

Chapter 10 Analyzing Multivariable Change: Optimization



10.2 Multivariable Optimization

As you might expect, multivariable optimization techniques on your calculator are very similar to those that were discussed in Chapter 5. The basic difference is that the algebra required to get the expression that comes from solving a system of equations in several unknowns down to one equation in one unknown is sometimes fairly difficult. However, once your equation is of that form, all the optimization procedures are the same as previously discussed.

10.2.1 FINDING CRITICAL POINTS To find critical points for a multivariable function (points at which maxima, minima, or saddle points occur), find the point or points at which the partial derivatives with respect to each of the input variables are zero. Once you use derivative formulas to find the partial derivative with respect to each input variable, set these partial derivatives formulas each equal to 0, and use algebraic methods to obtain one equation in one unknown input. Then, you can use the your calculator's solve routine to obtain the solution to that equation.

We illustrate these ideas with a model for the total daily intake of organic matter required by a beef cow grazing the Northern Great Plains rangeland:

$$I(s, m) = 8.61967 - 1.244s + 0.0897s^2 - 0.20988m + 0.035947m^2 + 0.214915sm \text{ kg per day}$$

Find the two partial derivatives and set each of them equal to 0 to obtain

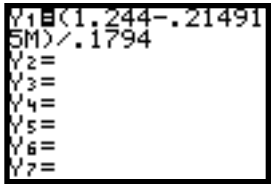
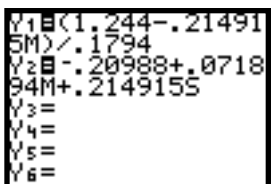
$$\frac{\partial I}{\partial s} = -1.244 + 0.1794s + 0.214915m = 0 \quad \text{and} \quad \frac{\partial I}{\partial m} = -0.20988 + 0.071894m + 0.214915s = 0$$

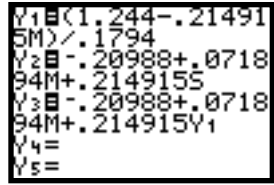
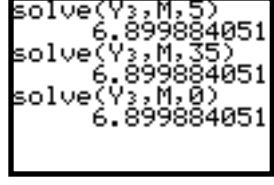
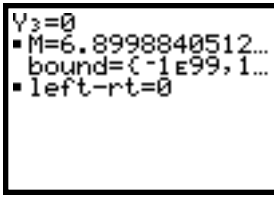
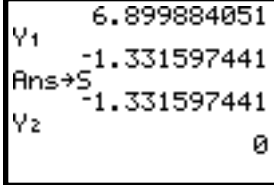
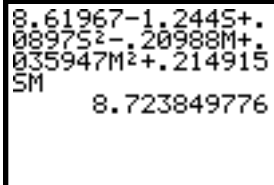
Solve one of the equations for one of the variables, say $\frac{\partial I}{\partial s}$ for s , to obtain

$$s = \frac{1.244 - 0.214915m}{0.1794}$$

(Remember that to solve for a quantity means that it must be by itself on one side of the equation and the other side of the equation cannot contain that letter at all.) Now, let your calculator do the rest of the work.

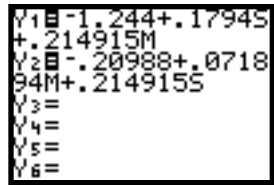
Warning: Everything that you do in your calculator depends on the partial derivative formulas that you have found using derivative rules. Double check your analytic work before you use any of the methods illustrated below.

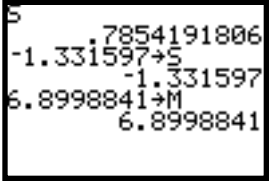
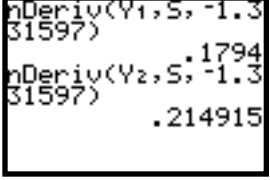
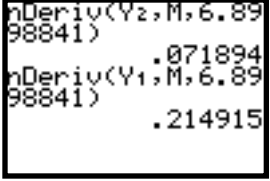
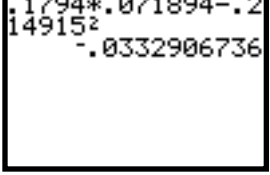
<p>Enter the expression for s, using the <i>same</i> letters that are in the problem situation, in Y1.</p> <p>Remember that any numerator, denominator, or power that consists of more than one number or letter should be enclosed in parentheses.</p>	
<p>Type the expression for the <i>other</i> partial derivative, $\frac{\partial I}{\partial m}$, exactly as you have it written on your paper, in Y2. (The expression you type should be equal to zero.)</p>	

<p>Now, put the cursor in the Y3 location. Type $R_{01} \cdot 2$ and press ENTER. [Access with 2nd STO (Rcl).]</p> <p>Next, replace every S in Y3 with Y1. What you have just done is substitute the expression for s in the other partial derivative!</p>	
<p>TI-82 The only thing that is left is to solve the equation $Y3 = 0$. Use the solve routine to do this.</p> <p>You can try several different guesses and see that they all result in the same solution. The best approach, however, is to note that Y3 is a linear equation and therefore has only one root.</p>	
<p>TI-83 The only thing that is left is to solve the equation $Y3 = 0$. Use the equation solver to do this.</p> <p>You can try several different guesses and see that they all result in the same solution. The best approach, however, is to note that Y3 is a linear equation and therefore has only one root.</p>	
<p>Both Return to the home screen and type M to recall the value you just found. Remember that you have the expression for s in Y1, so just type Y1 to find the value of s. The calculator, however, doesn't know this is s, so store the answer in memory location s.</p> <p>Note that if you now find the value of Y2, it should be 0.</p>	
<p>Either type the multivariable function $I(s, m)$ in one of the unused locations in the Y= list and evaluate it or just type $I(s, m)$ on the home screen. (You already have the values you found for s and m stored in those variables.)</p> <p>What all this has done is locate a critical point of $I(s, m)$ at $s \approx -1.33$ and $m \approx 6.90$.</p>	

10.2.2 THE DETERMINANT TEST Now that you have found one or more critical points, the next step is to classify those point(s) as points at which a maximum, a minimum, or a saddle point occurs. The Determinant Test will often give you the answer. Because this test uses derivatives, the calculator's numerical derivative nDeriv can help you.

For the beef cattle example in Section 10.2.1 with critical point $s \approx -1.33$ and $m \approx 6.90$, you now need to calculate some second partial derivatives evaluated at this point. Recall that we have $\frac{\partial I}{\partial m} = I_m$ in Y2. However, Y1 does not contain $I_s = \frac{\partial I}{\partial s}$.

<p>Clear Y1 and enter $\frac{\partial I}{\partial s} = -1.244 + 0.1794s + 0.214915m$.</p> <p>(We will not use Y3 again, so you can also clear that location if you wish.)</p>	
--	---

<p>Return to the home screen, and recall S and M to be certain that the critical point values are still stored in those locations. If not, store the critical values in those locations. This is a very important step!</p> <p>(You will have better accuracy with the Determinant Test if you use several decimal places for the values.)</p>	
<p>Now, if we take the derivative of $Y1 = I_s$ with respect to s, we will have I_{ss}, the second partial derivative of $I(s, m)$. Find $I_{ss} \approx 0.1794$.</p> <p>If we take the derivative of $Y2 = I_m$ with respect to s, we will have I_{ms}, the partial derivative of $I(s, m)$ with respect to m and then s. Find $I_{ms} \approx 0.2149$.</p>	 <p>Be certain the value you type after the variable matches the input that is varying.</p>
<p>Now, if we take the derivative of $Y2 = I_m$ with respect to m, we will have I_{mm}, the second partial derivative of $I(s, m)$. Find $I_{mm} \approx 0.0719$.</p> <p>If we take the derivative of $Y1 = I_s$ with respect to m, we will have I_{sm}, the partial derivative of $I(s, m)$ with respect to s and then m. Find $I_{sm} \approx 0.2149$.</p>	 <p>Be certain the value you type after the variable matches the input that is varying.</p>
<p>Now, find the value of $D \approx (0.1794)(0.0719) - (0.2149)^2$. Since $D < 0$, the Determinant Test tells us that the point $(s, m, I) \approx (-1.33, 6.90, 8.72)$ a saddle point.</p>	

- The values of the second partial derivatives were not very difficult to determine without the calculator in this example. However, with a more complicated function, we strongly suggest using the above methods to provide a check on your analytic work to avoid making simple mistakes.

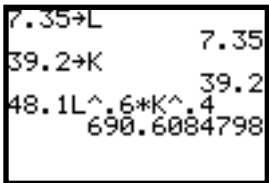


10.3 Optimization Under Constraints

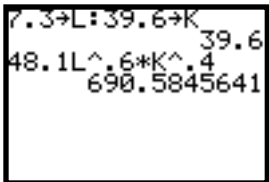
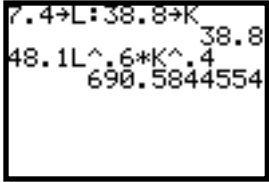
Optimization techniques on your calculator when a constraint is involved are exactly the same as those discussed in Sections 10.2.1 and Section 10.2.2 except that there is one more equation involved in the analytic process.

10.3.1 CLASSIFYING OPTIMAL POINTS UNDER CONSTRAINED OPTIMIZATION

Use the methods indicated in Section 10.2.1 to find the critical point $L = 7.35$, $K = 39.2$. We illustrate the procedure for determining if this is the point at which a maximum or minimum occurs for the Cobb-Douglas Production function $f(L, K) = 48.1L^{0.6}K^{0.4}$ subject to the constraint $g(L, K) = 8L + K = 98$.

<p>Find $f(L, K)$ at $L = 7.35, K = 39.2$.</p> <p>We now use the methods of Section 10.2.2 to test close points that are on the constraint.</p>	
---	---

We now test *close* points. You need to remember that whatever close points you test, they must be near the critical point and *they must be on the constraint* $g(L, K)$. Be wary of rounding during the following procedure. Rounding of intermediate calculations and/or critical points can give a false result when the “close” point is very close to the optimal point.

<p>Choose a new value of L that is less than L, say 7.3. Determine the value of K so that $8L + K = 98$:</p> $8(7.3) + K = 98, \text{ so } K = 98 - 58.4 = 39.6.$ <p>Now find $f(L, K)$ at $L = 7.3, K = 39.6$.</p>	
<p>Choose another nearby value of L, but this time choose one that is more than L, say 7.4. Determine the value of K so that $8L + K = 98$:</p> $8(7.4) + K = 98, \text{ so } K = 98 - 59.2 = 38.8.$ <p>Now find $f(L, K)$ at $L = 7.4, K = 38.8$.</p>	

Since $f(7.35, 39.2) \approx 690.608$ is greater than both $f(7.3, 39.6) \approx 690.585$ and $f(7.4, 38.8) \approx 690.585$, we conclude that $L = 7.35$ thousand worker hours, $K = \$39.2$ thousand is the point at which the maximum value of $f(L, K)$ occurs. Rounding this answer to make sense in the context of the problem, we find that 691 mattresses is a maximum production level.

- When you test close points, it is best to use all the decimal places provided by the calculator for the optimal values when classifying optimal points since rounding will, in many cases, give inaccurate or misleading results in the classification.