Chapter 6 Accumulating Change: Limits of Sums and the Definite Integral

6.1 Results of Change

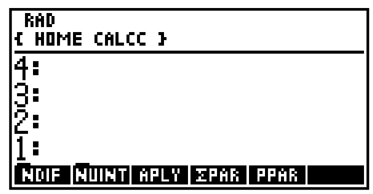
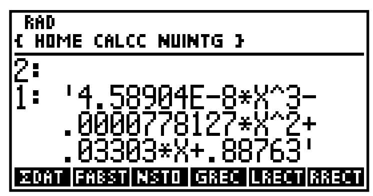
We have thus far seen how to use the calculator to work with rates of change. In this chapter we consider the results of change. Your calculator has many useful features that will assist you in your study of the accumulation of change.

6.1.1 APPROXIMATIONS WITH LEFT RECTANGLES Your calculator can be used to perform the calculations needed to approximate, using left rectangles, the area between the horizontal axis, a (non-negative) rate of change function, and two input values.

Consider, for example, a model for the number of customers per minute who came to a Saturday sale at a large department store between 9 a.m. and 9 p.m.:

$$c(m) = (4.58904*10^{-8})\, m^3 - (7.78127*10^{-5})\, m^2 + 0.03303\, m + 0.88763$$

customers per minute where m is the number of minutes after 9 a.m.

You need to have your data and equation in the directory containing the programs you need for this section of the text. So, first press NUINT .	
Enter the right-hand side of $c(m$[redacted] in EQ by pressing [redacted] (Press NXT [redacted]eral times to find EQ on the [redacted].) You must [redacted] input varia[redacted]nce we use a program that [redacted]upon program F.[redacted]emember that "10 to a p[redacted] is denoted by [redacted]tor. Access [redacted]	 Your HP may convert some of the scientific notation to decimal notation when you press ENTER.

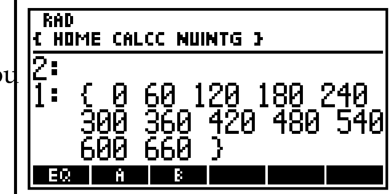
Suppose we want to estimate the total number of customers who came to the sale between $x = 0$ and $x = 660$ (12 hours) with 12 rectangles and $\Delta x = 60$. Now, you could type in the x-values 0, 60, 120, 180, …, 660 or you could have the calculator generate them for you.

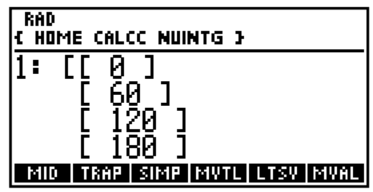
When an input list consists of many evenly-spaced values, there is a calculator command that will generate the list so that you do not have to type in the values in one by one. To use this sequence command, enter the following on the stack, one by one in this order: *the formula, the variable, the first value, the last value, the increment*.

When entering values that differ by a constant, the formula is the same as the variable and the increment is the constant. Any letter can be used for the variable -- we use X.

<table>
<tr><td>Generate the list of times beginning with 0, ending with 660, and differing by 60 by first pressing ' α X ENTER ENTER 0 ENTER 660 ENTER 60 ENTER .

(Note that there is also an X in level 5 of the stack that does not appear in the picture on the right.)</td><td>
</td></tr>
<tr><td>To generate the sequence, press PRG LIST NXT PROC NXT SEQ . (Instead of all these keystrokes, you could instead press and hold down α while typing S E Q. Release α and press ENTER .)

Notice that 9 p.m. is 720 minutes after 9 a.m. When using left-rectangle areas, the rightmost data point is <u>not</u> included.</td><td>
</td></tr>
<tr><td>Press VAR (maybe press NXT) LTSV .

Program LTSV converts the <u>l</u>ist of times <u>t</u>o a <u>s</u>ingle-<u>v</u>ariable matrix so that output can be included to form the data matrix ΣDAT.</td><td>
</td></tr>
</table>

Next, use program MVAL (found in the HP-48 Appendix) to calculate the outputs from the *c(m)* model. The program also puts these output values as the second column of ΣDAT.

- MVAL (model output values program)

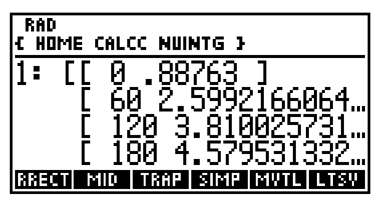
  A model must be stored in EQ before running program MVAL.

  Input: one-column ΣDAT matrix of x-data values in level 1 of stack

  Output: two-column ΣDAT matrix containing x-data values in column 1 and y-values calculated from the model in column 2

(If you have typed the programs in your calculator, program F.val must be in the same directory as program MVAL.)

<table>
<tr><td>Be sure you are in the directory containing program MVAL, (probably NUINT).

With the one-column matrix of input values in level 1 of the stack, press MVAL .</td><td>
</td></tr>
<tr><td>Column 2 of the matrix on the stack now contains the heights of the 12 rectangles.

Store this matrix as the current ΣDAT with ↰ ΣDAT .</td><td>You will probably need to press NXT to find ΣDAT on the menu.</td></tr>
</table>

We now need to use the heights of the rectangles that are in the second column of ΣDAT. Program MVTL will extract a column of ΣDAT and convert it to a list.

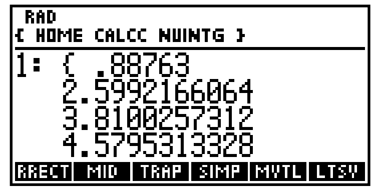
- MVTL (multivariable matrix to list)

  Input: two or more column ΣDAT on level 2 of the stack
  number of column to be extracted on level 1 of the stack

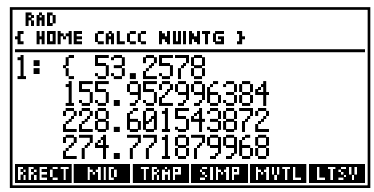
  Output: extracted column in the form of a list

ΣDAT needs to be in level 1 of the stack. If it is not, press [ΣDAT] .

Because we want to use the heights that are in column 2, press [2] [ENTER] . Press [MVTL] .

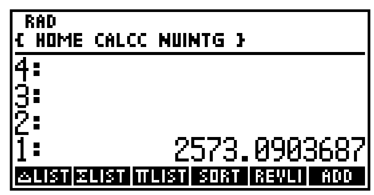


Because the *width* of each rectangle is 60, the area of each rectangle is 60**height*. Find the *areas* of the 12 rectangles by entering 6 0 on the stack and pressing [×] .



Find the sum of the areas of the rectangles by pressing [MTH] [LIST] [ΣLIST] . (Remember, if you make a mistake, you may be able to recover your last stack with [EVAL] (UNDO) .)

We estimate, using 12 left rectangles, that about 2,573 customers came to the Saturday sale.



Note: The values in Table 6.2 in your text and the final result differ slightly than those in your lists. This is because the unrounded model found with the data was used for computations in the text. If you have the unrounded model available, you should use it instead of the rounded model *c(m)* that is given above.

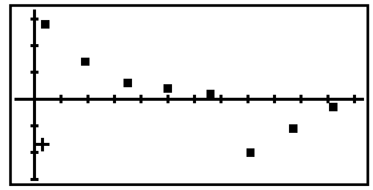
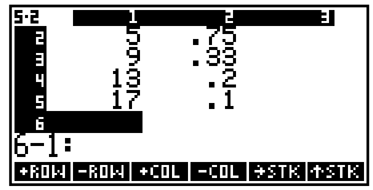
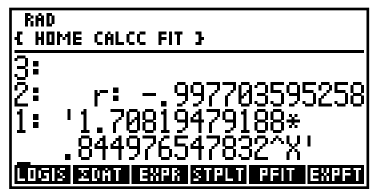
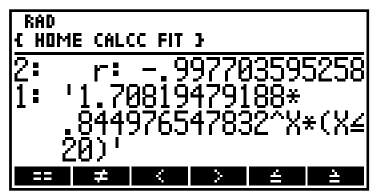
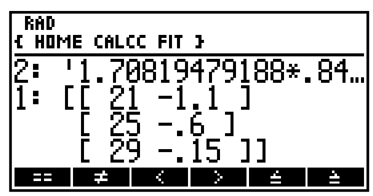
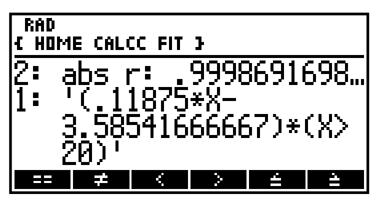
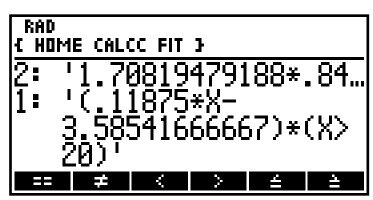
6.1.2 APPROXIMATIONS WITH RIGHT RECTANGLES When using left rectangles to approximate the results of change, the rightmost data point is not the height of a rectangle and is not used in the computation of the left-rectangle area. Similarly, when using right rectangles to approximate the results of change, the *leftmost* data point is not the height of a rectangle and is not used in the computation of the right-rectangle area.

The following data shows the rate of change of the concentration of a drug in the blood stream in terms of the number of days since the drug was administered:

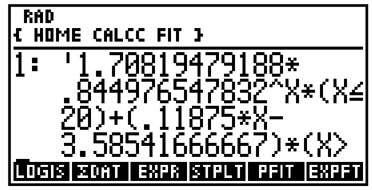
Day	1	5	9	13	17	21	25	29
Concentration ROC (μg/mL/day)	1.5	0.75	0.33	0.20	0.10	⁻1.1	⁻0.60	⁻0.15

First, we fit a piecewise model to the data. Recall from Section 1.3.1 of this *Guide* that a piecewise continuous function whose two pieces "break" at $x = a$ is entered in EQ in the form (first piece)*(X<*a*) + (second piece)*(X≥*a*). The parentheses are necessary. Note that the equality should be included with whichever piece is indicated by the application.

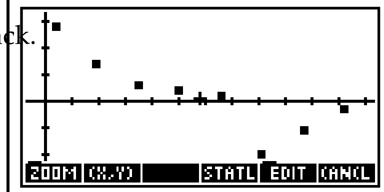
Because we will be fitting models, go into the FIT directory before entering your data.

Enter the days in the first column and the rate of change of concentration in the second column. Store the data in ΣDAT. Press ΣDAT to return the data to the stack, and draw a scatter plot of the data with STPLT . It is obvious that a piecewise model should be used with $x = 20$ as the "break" point.	
Press ΣDAT and ▽. Delete the last three data points (rows 6, 7, and 8) by highlighting the row you want to delete and pressing NXT and −ROW . Press ENTER to put the data on the stack, but do <u>not</u> store it as ΣDAT.	
Fit an exponential model to the day 1 through day 17 data by pressing EXPFT .	
Because this piece of the model is defined through day 20, press ' ALPHA X PRG TEST ≤ 20 ENTER and then press × . Leave this part of the equation on the stack.	
Next, enter the day 21 through day 29 data in a new matrix and have it in level 1 of the stack. (Do <u>not</u> store it as ΣDAT.)	
Press VAR 1 and PFIT to fit a linear model to the last 3 data points. Press ' ALPHA X PRG TEST > 20 ENTER and then press × .	
Press ▷ to swap rows 1 and 2 of the stack. Press ◀ to drop the unneeded information in level 1.	

Join the two pieces together by pressing [+]. Press [NXT] and store the piecewise model in EQ with [↰] [EQ].

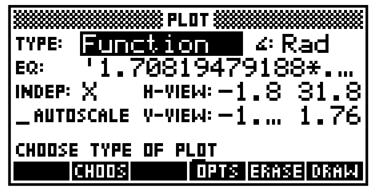


Press [NXT] and [ΣDAT] to put the original data on the stack. Press [STPLT] to redraw the scatter plot of all the data.

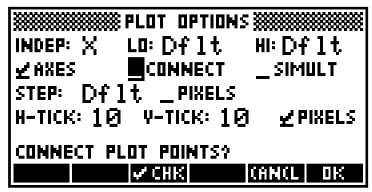


Go to the PLOT application, and choose Type: Function. (The horizontal and vertical views should already be set from drawing the scatter plot.)

(If the correct EQ does not appear, you did not store it in EQ in the directory you were in when you went to the PLOT application.)



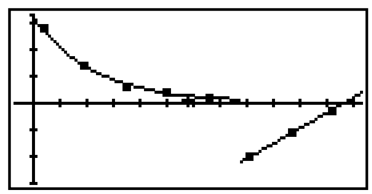
To draw the plot in "dot" mode -- that is, the points are not connected as the graph draws, press [OPTS] on the plot screen menu and be sure that CONNECT is <u>not</u> checked. Press [OK] to return to the previous screen.



Press [DRAW].

(Do <u>not</u> press [ERASE] before drawing because you want to see the curve graphed over the scatter plot.)

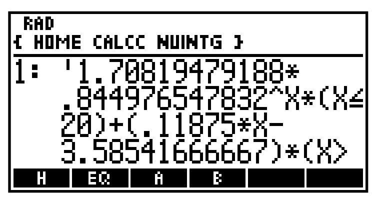
Press [ON] and [OK] to return to the FIT directory.



Now, we determine the right-rectangle area for $0 \le x \le 20$:

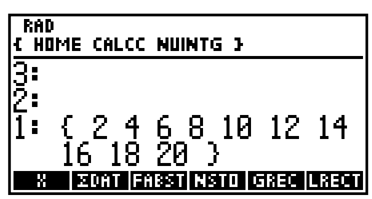
Press [EQ] to put the piecewise model on the stack.

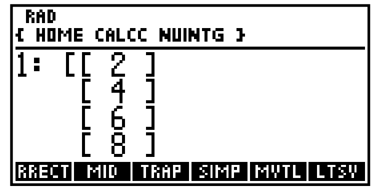
To find the rectangle areas, we need programs in the NUINT directory. Go in that directory and store the piecewise model in EQ in that directory.

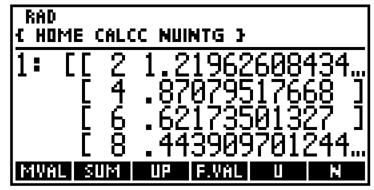
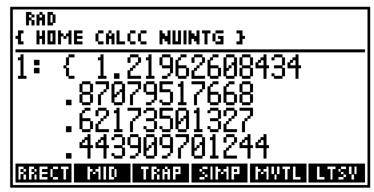
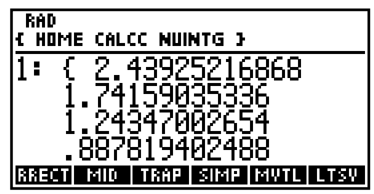


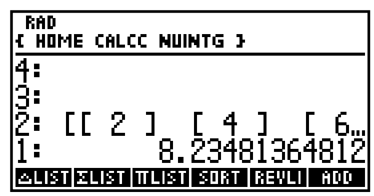
Now, we need to enter the *right* endpoints of the rectangles. Because 0 is not the right endpoint and $\Delta x = 2$, enter the values 2, 4, 6, …, 20 in the first column of ΣDAT or use the sequence command to generate these values:

['] [α] X [ENTER] [ENTER] 2 [ENTER] 20 [ENTER] 2 [ENTER] [PRG] [LIST] [NXT] [PROC] [NXT] [SEQ].



<table>
<tr><td>Press VAR (maybe press NXT) LTSV .</td><td>
</td></tr>
<tr><td>With the one-column matrix of input values in level 1 of the stack, press NXT MVAL .

The heights of the right rectangles are now in column 2 of ΣDAT.</td><td>
</td></tr>
<tr><td>Because we want to extract and use the heights that are in column 2, press 2 ENTER NXT NXT NXT MVTL .</td><td>
</td></tr>
<tr><td>Each rectangle has width 2. The heights of the rectangles are in the list in level 1 of the stack. Find the rectangle areas by entering 2 and pressing × .</td><td>
</td></tr>
<tr><td>Find the sum of the areas of the rectangles by pressing MTH LIST ΣLIST .

We estimate, using 10 right rectangles, that the change in concentration was approximately 8.24 μg/mL.</td><td>
</td></tr>
</table>

To estimate, using right rectangles, the change in drug concentration for $20 \le x \le 29$ days with $\Delta x = 1$, follow the same procedure as above. The values in the first column of ΣDAT begin with 21 because we must eliminate the leftmost value (20) when using right rectangles. You should use the absolute value of the model in EQ to generate the rectangle heights in the second column of ΣDAT to find that the change in drug concentration is about 5.55 μg/mL.

6.2 Trapezoid and Midpoint Rectangle Approximations

You can compute areas of trapezoids using the stack or the fact that the trapezoid approximation is the average of the left- and right-rectangle approximations. Areas of midpoint rectangles are found in the same manner as left and right rectangle areas except that the midpoint of the base of each rectangle is in the first column of ΣDAT and no data values are deleted. However, such procedures can become tedious when the number, n, of subintervals is large.

6.2.1 SIMPLIFYING AREA APPROXIMATIONS When the number of subintervals, n, is large, it is impractical to calculate areas using the methods previously given. In such cases, the programs GRECT, LRECT, RRECT, MID, and TRAP in the HP-48 Appendix are used to calculate the corresponding areas and draw the graphs of the approximating rectangles or trapezoids. (The other programs that are indicated to go in the directory containing these programs are needed as subroutines and must also be entered in or transferred to your calculator.)

We illustrate using this program with a model for the Carson River flow rates:

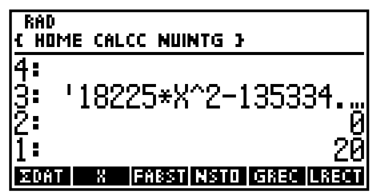
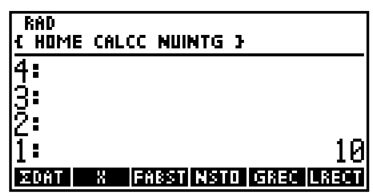
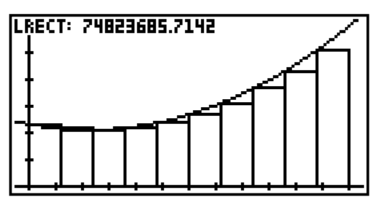
$$f(h) = 18{,}225h^2 - 135{,}334.3h + 2{,}881{,}542.9 \text{ cubic feet per hour}$$

h hours after 11:45 a.m. Wednesday.

The complete model found from the data in your text is used for all the following calculations You should enter the data in this example, fit a quadratic model, and store the quadratic equation in EQ in the NUINT directory before continuing on with the following.

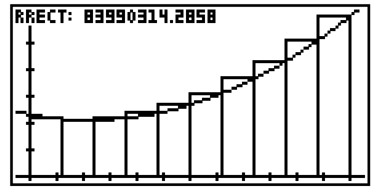
Before using any of these programs, store the model, the endpoints of the interval, and the number of approximating rectangles by placing on the stack, in this order:

- the model, using x as the input variable
- a, the left endpoint of the interval
- b, the right endpoint of the interval
  Press FABSTO .
- n, the number of subintervals
  Press NSTO .

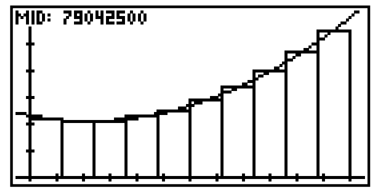
To approximate the area, from 0 to 20, of the region beneath the graph of $f(h)$ and the horizontal axis using n subintervals, prepare to use the calculator's programs. With the information shown to the right on the stack, press FABSTO .	
Let us first use $n = 10$ subintervals and left rectangles to illustrate. Place 10 on level 1 of the stack and press NSTO .	
Press GREC . When the screen listing your choices appears, choose Left with 1 . The left-rectangle area with 10 rectangles is displayed. This value is copied to the stack when you press ON after viewing the graph.	

Suppose we now want to find the approximating area and see the figure using 10 right rectangles.

Press GREC . When the screen listing your choices appears, choose Right with 3 . The right sum with 10 rectangles is displayed.



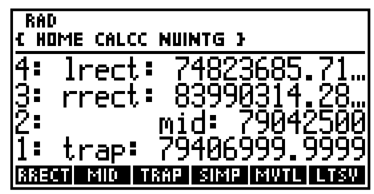
Repeat the process to find the approximating area and see the figure using 10 midpoint rectangles. Press GREC . When the screen listing your choices appears, choose Mid with 2 .



- Notice that program GREC draws an autoscaled graph of the model. You need *not* do this before executing the program.

For the trapezoidal approximation using 10 subintervals, press TRAP .

The programs do not draw a graph of the approximating trapezoids.

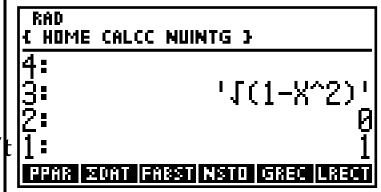


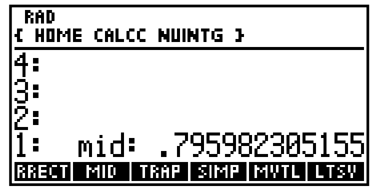
- When n, the number of subintervals, is large, it is not advisable to draw pictures. For area approximations only (without pictures), press LREC , RREC , and MID instead of GREC after you have specified the value of n.
- To change the number of subintervals, n, place the new value for n on the stack and press NSTO .
- These area approximation programs cannot be used with data unless you use a model you have fit to the data.

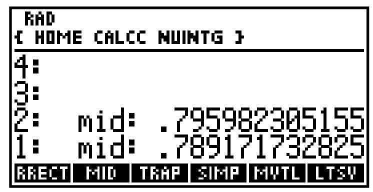
6.3 The Definite Integral as a Limit of Sums

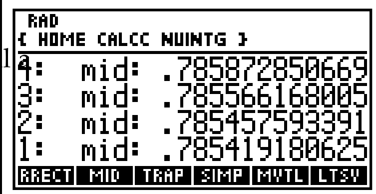
This section introduces you to a very important and useful concept of calculus -- the definite integral. Your calculator can be very helpful as you study definite integrals and how they relate to the accumulation of change.

6.3.1 LIMITS OF SUMS When you are looking for a trend in midpoint-rectangle approximations to the area between a non-negative function and the horizontal axis between two values of the input variable, program MID is extremely helpful! However, for this use of the program, it is not advisable to draw pictures when n, the number of subintervals, is large.

<table>
<tr>
<td>To construct a chart of midpoint approximations for the area between $f(x) = \sqrt{1 - x^2}$ and the x-axis from x=0 to x=1, first follow the instructions in 6.2.1 of this Guide to store f(x) and the two values of x with FABSTO . (Don't forget to enclose $1 - x^2$ in parentheses.)</td>
<td>
</td>
</tr>
<tr>
<td>Input some number of subintervals, say N = 4 by entering 4 on the stack and pressing NSTO . Press MID .

Record on paper the midpoint area approximation 0.795982305155. (If you want an answer accurate to the thousandths position, record at least 4 decimal places.)</td>
<td>
</td>
</tr>
<tr>
<td>Double the number of subintervals by placing 8 on level 1 of the stack and pressing NSTO . Press MID .

Record the midpoint approximation 0.7891717328 (again, to at least 4 decimal places).</td>
<td>
</td>
</tr>
<tr>
<td>Continue on in this manner, each time doubling the number of subintervals and pressing NSTO and MID until a trend is evident.

(Finding a trend means that you can tell what value the approximations are getting closer and closer to, within a specified accuracy, without having to run the program ad infinitum!)</td>
<td>
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</tr>
</table>

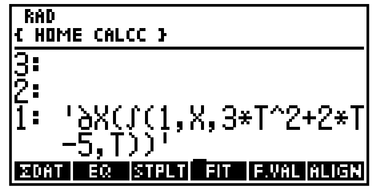
- Remember that the trend indicated by the limit of sums can be interpreted as the area of the region between the function and the input axis *only* when that region lies above the input axis. When the region lies below the input axis, the trend is the negative of the area of the region.

6.5 The Fundamental Theorem

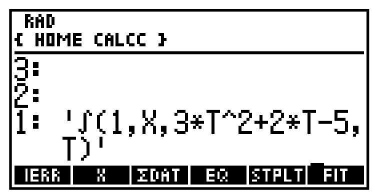
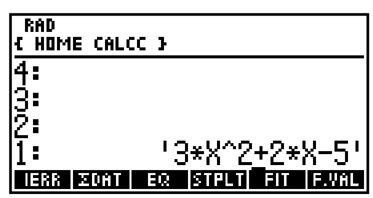
As it did when taking the derivative of a function, your calculator can give you a numerical approximation for a definite integral of a function. This numerical integrator is found in the SYMBOLIC application or can be accessed from the keyboard when working in the stack.

6.5.1 THE FUNDAMENTAL THEOREM OF CALCULUS This theorem tells us that the derivative of an antiderivative of a function is the function itself. Let us consider this theorem numerically. We also illustrate the FTC using the Hewlett-Packard's unique capability of determining the symbolic form of certain derivatives and integrals.

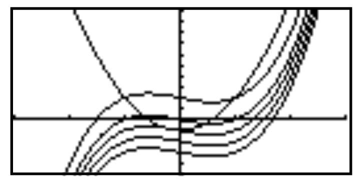
Recall that when using the stack, the HP-48's derivative entry form for $\frac{d(f(x))}{dx}$ is '∂X($f(x)$)'. The stack entry form for the integral $\int_a^x f(t)\,dt$ is '∫ (A, X, $f(T)$, T) '.

Consider $F'(x) = \frac{d}{dx}\left(\int_1^x 3t^2 + 2t - 5\, dt\right)$. The FTC tells us that $F'(x)$ should equal $f(x) = 3x^2 + 2x - 5$. Enter $F'(x)$ as shown to the right. Access ∂ with [↱] [SIN] and $\int$ with [↱] [COS].	 Store this expression in EQ.
Since $F'(x)$ is a function of X, use program F.val to evaluate it at several different values of X. (Remember to enter the value of X before pressing [F.val] and to press [↰] [EVAL] (→NUM) after pressing [F.val].)	X: ⁻2 0 3.5 10 EQ: 3 -5 38.75 315
Now, store $f(x) = 3x^2 + 2x - 5$ in EQ. Evaluate this function at the same values of x. Compare them to the values obtained for $F'(x)$.	X: ⁻2 0 3.5 10 EQ: 3 -5 38.75 315 Compare these to the $F'(x)$ values above.

Let us now consider the symbolic representation and see the FTC in action!

Enter $\int_a^x f(t)\, dt$ on the stack as indicated on the right. Store the integral in EQ.	
Enter $F'(x) = \frac{d}{dx}\left(\int_1^x 3t^2 + 2t - 5\, dt\right)$ on the stack as indicated on the right. *(Purge all X from your menus!)*	
Press [EVAL] repeatedly until the HP has finished applying the derivative formulas and simplified the "answer" to the expression on the right. Repeat this exploration for other functions and different lower limits. Are you convinced?	

6.5.2 DRAWING ANTIDERIVATIVE GRAPHS All antiderivatives of a specific function differ only by a constant. We explore this idea using the function $f(x) = 3x^2 - 1$ and its antiderivative $F(x) = x^3 - x + C$.

Enter $f(x)$ on the stack. To simplify entry of the remaining functions, store $f(x) = 3x^2 - 1$ in some variable, say A. Store 'X^3 – X' in another variable, say B. Enter the functions, as they appear to the right, on the stack.	1: 'A' 2: ' ∫ (0, X, A, X) ' 3: ' B + 0 ' 4: ' B – 2 ' 5: ' B – 1 ' 6: ' B + 2 ' 7: ' B – 3 '
Form a list of the seven functions on the stack by entering 7 and pressing PRG LIST →LIST .	Store the list as the current EQ in the PLOT application.
Find a suitable view and graph all the functions. (Try x between $^-3$ and 3 and y between $^-5$ and 10.) It seems that the only difference in the graphs (other than the graph of A) is that the y-intercept is different. But, isn't C the y-intercept of the antiderivative?	

- Trace the graphs and then jump between them with ▽. It appears that the graph of $F(x)$ with $C = 0$ and $\int_0^x (3t^2 - 1)\,dt$ are the same. If it becomes difficult to tell on which graph you are tracing, have TRACE activated, press FCN NXT and VIEW . The equation of the graph the trace cursor is on is displayed on the screen.
- What if you changed the function in level 2 of the stack to '∫ (2, X, A, X)'? Would find the graphs of ∫ (2, X, A, X) and $x^3 - x + 2$, or maybe ∫ (2, X, A, X) and $x^3 - x - 2$, the same? Can you justify your answer with antiderivative formulas? Explore!

6.6 The Definite Integral

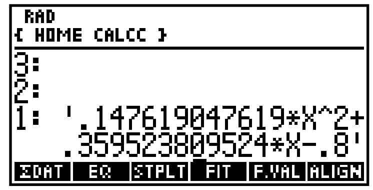
As was illustrated in Section 6.5.1, you can evaluate definite integrals on the stack. All you have to do is type ∫(a, b, $f(x)$, X) for a specific function $f(x)$ and specific values of a and b, and press ENTER ↰ EVAL →NUM . The function can be stored in another variable such as A, EQ, etc., or it can be entered directly into the integral form.

- If you evaluate a definite integral using antiderivative formulas and check you answer using the calculator, you may sometimes find a slight difference in the trailing decimal places. Remember, the HP-48 is sometimes evaluating the definite integral using approximation techniques.

6.6.1 EVALUATING A DEFINITE INTEGRAL USING SYMBOLIC MODE We illustrate another way to use your calculator's numerical integrator with the model for the rate of change of the average sea level in meters per year during the last 7000 years. A model for these data is $r(t) = 0.14762t^2 + 0.35952t - 0.8$ meters per year t thousand years from the present. (Note that t is negative since we are talking about the past.)

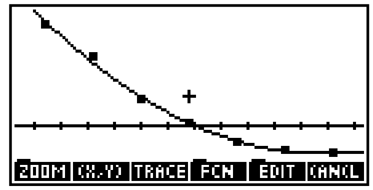
As we have previously mentioned, you should always use the full model (not rounded) for any model you find from data.

Find a quadratic to fit the data shown in the *Changing Sea Levels* example in your text.

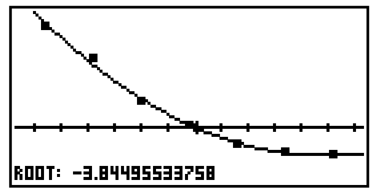


The model provides a good fit to the data.

Because we are asked for *area* between the input axis and the function, we must find where the function be- comes negative.

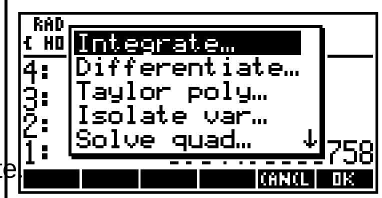


Press ◁ to return the graph to the screen, TRACE to mark a guess for the intercept, and FCN ROOT . (See Section 1.2.3 of this *Guide* .)



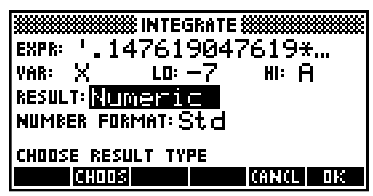
Return to the home screen and store this value in some location, say A, with the keystrokes ' ALPHA A STO .

Press ↱ 9 (SYMBOLIC) and press OK to choose Integrate.



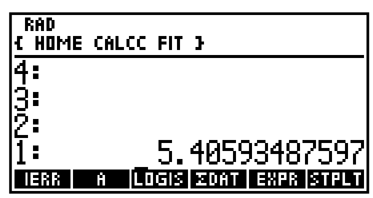
Press CHOOS and select EQ for the expression location, enter X in the variable location, enter -7 in the "low" location, and enter A in the "high" location.

Press OK after each entry. Highlight the RESULT location and press +/– to choose NUMERIC. Press OK when finished.



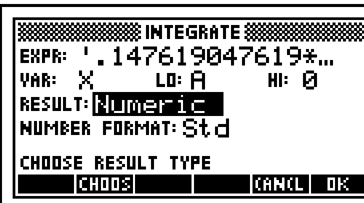
The value of the definite integral appears on the stack.

$$\int_{-7}^{-3.845} r(t)\,dt \approx 5.4 \text{ meters}$$

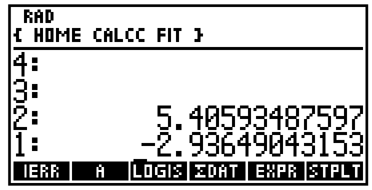


The area of the region above the x-axis is about 5.4 meters. Now, find the area of the region below the x-axis. This area equals the negative of the definite integral of the function over that region.

Evaluate $\int_{-3.845}^{0} r(t)\,dt$ using the same procedure as illus- trated above. The only difference is that you have a different lower and upper limit.

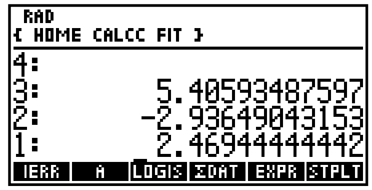


Press OK to find the value of the definite integral which is the negative of the area of the region below the horizontal axis.



Now, find $\int_{-7}^{0} r(t)\, dt$.

Note that this value is *not* the sum of the two *areas*. It is their difference.



Note: When you choose a numeric result, a NUMBER FORMAT field appears. Press CHOOS and use ▽ to select Fixed. Press OK ▷ and enter 5 (or the number of decimal places required by your instructor). Press OK . If you use Std as the number format, the HP may take quite a while on some applications to calculate the result to twelve-digit precision.

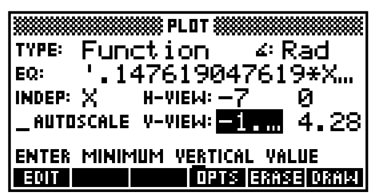
6.6.2 EVALUATING A DEFINITE INTEGRAL FROM THE GRAPHICS SCREEN

Provided that a and b are possible x-values when you trace the graph, you can find the value of the definite integral $\int_a^b f(x)\, dx$ from the graphics screen. (For "nice" numbers, you can often find the exact x-value you need if you graph in the RESET screen or set the viewing window so that the horizontal view is a multiple of 13.)

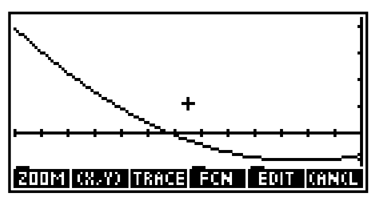
Have the model for the average sea level in EQ (see Section 6.6.1).

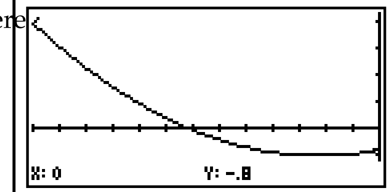
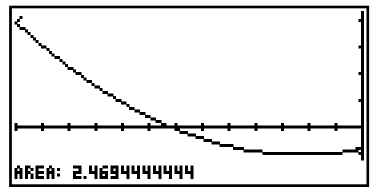
Suppose we want to find $\int_{-7}^{0} r(t)\, dt$.

Choose the PLOT application settings shown to the right. (Either check AUTOSCALE or use the vertical view that was set when drawing the scatter plot.)



Graph the function with ERASE DRAW .



Press TRACE + and use ◁ to move the cursor to where X= ⁻7 (the left edge of the screen). (The *y*-value at this location is not important.) Press X to mark the location. Next, use ▷ to move the cursor to where X=0 (the right edge of the screen). Press + or one of the white keys to return the menu to the screen.	
Press FCN and then AREA . The value of the definite integral of the function between the two marks (⁻7 and 0) appears on screen and on level 1 of the stack.	
Bring back the menu, and press SHADE to shade the area. Note that your HP does not shade the region between ⁻7 and 0. The only region that is shaded is where the function is above the horizontal axis because that is the only place that the definite integral equals the *area* of the region.