Chapter 5 Analyzing Change: Extrema and Points of Inflection

5.1 Optimization

Your calculator can be very helpful in checking your analytic work when you find optimal points and points of inflection. When you are not required to show work using derivatives or when a very good approximation to the exact answer is all that is required, it is a very simple process to use your calculator to find optimal points and inflection points.

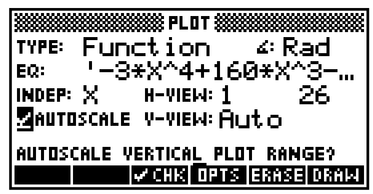
5.1.1 FINDING *X*-INTERCEPTS OF SLOPE GRAPHS Where the graph of a function has a local maximum or minimum, the slope graph has a horizontal tangent. Where the tangent line is horizontal, the derivative of the function is zero. Thus, finding where the slope graph *crosses* the input axis is the same as finding where a relative maximum or a relative minimum occurs.

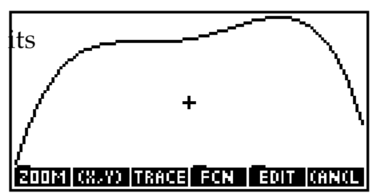
Consider, for example, the model for a cable company's revenue for the 26 weeks after it began a sales campaign:

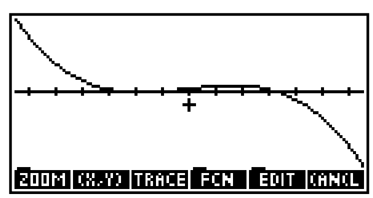
$$R(x) = {}^{-}3x^4 + 160x^3 - 3000x^2 + 24{,}000x \text{ dollars}$$

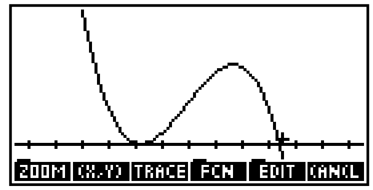
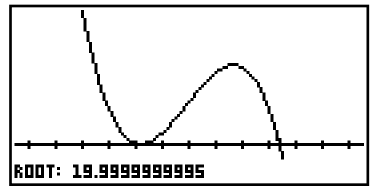
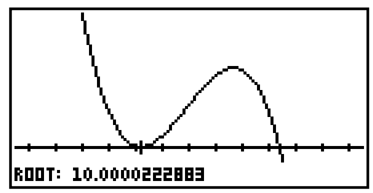
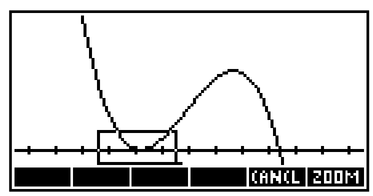
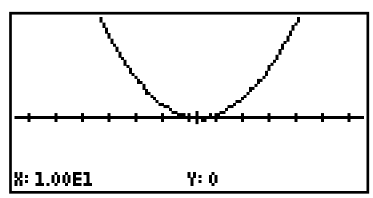
where x is the number of weeks since the cable company began sales.

<table>
<tr><td>Go to the PLOT application and enter R(x) , the revenue model, as the current EQ. (Be sure FUNCTION is selected as TYPE and the correct input variable, here X, is entered as INDPT.)

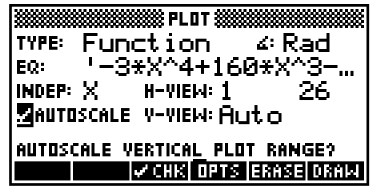
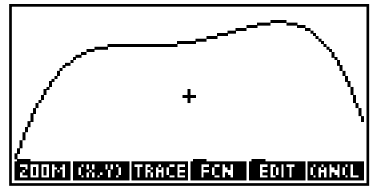
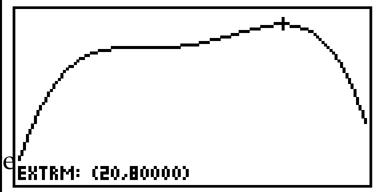
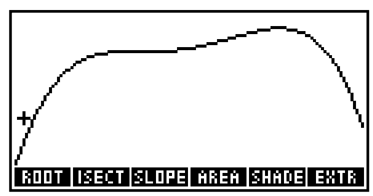
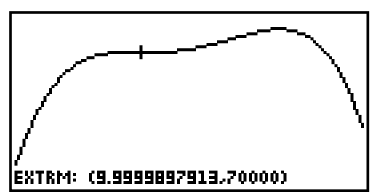
The statement of the problem indicates that x should be graphed between 1 and 26. Graph R(x) with the AUTOSCALE feature.</td><td>
</td></tr>
<tr><td>Press [FCN] [NXT] [F'] to have the HP graph the model and its derivative.

Evidently the view is not good for the slope graph. [We should see the graph of a cubic equation for R´(x) .]</td><td>
</td></tr>
<tr><td>Because the HP has now created a list {R´(x) R(x)}, you can set an appropriate view for the derivative graph by returning to the plot application with [ON] and checking AUTOSCALE.

Redraw the graphs with [ERASE] [DRAW] .</td><td>
</td></tr>
</table>

If you find it difficult to see the x-intercepts, reset the vertical view to a narrower interval, say $^-800$ to 3000. Redraw with ERASE DRAW . Next, move the cursor near one of the x-intercepts. (Either trace the graph or use the cursor keys to move the cursor.)	
Find this x-intercept of the slope graph using FCN ROOT . The rightmost x-intercept (root) is at 20. That is, $R'(x) = 0$ at $x = 20$. (Remember that the calculator is using a numerical derivative that does not always give exact values.)	
Press one of the white keys to return the menu to the screen. Repeat the above process by using ◁ to move the cursor near the location of the other x-intercept and then press FCN ROOT . The leftmost x-intercept (root) is at 10. That is, $R'(x) = 0$ at $x = 10$.	
It is obvious that the derivative graph crosses the input axis at $x = 20$. However, you must now determine if the derivative graph crosses or just touches the x-axis at $x = 10$. Press any white key and NXT PICT to return to the original graphics menu. Use ZOOM BOXZ to magnify the portion of the graph near $x = 10$ to see what happens there. (See Section 3.3.1 of this *Guide*.)	
After using ZOOM BOXZ several times, we see that the graph just touches and does not cross the x-axis near $x = 10$. (Try to draw wide, narrow boxes.) Using the information gained from $R'(x)$ and the graph of $R(x)$, we find that revenue was greatest 20 weeks after the cable company began sales.	

5.1.2 FINDING OPTIMAL POINTS Once you draw a graph of a function that clearly shows any optimal points, finding the location of those high points and low points is an easy task for your calculator. When a relative maximum or a relative minimum exists at a point, your calculator can find it in a few simple steps. We again use the cable company revenue equation, $R(x)$, from Section 5.1.1.

Re-enter $R(x) = {}^{-}3x^4 + 160x^3 - 3000x^2 + 24000x$ in EQ. Remember that the input x should be graphed between 1 and 26. Next, use AUTOSCALE to draw a graph of the revenue, $R(x)$.	
Reset the upper vertical view value to a slightly larger number, say 85,000, to give a little more room at the top of the screen.	
Prepare to find the relative maximum by pressing ◁ (and maybe △ and/or ▽) to move the cursor to your *estimate* of the high point. Press FCN EXTR and the coordinates of the maximum are displayed at the bottom of the graph and are also copied to the stack. We see that the revenue is greatest at 20 weeks with a value of $R(20) = \$80{,}000$.	
What about the other part of the curve that may contain a peak? Follow the same procedure as indicated above, but this time put the cursor far to the left side of the graph. (We are trying to see if there is a local maximum somewhere around $x = 10$.)	
Notice that the calculator returns an extreme point at $x \approx 10$. However, this is NOT the location of a relative maximum. Remember that we found that $R'(x)$ only touched and did not cross the x-axis at $x = 10$. As we will see in the next section, $x = 10$ is the location of another type of extreme point.	

To find a relative minimum value, follow the same procedure. Be certain that you have moved the cursor near the location of the minimum value before using EXTR .

WARNING: Whenever you use your calculator's EXTR function to find the location of relative maxima and/or minima, you must know that the graph of the derivative of the function *crosses* (not just touches) the input axis at the critical point before classifying it as a maximum or minimum point.

5.2 Inflection Points

As was the case with optimal points, your calculator can be very helpful in checking your analytic work when you find points of inflection. You can also use the methods illustrated in Section 5.1.2 of this *Guide* to find the location of any maximum or minimum points on the graph of the first derivative to find the location of any inflection points for the function. In fact, your calculator offers three graphical methods for finding inflection points. We investigate these as well as the analytic method in the following discussions.

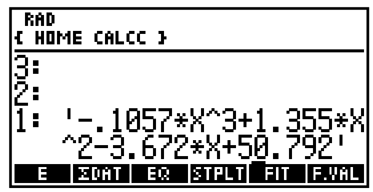
5.2.1 FINDING *X*-INTERCEPTS OF A SECOND DERIVATIVE GRAPH We first look at using the analytic method of finding inflection points -- finding where the graph of the second derivative of a function *crosses* the input axis.

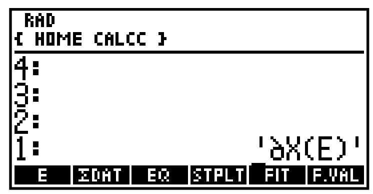
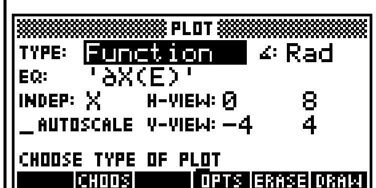
To illustrate, consider a model for the percentage of students graduating from high school in South Carolina from 1982 through 1990 who entered post-secondary institutions:

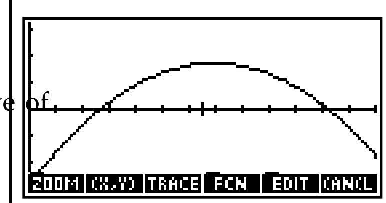
$$f(x) = {}^{-}0.1057x^3 + 1.355x^2 - 3.672x + 50.792 \text{ percent}$$

where $x = 0$ in 1982.

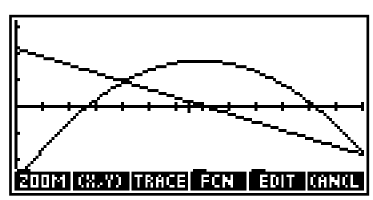
<table>
<tr><td>Enter $f(x)$ on the stack and store it in memory location E.

Next, enter ' ∂X(E)' as the current EQ and go to the plot application.</td><td>
</td></tr>
<tr><td>Instead of ' ∂X($f(x)$)', you could enter your derivative.

(Hopefully, you have checked that your derivative and the calculator's derivative are the same!)</td><td>
</td></tr>
<tr><td>The problem says the model is for 1982 through 1990 which corresponds to $0 \le x \le 8$. Thus, you are told the horizontal view. Choose an appropriate vertical view -- possibly y between ⁻4 and 4.</td><td>
</td></tr>
<tr><td>Graph $f''(x)$

Press FCN NXT F' to have the HP graph the derivative of the model and its derivative.</td><td>
</td></tr>
<tr><td>Remember that you should be able to clearly see any optimal points.

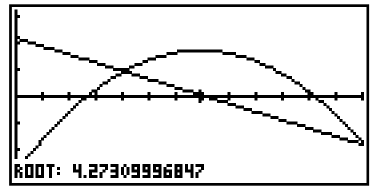
Leave room at the bottom of the screen so that the menu does not block your view of any important points on the graph.

Press TRACE and be sure you are on the graph of the second derivative, the line. (If not, move to it with ▽.)</td><td>

Since the second derivative is a line and we need to find the x-intercept, this is a "good" graph.</td></tr>
</table>

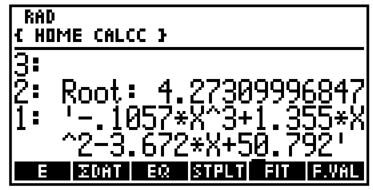
Use the methods illustrated in 5.1.1 of this *Guide* to find where the second derivative graph crosses the x-axis.

If you are asked to give the inflection *point* of $f(x)$, you should give both an x-value and a y-value.



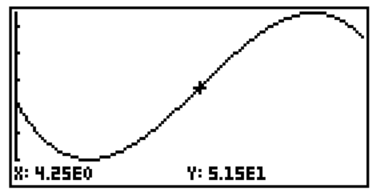
Return to the stack and press [E] to put *f(x)* on the stack. Store *f(x)* in EQ.

With only the root on the stack, press [F.VAL] to substitute the input value of the inflection point into the function. The inflection point is ≈ (4.3, 51.6).

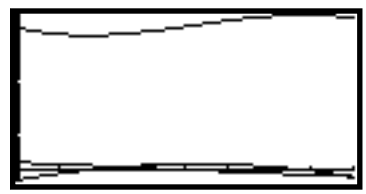


Next, look a graph of the function and verify that there does indeed appear to be an inflection point at $x \approx 4.3$.

To do this, have *f(x)* in EQ and draw the graph with a horizontal view of 0 8 and the vertical view set with AUTOSCALE.



In this problem, it is difficult to find a window that shows a good graph of both the function and its derivatives. However, if you draw a graph of all three, you can roughly see that the location of the inflection point of the function occurs at the location of the maximum of the first derivative and at the location of the x-intercept of the second derivative.



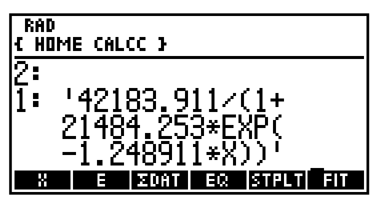
5.2.2 FINDING INFLECTION POINTS WITH YOUR CALCULATOR

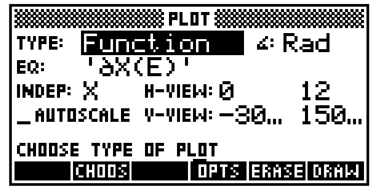
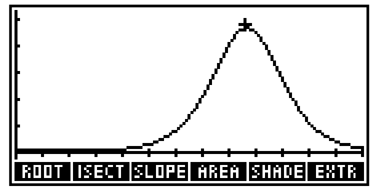
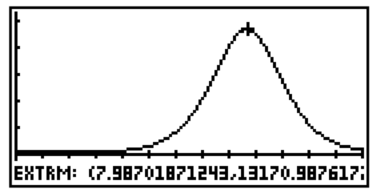
Remember that an inflection point is a point of greatest or least slope. Whenever finding the second derivative of a function is tedious or you do not need an exact answer from an analytic solution, you can very easily find an inflection point of a function by finding where the first derivative of the function has a maximum or minimum value.

We illustrate this process with the function giving the number of polio cases in 1949: $y = \dfrac{42183.911}{1 + 21484.253e^{-1.248911t}}$ where $t = 1$ on January 31, 1949, $t = 2$ on February 28, 1949, etc.

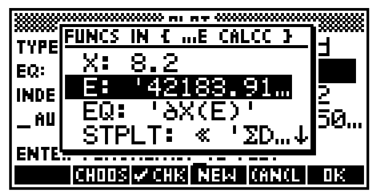
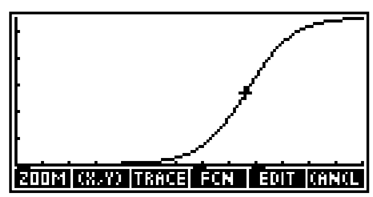
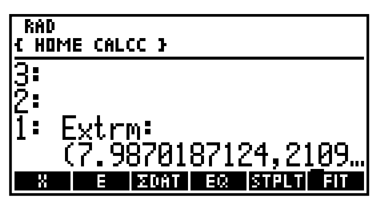
Enter $f(x)$ on the stack and store it in memory location E.

Next, enter '∂X(E)' as the current EQ and go to the plot application.



Graph the slope function in an appropriate view; for instance, a horizontal view of 0 12 and a vertical view of −3000 15,000.	
Remember that you should be able to clearly see any optimal points on the resulting graph. If not, reset the view and redraw the graph.	
Use the methods illustrated in 5.1.2 of this *Guide* to find the maximum of the slope graph. The *x*-value of the maximum of the slope graph is the *x*-value of the inflection point of the function.	

Our final method is the simplest -- just be certain if you use it that the function does have an inflection point at the location indicated by the calculator.

If you are still in the plot application, move the cursor to the EQ location, press CHOOS and pick E as the new EQ with ▽ OK . (If not, return to the stack and press E to return the original function to the stack. Store this logistic model, not its derivative, in EQ. Return to the plot application.)	
Have 0 ≤ *x* ≤ 12 and use AUTOSCALE to set the vertical view. Move the cursor near the location of the inflection point.	
Press FCN EXTR . The inflection point is displayed, marked on the graph, and copied to the stack.	

Now you know that EXTR gives the location of relative maxima, relative minima, and/or inflection points. You must determine which type of point is being displayed by EXTR by looking at a graph of the function or by looking at the relationship between the function, its derivative, and/or its second derivative.