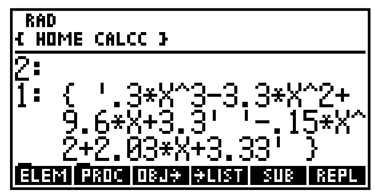
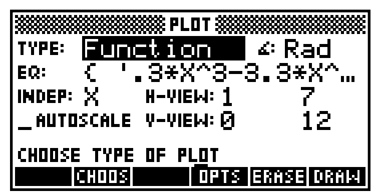
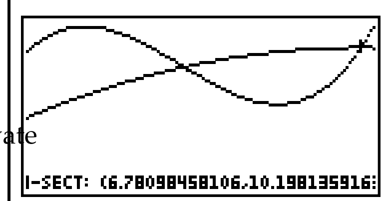
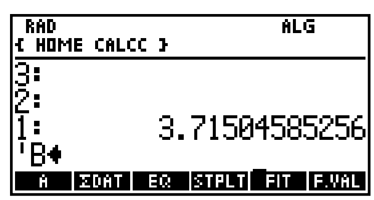
Chapter 7 Analyzing Accumulated Change: More Applications of Integrals

7.1 Differences of Accumulated Changes

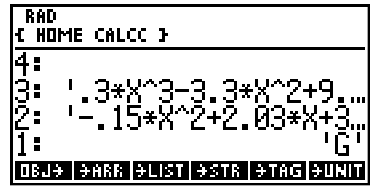
This chapter helps you effectively use your calculator's numerical integrator with various applications pertaining to the accumulation of change. In this first section, we focus on the differences of accumulated changes.

7.1.1 FINDING THE AREA BETWEEN TWO CURVES Finding the area of the region enclosed by two functions uses many of the techniques presented in preceding sections.

Suppose we want to find the difference between the accumulated change of *f(x)* from *a* to *b* and the accumulated change of *g(x)* from *a* to *b* where $f(x) = 0.3x^3 - 3.3x^2 + 9.6x + 3.3$ and $g(x) = {}^-0.15x^2 + 2.03x + 3.33$. The input of the leftmost point of intersection of the two curves is *a* and the input of the rightmost point of intersection is *b*.

Enter *f(x)* and *g(x)* on the stack. Put 2 on the stack and press PRG LIST →LIST. Store this list in EQ. (You are forming a list of the two functions so that they can both be graphed at the same time.)	
From Figure 7.9 in the text, we see that the horizontal view is 1 to 7 and the vertical view is 0 to 12. Set these views in the plot application and then graph the two functions.	
Next, find the two intersection points by shown on the screen and tracing to each and pressing FCN ISECT. (NXT PICT brings you back to the original menu to activate TRACE again.) The *x*-values of these points of intersection will be the limits on the integrals we use to find the area.)	
Press ON NXT OK to return to the stack and store the leftmost *x*-value in *A* and the rightmost *x*-value in *B* by putting the name, in ticks, under the value and pressing STO. *Note:* You can either retype these values or use PRG TYPE OBJ→ to disassemble the points and DROP everything except the *x*-values of the intersections.	

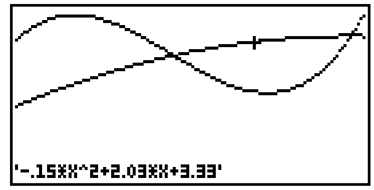
Instead of having to retype the functions when you evaluate the integrals, press EQ PRG TYPE OBJ→ , drop the 2, and store the functions in G and F, respectively.



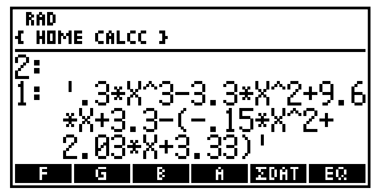
The combined area of the three regions is

$$\int_1^A (f-g)\,dx + \int_A^B (g-f)\,dx + \int_B^7 (f-g)\,dx.$$

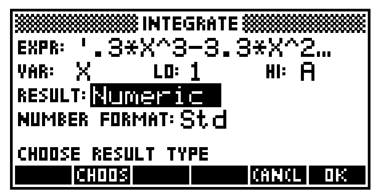
(If you are not sure which curve is on top in each of the various regions, trace a curve, press FCN NXT NXEQ .)



Before you go to the symbolic mode to find the value of each the definite integral, use your stored F and G to store the proper function for each integral in EQ. For instance, to form *(f–g)(x)*, press F, press G, and then press – . Store this in EQ.

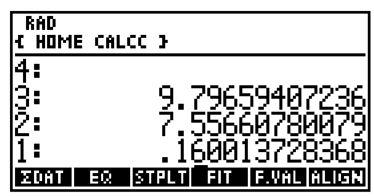


Press ↱ 9 (SYMBOLIC) , choose EQ as the EXPR, and find the value of the first integral.



Repeat the process for the other two integrals. (The value of each is copied to the stack when you find the value of the definite integral.)

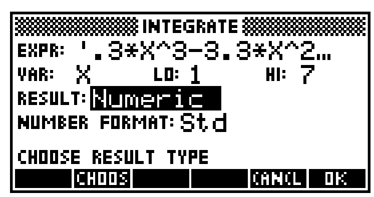
Press + + to find a total area of 17.5132156015.



Use either the home screen or the graphics screen to find the value of the definite integral

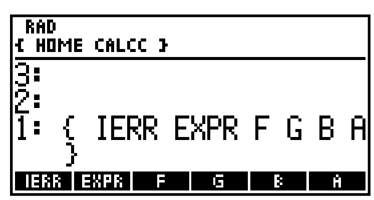
$$\int_1^7 (f-g)\,dx = \int_1^7 \big(f(x)-g(x)\big)\,dx = 2.39999999989$$

Note that the value of the integral is *not* the same as the area between the two curves.



When you have finished with this section of the text, you may not want the extra items cluttering up your menu. To delete menu items that you have input and no longer need, press ↰ + followed by the menu keys corresponding to the unwanted menu items.

Press ENTER ↰ EEX (PURGE) .



7.3 Streams in Business and Biology

You will find your calculator very helpful when dealing with streams that are accumulated over finite intervals.

7.3.1 FUTURE VALUE OF A DISCRETE INCOME STREAM

We use the sequence command to find the future value of a discrete income stream. The change in the future value at the end of T years that occurs because of a deposit of \$$A$ at time t where interest is earned at an annual rate of $100r\%$ compounded n times a year is

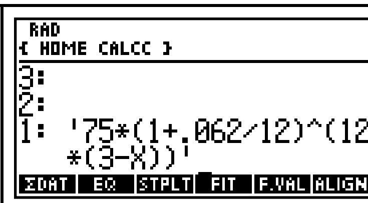
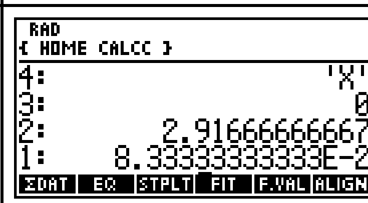
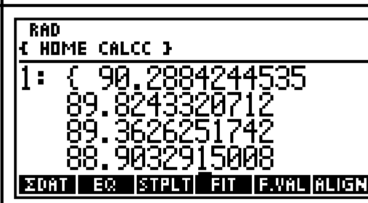
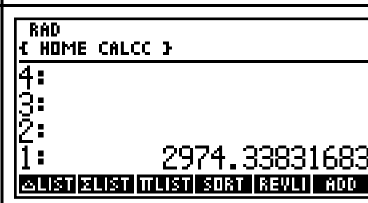
$$f(t) = A\left(1 + \frac{r}{n}\right)^{n(T-t)} \text{ dollars per compounding period}$$

where t is the number of years since the first deposit was made. We assume the initial deposit is made at time $t = 0$ and the last deposit is made at time $T - \frac{1}{n}$. The increment for a discrete stream involving n compounding periods and deposits of \$$A$ at the beginning of each compounding period is $\frac{1}{n}$. Thus, to find the future value of the discrete income stream, enter in this order, on the stack *f(x)*, *x*, *0*, *T–1/n*, *1/n* and activate the sequence command.

Suppose that you invest \$75 each month in a savings account yielding 6.2% APR compounded monthly. What is the value of your savings in 3 years? To answer this question, note that the change in the future value that occurs due to the deposit at time t is

$$f(t) = 75\left(1 + \frac{0.062}{12}\right)^{12(3-t)} \approx 75\left(1.005166667^{12}\right)^{(3-t)} \approx 75(1.06379)^{(3-t)}$$

Now, find the future value of this stream with $A = 75$, $r = 0.062$, $T = 3$, and $n = 12$.

Enter *f(t)*, using X as the input variable, in EQ. Note that if you want an exact answer you should enter 75(1+ 0.062/12)^(12(3–x)) (You must carefully use parentheses with this form of the equation.)	
Use the sequence command to generate the list of outputs with formula EQ, variable X, beginning value 0, ending value 3 –1/12, and increment 1/12. (See Section 6.1.1 of this *Guide*.) '75(1+ 0.062/12)^(12(3–x))' is in level 5 of the stack.	
The sequence is generated. (Note that since you start counting at 0, the ending value will be one increment less than the number of years the money accumulates in the account.)	
This list contains the heights of the 36 left rectangles. The future value we seek is the sum of the heights of the rectangles. Press MTH LIST ΣLIST .	

- When using definite integrals to approximate either the future value or the present value of a discrete income stream or to find the future value or the present value of a continuous stream, use the SYMBOLIC application to find the area between the appropriate continuous rate of change function and the *t*-axis from 0 to T.

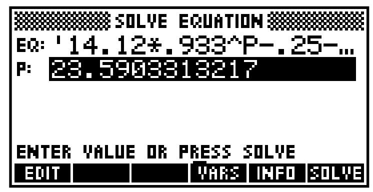
7.4 Integrals in Economics

Consumers' and producers' surplus, being defined by definite integrals, are easy to find using the calculator. You should always draw graphs of the demand and supply functions and think of the surpluses in terms of area to better understand the questions being asked.

7.4.1 CONSUMERS' SURPLUS Suppose that the demand for mini-vans in the United States can be modeled by $D(p) = 14.12(0.933)^p - 0.25$ million mini-vans when the market price is p thousand dollars per mini-van.

At what price will consumers purchase 2.5 million mini-vans? We solve $D(p) = 2.5$ to find the price.

Enter $14.12(0.933)^p - 0.25 - 2.5$ in EQ. You can use either X or P as the input variable. Press [↱] [7] (SOLVE), choose Solve equation, use [▽] to move to the input line, and press [SOLVE] to find that $p \approx \$23.6$ thousand.

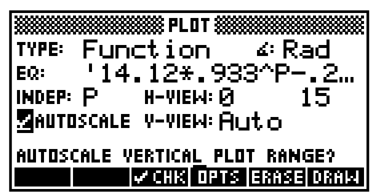


You should always know how many solutions to expect. Because this is an exponential equation, it can have no more than one solution. Refer to Section 1.2.2 of this *Guide* for instructions on using the solver.

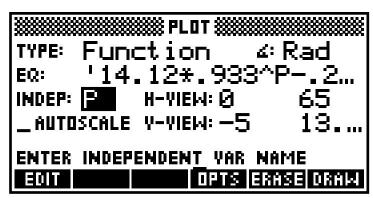
Now, let's find if the model indicates a possible price above which consumers will purchase no mini-vans. If so, we will find that price.

Edit the current EQ so that it is the demand function $14.12(0.933)^p - 0.25$. Store this in EQ. Go to the PLOT application and prepare to graph EQ.

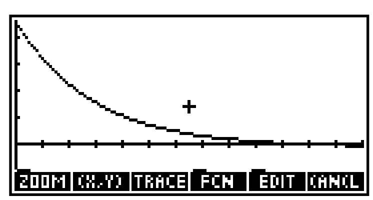
Since the output is mini-vans, we set a lower value of 0 for the horizontal view and consider the graph for different values of the upper value. First, set 15 (or some other value) and use AUTOSCALE to draw the graph.



Not much can be observed from this graph. Because we are trying to see if the demand function crosses or touches the input axis, reset the lower vertical view value to $^-5$ and the upper horizontal view value to a larger value, say 65. Redraw the graph.

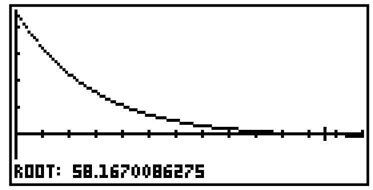


It is difficult to tell if there is an intercept. So, use [FCN] [ROOT] to find, should it exist, the root of the demand function.



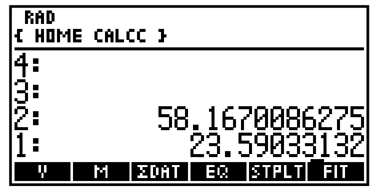
Consumers will not pay more than about $58,200 per mini-van.

(Instead of drawing the graph, you could have used SOLVE to find the location of the x-intercept. Notice that here we are looking for where EQ = 0.)



What is the consumers' surplus when 2.5 million mini-vans are purchased? We earlier found that the market price for this quantity is M ≈ 23.59033 thousand dollars.

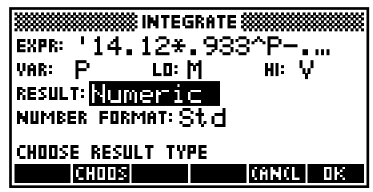
(Remember that you can avoid retyping the long decimal numbers by storing them to various memory locations as you find them.)



Find the consumers' surplus by finding the value of

$$\int_{M}^{V} D(p)\,dp \;=\; \int_{23.59033132}^{58.1670086275} \left[14.12(0.933)^{p}\,dp\right].$$

Press OK to find the value 27.4048167387.



- Carefully watch the units involved in your computations. Refer to the statement of the problem and note that the height is measured in thousand dollars per mini-van and the width in million mini-vans. Thus, the area should be written in units that make sense in the context of this problem:

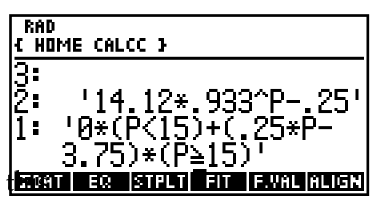
$$\text{height} * \text{width} \approx 27.4\left(\frac{\text{thousand dollars}}{\text{minivan}}\right)(\text{million minivans}) = \$\,27.4 \text{ thousand million}$$

$$= \$27.4\ (1{,}000)(1{,}000{,}000) = \$27.4(10^9) = \$27.4 \text{ billion dollars}$$

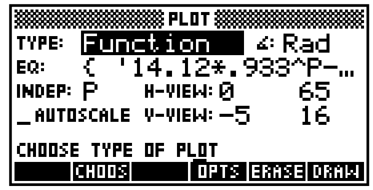
7.4.2 PRODUCERS' SURPLUS When dealing with supply functions, use definite integrals in a manner similar to that for consumers' quantities to find producers' revenue, producers' surplus, and so forth. To illustrate, suppose the demand for mini-vans in the United States can be modeled by $D(p) = 14.12(0.933)^p - 0.25$ million mini-vans and the supply curve is $S(p) = 0$ million mini-vans when $p < 15$ and $S(p) = 0.25p - 3.75$ million mini-vans when $p \geq 15$ where the market price is p thousand dollars per mini-van.

Press EQ to place the demand curve on the stack. Enter the supply curve as a piecewise function on the stack. (Remember to use the same input variable symbol in both equations -- either X or P is okay.)

Put 2 on the stack and press PRG LIST →LIST . Store the list in EQ.

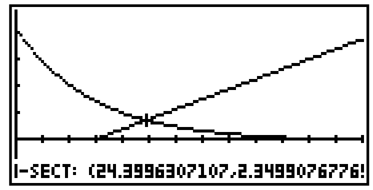


Draw a graph of the demand and supply functions in an appropriate view, say $0 \le \text{input} \le 65$ and $^{-}5 \le \text{output} \le 16$.



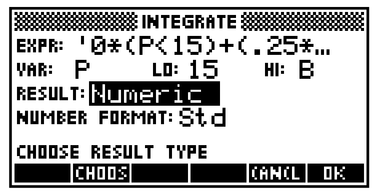
Market equilibrium occurs when $D(p) = S(p)$. Use the methods of Section 7.1.1 of this *Guide* to find the intersection of the two curves.

Store the x-value of the intersection as B.



Press [EQ] [PRG] [TYPE] [OBJ→], drop the 2, and store the supply function as EQ.

Producers' surplus is found as $\int_{15}^{B} S(p)\,dp$ = 11.0441321872.



7.5 Average Value and Average Rates of Change

You need to carefully read any question involving average value in order to determine which quantity is involved. Considering the units of measure in the situation can be of tremendous help when trying to determine which function to integrate when finding average value.

7.5.1 AVERAGE VALUE OF A FUNCTION

Suppose that the hourly temperatures shown below were recorded from 7 a.m. to 7 p.m. one day in September.

Time	7am	8	9	10	11	noon	1pm	2	3	4	5	6	7
Temp. (ºF)	49	54	58	66	72	76	79	80	80	78	74	69	62

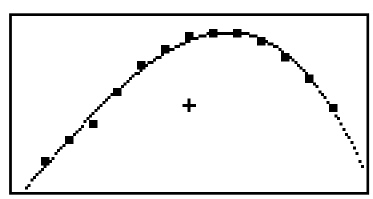
Enter the input data in ΣDAT as the number of hours after midnight: 7 am is 7 and 1 p.m. is 13, etc.

First, we fit a cubic model

$$t(h) = {}^{-}0.03526h^3 + 0.71816h^2 + 1.584h + 13.689 \text{ ºF}$$

where h is the number of hours after midnight.

Store the cubic equation in EQ. Notice when the model graphs on the scatter plot of the data that it provides a good fit.



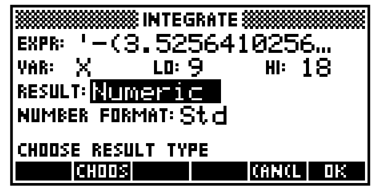
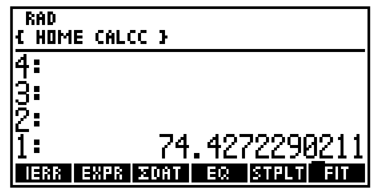
Next, you are asked to calculate the average temperature. Because temperature is measured in this example in degrees Fahrenheit, the units on your result should be ºF. When evaluating integrals, it helps to think of the units of the integration result as (height)(width) where the height units are the output units and the width units are the input units of the function that you are integrating. That is,

$$\int_{9\text{ hours}}^{18\text{ hours}} t(h)\text{ degrees } dh \quad \text{has units of (degrees)(hours).}$$

When we find the average value, we divide the integral by (upper limit – lower limit). So,

$$\text{average value} = \frac{\int_{9\text{ hours}}^{18\text{ hours}} t(h)\text{ degrees } dh}{18\text{ hours} - 9\text{ hours}} = \frac{\left(T(18) - T(9)\right)\text{ degrees}\cdot\text{hours}}{9\text{ hours}}$$

where $T(h)$ is an antiderivative of $t(h)$. Because the "hours" cancel, the result is in degrees as is desired. Remember, when finding average value, *the units of the average value are always the same as the output units of the quantity you are integrating.*

First, find the value of $\int_{9\text{ hours}}^{18\text{ hours}} t(h)\text{ degrees } dh$ to be 669.84506119.	
Then, divide by (18-9) = 9 to find the average value of the temperature between 9 a.m. and 6 p.m. to be approximately 74.4 °F.	

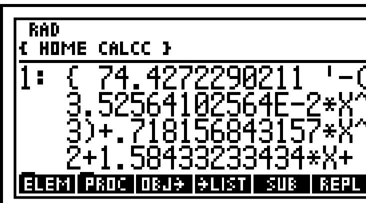
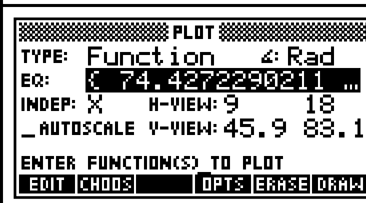
The third part of this example asks you to find the average rate of change of temperature from 9 a.m. to 6 p.m. Again, let the units be your guide. Because temperature is measured in °F and input is measured in hours, the average rate of change is measured in output units per input units = °F per hour. Thus, we find the average rate of change to be

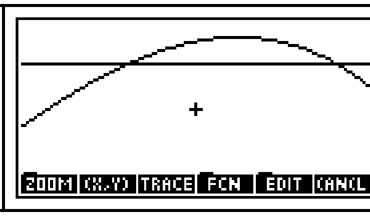
$$\begin{array}{c}\text{average rate of change}\\ \text{of temperature}\end{array} = \frac{\left(t(18) - t(9)\right)\ ^{o}\text{F}}{(18-9)\text{ hours}} = 0.98\ ^{o}\text{F per hour}$$

7.5.2 GEOMETRIC INTERPRETATION OF AVERAGE VALUE

What does the average value of a function mean in terms of the graph of the function? Consider the model we found above for the temperature one day in September between 7 a.m. and 7 p.m.:

$$t(h) = -0.03526h^3 + 0.71816h^2 + 1.584h + 13.689 \text{ degrees } \quad t \text{ hours after 7 a.m..}$$

You should already have the unrounded model for *t(h)* in EQ. If not enter it now. If the average value found in Section 7.5.1 of this Guide is not on the stack, also enter it now. Press [EQ] to put both the model and the average value on the stack. Form a list of *t(h)* and the average value. Store this in EQ.	
Reset the views in the PLOT application to those given on the right. (Notice the input is between 9 a.m. and 6 p.m.)	

Draw the graph. Notice that the area of the rectangle whose height is the average temperature is (74.42722902)(18 – 9) ≈ 669.8.	

- The area of the rectangle equals the area of the region below the temperature function *t(h)* and above the *t*-axis between 9 a.m. and 6 p.m. In this application, the area does not have a meaningful interpretation because its units are (degrees)(hours).