

Clear any old data, and enter the following in lists L1 and L2:

Aligned end of month	1	2	3	4	5	6	7	8	9
Total swimsuits sold	4	12	25	58	230	439	648	748	769

Construct a scatter plot of the data. A logistic model seems appropriate.	
TI-82 Use program[1] LOGISTIC to fit the logistic model. **Note:** To use this program, the input data must be in order, from smallest to largest, in list L1. Have the corresponding output data in L2. Run program LOGISTIC with PRGM followed by the number of the location of the program. Press ENTER .	 Your program list may not look exactly like this.
The first message you see reminds you that the input data should be in list L1 and the output data in list L2 . If you have not done this, press ON and choose 2 (QUIT) . Enter the data and then rerun the program.	
After pressing ENTER to continue, the program displays several messages to let you know it is working. (These messages can be ignored. You can see *SSE* being reduced as the model is fit.)	

[1]The authors express their sincere appreciation to Dr. Dan Warner and Robert Simms of the Mathematical Sciences Department at Clemson University for their invaluable help with program LOGISTIC .

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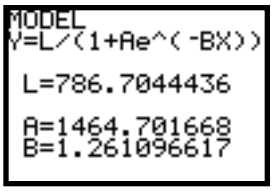
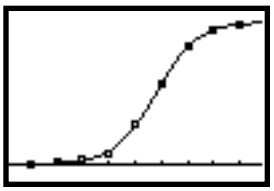
After a short time, the best-fit logistic model is displayed on the screen. The program also places the equation of the model in the Y4 location of the Y= list.

(CLEAR clears the screen when you are finished.)



Press GRAPH to see the model overdrawn on the scatter plot.

(The next section of this *Guide* shows you how to recall the model parameters if they are needed.)

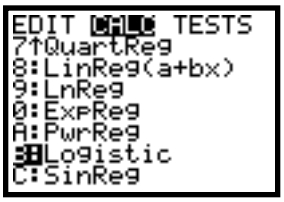


TI-83 Fit a logistic model to the data and copy the model to the Y1 location of the Y= list by pressing

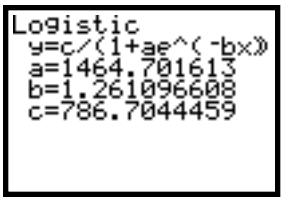
STAT ▶ (CALC) ▲ ▲ (Logistic) ENTER

2nd 1 (L1) , 2nd 2 (L2) , VARS ▶

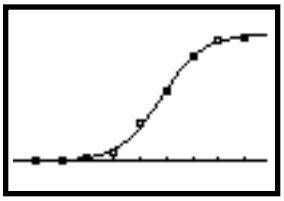
(Y-VARS) 1 (Function) 1 (Y1). Press ENTER .



This model will take longer to generate than the other models. Notice that the TI-83 uses the variable *c* for the limiting value that your text calls *L*.



Press GRAPH to see the graph of the model on the scatter plot.



- **Both** Provided the input values are evenly spaced, program DIFF might be helpful when you are trying to determine if a logistic model is appropriate for certain data. If the first differences (in list L3 after running program DIFF) *begin small, peak in the middle*, and *end small*, this is an indication that a logistic model may provide a good fit to the data. Such is true for this data set because the first differences are 8, 13, 33, 172, 209, 209, 100, and 21.

2.1.5 RECALLING MODEL PARAMETERS

Rounding of model parameters can often lead to incorrect or misleading results. You may find that you need to use the full values of model parameters after you have found a model.

It would be tedious to copy all these digits into another location of your calculator. You don't have to! The following procedure applies for any model you find using one of the built-in regressions (*i.e.*, from the STAT CALC menu) in your calculator. Of course, once another model is found, previous parameters are no longer stored in the calculator's memory.

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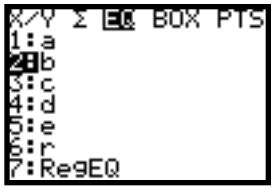
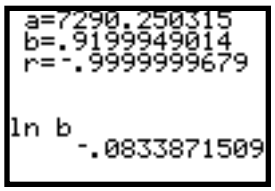
TI-82 As an example, we use the parameter *b* for the exponential model found in Section 2.1.3. However, this same procedure applies to any model you have found using the STAT CALC menu.

To recall the value of *b* in the model $y = ab^x$, press
VARS 5 (Statistics) ▷ ▷ (EQ) 2 (b) ENTER .



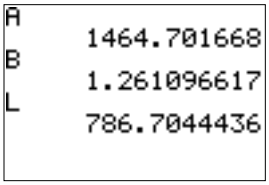
The full value of *b* will be "pasted" where ever you had the cursor before beginning the above keystrokes.

Note: Remember that you can only recall the parameters before you use the menu to find a different model. Once a different model is found, the parameters are given for the new model.



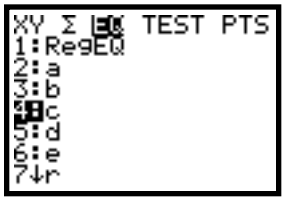
The above procedure does not apply to models found with programs you have entered in the calculator.

If you need to recall the values of A, B, or L for the logistic model, return to the home screen and type
ALPHA MATH (A) ENTER , ALPHA MATRIX
(B) ENTER , and then ALPHA) (L) ENTER .

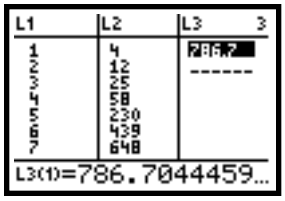


TI-83 As an example, we use the parameter *c* for the logistic model found in Section 2.1.4. However, this same procedure applies to any model you have found using the STAT CALC menu.

To recall the value of *c* in $y = c/(1+ae^{-bx})$, press
VARS 5 (Statistics) ▷ ▷ (EQ) 4 (c) ENTER .



The full value of *c* will be "pasted" where ever you had the cursor before beginning the above keystrokes.



2.1.6 RANDOM NUMBERS Imagine all the real numbers between 0 and 1, including the 0 but not the 1, written on identical slips of paper and placed in a hat. Close your eyes and draw one slip of paper from the hat. You have just chosen a number "at random."

Your calculator doesn't offer you a random choice of all real numbers between 0 and 1, but it allows you to choose, *with an equal chance of obtaining each one*, any of 10^{14} different numbers between 0 and 1 with its random number generator called rand.

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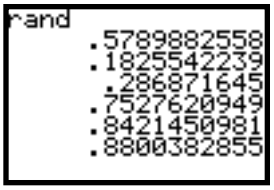
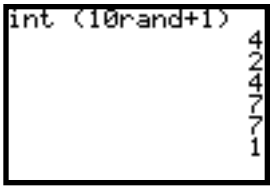
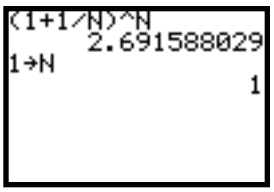
First, "seed" the random number generator. (This is like mixing up all the slips of paper in the hat.)

Pick some number, not the one shown on the right, and store it as the "seed". (Everyone needs to have a different seed, or the choice will not be random.) The random number generator is accessed with MATH ◀ (PRB) 1 (rand) .	
Enter rand again, and press ENTER several times. Your list of random numbers should be different from the one on the right if you entered a different seed.	
If you want to choose, at random, a whole number between 1 and N, enter int(N rand + 1) by pressing MATH ▶ 4 (int) (N MATH ◀ (PRB) 1 (rand) + 1) ENTER f or a specific value of N. Repeatedly press ENTER to choose more random numbers. For instance, the screen to the right shows several values that were chosen with $N = 10$.	

2.2 Exponential Models in Finance

You are probably familiar with the compound interest formulas. This section introduces you to some new methods of using your calculator with familiar formulas.

2.2.1 REPLAY OF PREVIOUS ENTRIES TO FIND FORMULA OUTPUTS You can recall expressions previously typed by repeatedly using the calculator's last entry feature. Learn to use this time-saving feature of your calculator.

On the home screen, type the formula $\left(1+\frac{1}{n}\right)^n$ and press ENTER . The output depends on the value of n. You probably obtained a different output value because you have a different value stored in N. Store 1 in N.	

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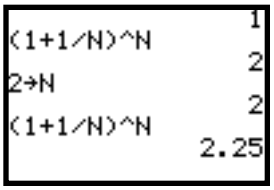
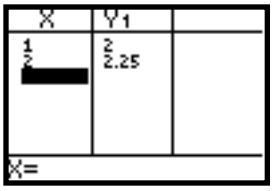
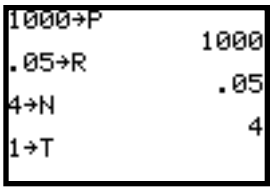
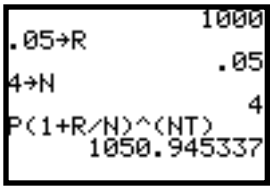
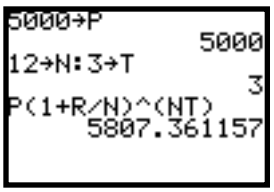
Press 2nd ENTER (ENTRY) twice (or as many times as it takes to again display the amount formula on the screen), and then press ENTER. The formula is now evaluated at N = 1. Store 2 in N and repeat the procedure.	

Because this formula contains only one input variable, you could enter it in the Y= list, using X as the input variable, and find the outputs using the TABLE.

Enter Y1 = (1 + 1/X)^X. Refer to 1.2.1 of this *Guide* to review the information about evaluating outputs using the TABLE.	

2.2.2 DETERMINING FUTURE VALUE You can save a lot of keystrokes by recalling expressions previously typed by repeatedly using the calculator's last entry feature. When a formula contains more than one input variable, it is easier to recall the last entry on the home screen than to try to use the TABLE. To illustrate, consider the compound interest formula -- one that contains several input variables.

On the home screen, type in the formula for the amount in an account paying $r\%$ interest (compounded n times a year) on an initial deposit of $\$P$ over a period of t years: $$Amount = P\left(1+\frac{r}{n}\right)^{nt} \text{ dollars}$$ The value you obtain with ENTER depends on the values your calculator has stored in P, R, and T.	 Store 1000 in P, 0.05 in R, 4 in N, and 1 in T.
Press 2nd ENTER (ENTRY) three times (or as many times as it takes to again display the formula on the screen) and then press ENTER. The formula has been evaluated at the stored values of the variables.	
Determine the accumulated amount in an account if \$5000 is invested at 5% interest compounded monthly for 3 years by repeating the procedure described above. (Note that since the value of R has not changed, it is not necessary to again store 0.05 to R.) The 3-year future value of the amount in the account is \$5807.36.	

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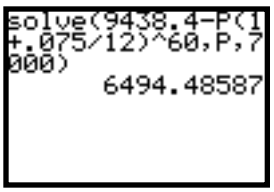
2.2.3 FINDING PRESENT VALUE The present value of an investment is easily found with the calculator's solve routine. For instance, suppose you want to solve the equation $9438.40 = P\left(1 + \frac{0.075}{12}\right)^{60}$ for the present value P.

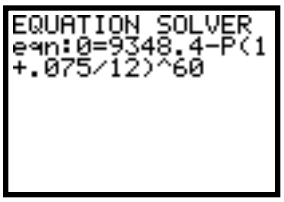
TI-82 Refer to Section 1.2.2 of this *Guide* for instructions on using the TI-82's solve routine and enter the expression on the right.

Remember that a *guess*, here entered as 7000, can be obtained from viewing a graph of `Y1 = 9438.4 – X(1+.075/12)^60` and tracing to the approximate location where the graph crosses the horizontal axis.



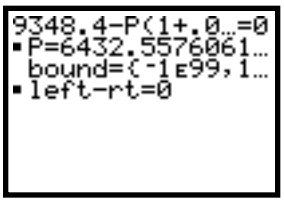
TI-83 For this illustration, we use solver Method 2 discussed in Section 1.2.2 of this *Guide*.

Press MATH 0 to access the solver, ▲ CLEAR to clear any previously-entered equation, and enter the expression on the right.



Press ENTER. With the blinking cursor on the line with P, press ALPHA ENTER to find P = \$6432.56.

Remember that a *guess* for P should be entered before solving if there is more than one solution. The number of possible solutions as well as guesses for their values can be obtained from viewing a graph of `Y1 = 9438.4 – X(1 + 0.075/12)^60` and tracing to the approximate locations where the graph crosses the horizontal axis.



- If you prefer, you could find the present value by graphically locating the x-intercept of `Y1 = 9438.4 – X(1 + 0.075/12)^60`. Refer to Section 1.2.3 of this *Guide* for more detailed instructions.

2.3 Polynomial Functions and Models

You will in this section learn how to fit models to data that have the familiar shape of a parabola or a cubic. Using your calculator to find these models involves basically the same procedure as when using it to find linear and exponential models.

2.3.1 FINDING SECOND DIFFERENCES When the input values are evenly spaced, use program DIFF to compute second differences in the output values. If the data are perfectly quadratic (*i.e.*, every data point falls on a quadratic model), the second differences in the output values are constant. If the second differences are "close" to constant, this is an indication that a quadratic model *may* be appropriate.

Clear any old data, and enter the roofing job data in lists L1 and L2:

Months after January	1	2	3	4	5	6
Number of jobs	12	14	22	37	58	84

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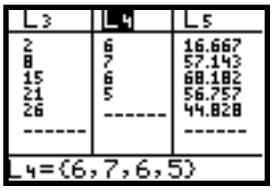
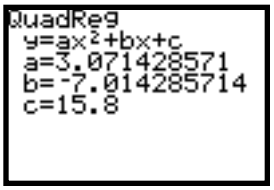
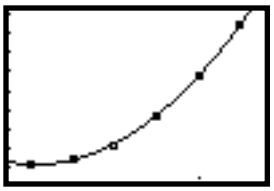
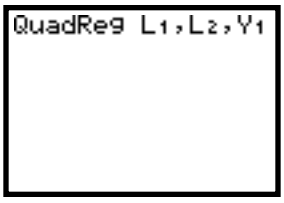
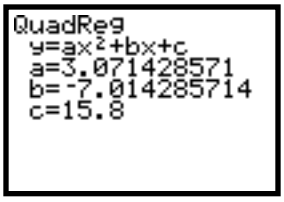
The input values are evenly spaced, so we can see what information is given by viewing the second differences.

Run program DIFF and observe the first differences in list L3, the second differences in L4, and the percentage differences (changes) in list L5. The second differences are close to constant, so a quadratic model may be a good fit. Construct a scatter plot of the data. A quadratic model seems appropriate!	

2.3.2 FINDING A QUADRATIC MODEL Use your calculator to obtain a quadratic model that fits the data. Your calculator's quadratic model is of the form $y = ax^2 + bx + c$.

TI-82 Press [STAT] [▶] (CALC) [6] (QuadReg) [ENTER] .	
The best-fitting quadratic model is displayed on the home screen.	
Copy the model to the Y= list, overdraw the graph on the scatter plot, and see that it gives a good fit to the data.	
TI-83 Fit a quadratic model and copy it to the graphing list by pressing [STAT] [▶] (CALC) [5] (QuadReg) [2nd] [1] (L1) [,] [2nd] [2] (L2) [,] [VARS] [▶] (Y-VARS) [1] (Function) [1] (Y1) .	
Press [ENTER] to see the model. Overdraw the graph of the model on the scatter plot.	

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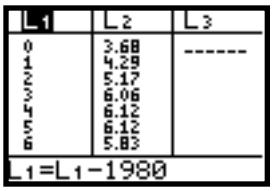
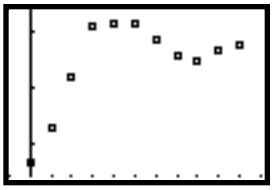
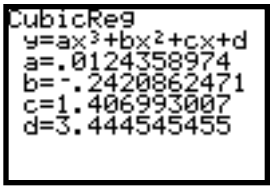
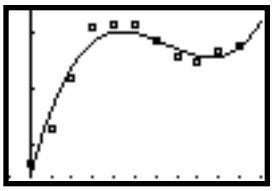
2.3.3 FINDING A CUBIC MODEL Whenever a scatter plot of the data shows a single change in concavity, a cubic or logistic model is appropriate. If a limiting value is apparent, use the logistic model. Otherwise, a cubic model should be considered. When appropriate, use your calculator to obtain the cubic model that best fits data. The calculator's cubic model is of the form $y = ax^3 + bx^2 + cx + d$.

Clear any old data, and enter the average price in dollars per 1000 cubic feet of natural gas for residential use in the U.S. from 1980 through 1990 in lists L1 and L2:

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Price	3.68	4.29	5.17	6.06	6.12	6.12	5.83	5.54	5.47	5.64	5.77

First, clear your lists, and then enter the data. In order to work with smaller coefficients, align the data so that x represents the number of years since 1980.	
Draw a scatter plot of these data. Notice that a concavity change is evident, but there do not appear to be any limiting values. Thus, a cubic model is appropriate to fit the data.	
TI-82 Press STAT ▷ (CALC) 7 (CubicReg) ENTER .	
The best-fitting cubic model is displayed on the home screen.	
Copy the model to the Y= list, overdraw the graph on the scatter plot, and see that it gives a reasonably good fit to the data.	

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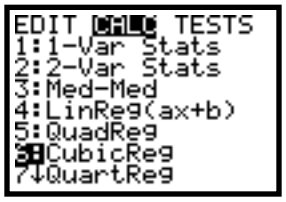
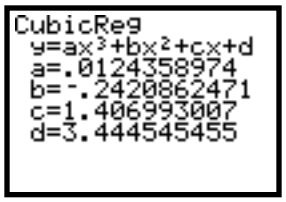
TI-83 Fit a cubic model and copy it to the graphing list by pressing STAT ▶ (CALC) 6 (CubicReg) 2nd 1 (L1) , 2nd 2 (L2) , VARS ▶ (Y-VARS) 1 (Function) 1 (Y1) .	
Press ENTER to see the model and to overdraw the graph of the model on the scatter plot.	

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