Chapter 4 Determining Change: Derivatives

4.1 Numerically Finding Slopes

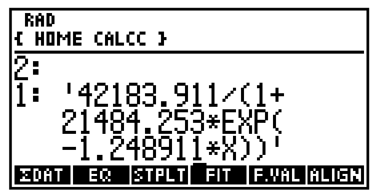
Using your calculator to find slopes of tangent lines does not involve a new procedure. However, the techniques in this section allow you to repeatedly apply a method of finding slopes that gives quick and accurate results.

4.1.1 NUMERICALLY INVESTIGATING SLOPES ON THE HOME SCREEN

Finding slopes of secant lines joining the point at which the tangent line is drawn to increasingly close points on a function to the left and right of the point of tangency is easily done using your calculator. Suppose we want to find the slope of the tangent line at $t = 8$ to the graph of the function giving the number of polio cases in 1949: $y = \dfrac{42183.911}{1 + 21484.253e^{-1.248911t}}$ where $t = 1$ on January 31, 1949, $t = 2$ on February 28, 1949, and so forth.

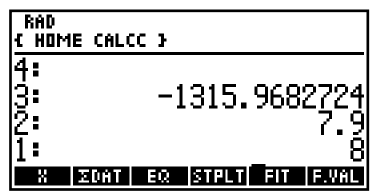
Enter the directory containing program F.VAL. Using X as the input variable, store the right-hand side of the above equation in EQ.

(Carefully check the entry of your equation, especially the location of the parentheses. Parentheses are needed around the denominator and around the exponent.)



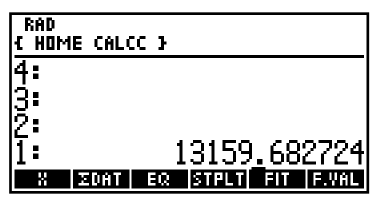
We now evaluate the slopes joining nearby points to the *left* of $x = 8$. Enter 7.9 on the stack. Press F.VAL . Enter 8 on the stack. Press F.VAL . Press – .

Enter the two input values, *in the same order as you entered them in the previous step*, on the stack.



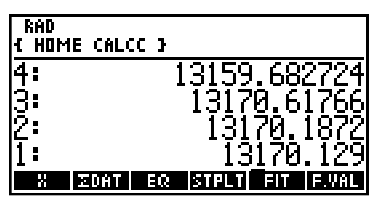
Compute the slope of the secant line joining the points where $x = 7.9$ and $x = 8$ by pressing – ÷ .

Record each slope on your paper as it is computed. You are trying to find what these slopes are approaching.



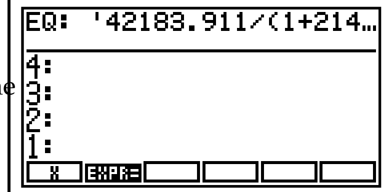
Continue in this manner, changing the 7.9 to 7.99, 7.999, 7.9999, etc. , until you can determine the value to which the slopes from the left are getting closer and closer.

(Note: The 8 does not change because that is the input value of the point of tangency.)



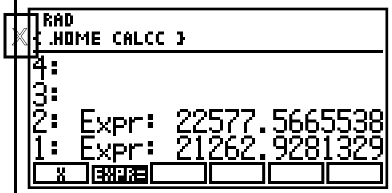
We now evaluate the slopes joining nearby points to the *right* of $x = 8$. You could use the same procedure as was illustrated with points to the left of $x = 8$. However, we use the SOLVE application to illustrate another method for determining the slopes.

Clear the screen with [↰] [DEL] (CLEAR) . Press [↰] [7] (SOLVE) [ROOT] . Press [EQ] to be certain you have the correct equation. If not, store it in EQ. Press [SOLVR] .



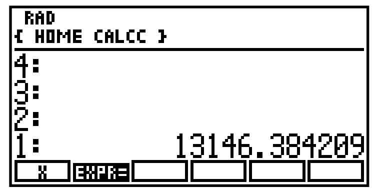
Type 8.1 and press [X] . Press [EXPR=] . Type 8 and press [X] . Press [EXPR=] .

Press [−] to find the change in outputs.



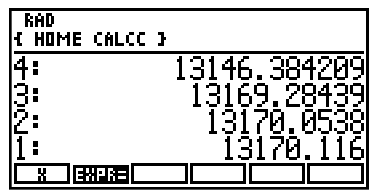
Enter the two input values, *in the same order as you entered them in the previous step*, on the stack.

Compute the slope of the secant line joining the points where $x = 8.1$ and $x = 8$ by pressing [−] [÷] .



Continue in this manner, changing the 8.1 to 8.01, 8.001, 8.0001, etc. , until you can determine the value to which the slopes from the left are getting closer and closer. (Record your results on paper as you progress.)

When the slopes from the left and the slopes from the right approach the same value, that value is the slope of the tangent line at $x = 8$.



- The slopes from the left and from the right appear to be getting closer and closer to 13,170. (The number of polio cases makes sense only as a whole number.)

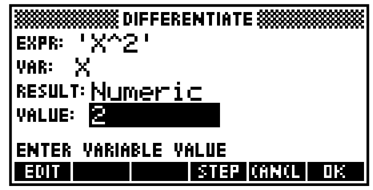
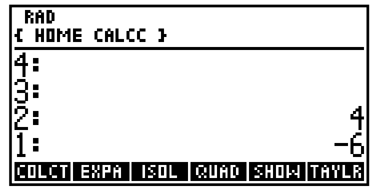
4.3 Slope Formulas

Your calculator can draw slope formulas. However, to do so, you must first enter a formula for the function whose slope formula you want the calculator to draw. Because you will probably be asked to draw slope formulas for functions whose equations you are not given, you must not rely on your calculator to do this for you.

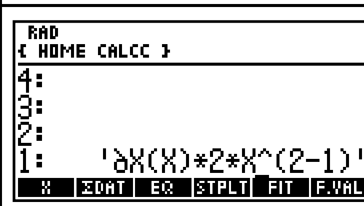
You should instead use technology to check your hand-drawn graphs and to examine the relationships between a function graph and its slope graph. It is very important in both this chapter and several later chapters that you know these relationships.

4.3.1 DERIVATIVE NOTATION AND CALCULATOR NOTATION You can often see a pattern in a table of values for the slopes of a function at indicated values of the input variable and discover a formula for the slope (derivative). The process of calculating the slopes uses the calculator's symbol for finding slopes, ∂.

Suppose you are asked to construct a table of values of $f'(x)$ where $f(x) = x^2$ evaluated at different values of x. Two methods of doing this are illustrated below:

Press [↱] [9] (SYMBOLIC) and use [▽] [OK] to choose Differentiate. Enter ' X^2 ' in the expression location. Press [OK] . Enter X in the variable location. Press [OK] . Press [+/–] if NUMERIC is not in the RESULT location. Enter 2 in the VALUE location, and press [OK] .	
Use [▽] to return to the VALUE location, and press [OK] . The slope at $x = 2$ is placed on the stack. Repeat the entire process, but this time enter ¯3 in the VALUE location, and press [OK] . Use [▽] to return to the VALUE location, and press [OK] . The slope at $x = {}^{-}3$ is placed on the stack.	

Suppose you are asked to construct a table of values for $y = x^2$ evaluated at different values of x. You could find the slope as just illustrated, changing VALUE each time. However, you probably will find the following method takes less time.

Press [↰] [ENTER] (EQUATION) to enter the EQUATION WRITER. Press [↱] [SIN] (∂) to enter the calculator's slope symbol. Type the input variable in the box that appears. Press [▷] and enter the function. (Press [▷] to move out of the power.) Press [▷] when you have finished typing the function.	
Press [ENTER] and ' ∂X(X^2) ' is placed on the stack. (Note: You could enter the formula directly on the stack instead of using the equation writer. However, you must carefully watch the parentheses placement.)	
Change to the directory containing program F.VAL (probably CALCC). Press [↰] [EQ] to store this expression as the current equation. Place the value of X on the stack, say X = 2, and press [F.VAL] . Press [EVAL] to find the slope formula value.	

Repeat the process to evaluate the calculator's slope formula at $x = {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2,$ and 3.

Record these values as they are displayed and try to determine a pattern between the calculated slopes and the values of x.

- If you have difficulty determining a pattern, enter the x-values at which you are evaluating the slope in the first column of ΣDAT and the evaluated values of '∂X($f(x)$)' in the second column of ΣDAT. Use program STPLT to draw a scatter plot of the x-values and the calculated slope formula values. The shape of the scatter plot should give you a clue as to the equation of the slope formula. If not, try drawing another scatter plot where the first column of ΣDAT contains the values of $f(x)$ and the second column contains the calculated slope formula values. Note that this method might help only if you consider a variety of values for x.
- The HP-48 finds numeric and symbolic slopes. Using your calculator to find the symbolic derivative (i. e., the formula form) is discussed in a later section.

4.3.2 DRAWING TANGENT LINES FROM THE GRAPHICS SCREEN

Chapter 3 of this *Guide* (specifically, Section 3.3.2) presented a method of drawing tangent lines from the graphics screen. You will find it helpful to review that method at this time.

4.3.3 CALCULATING $\frac{dy}{dx}$ FROM THE GRAPHICS SCREEN

Section 4.3.1 of this *Guide* examined several methods for finding values of your calculator's numerical derivative. This section illustrates another method using the function $f(x) = 2\sqrt{x-5}$.

Without the context of a real-world situation, how do you know what input values to consider? The answer is that you need to call upon your knowledge of functions. Remember that we graph only real numbers. If the quantity under the square root symbol is negative, the output of $f(x)$ is not a real number. We therefore know that x must be greater than or equal to 5. Many different horizontal views will do, but we choose to use $0 \le x \le 15$.

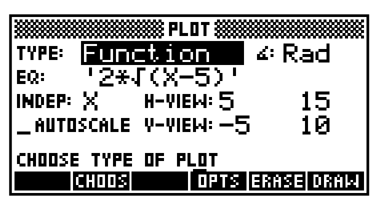
You can use previously-discussed methods to set height of the window, or you can follow the directions below.

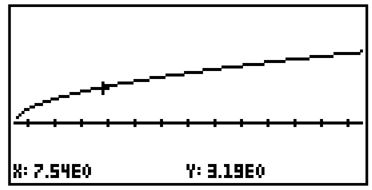
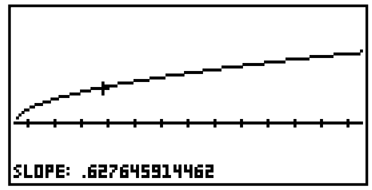
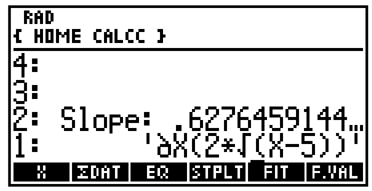
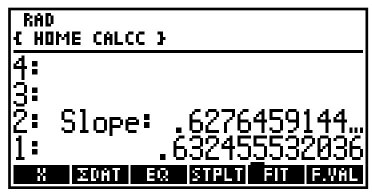
Enter $f(x) = 2\sqrt{x-5}$ in EQ.

The parentheses around the $x - 5$ are necessary to include the entire quantity under the square root.

Go to the PLOT application and enter the horizontal and vertical views shown to the right.

Draw the graph of $f(x) = 2\sqrt{x-5}$.



Press TRACE (X,Y) and move as close as you can to the point at which you want to evaluate the derivative, say, X = 7.5.	
Press any white key to bring back the menu, and then press FCN SLOPE .	
Return to the stack, and notice that the value of the derivative is also there. Now enter the expression shown to the right and store it in EQ.	
Enter 7.5 on the stack and press F.VAL . Press EVAL as many times as necessary until a numerical value appears. Note that the slope obtained from the graphics screen is close to, but not the same, value because the input value on the graphics screen was 7.54, not 7.5.	

4.4 The Sum Rule, 4.5 The Chain Rule, and 4.6 The Product Rule

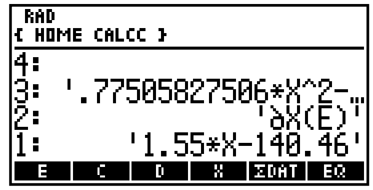
If you have time, it is always a good idea to check your answer. Although your calculator cannot give you a general rule for the derivative of a function, you can use graphical and numerical techniques to check your derivative formula answers. These same procedures apply when you check your results after applying the Sum Rule, the Chain Rule, or the Product Rule.

4.4.1 NUMERICALLY CHECKING SLOPE FORMULAS When you use a formula to find the derivative of a function, it is possible to check your answer using the calculator's numerical derivative. The basic idea of the checking process is that if you evaluate your derivative and the calculator's numerical derivative at several randomly chosen values of the input variable and the output values are very close to the same values, your derivative is *probably* correct. We illustrate with the following activity.

The average yearly fuel consumption per car in the United States from 1980 through 1990 can be modeled by $g(t) = 0.775t^2 - 140.460t + 6868.818$ gallons per car where t is the number of years since 1900. Applying the sum, power, and constant multiplier rules for derivatives, suppose you determine $g'(t) = 1.55t - 140.460$ gallons per year per car. We now numerically check this answer. (As we have mentioned several times, if you have found a model from data, you should have the complete model, not the rounded one given by $g(t)$, in EQ.)

Have the quadratic model you fit to the data in EQ. Because we want to use this location for a different equation but do not want to have to retype the model, press [EQ] ['] [ALPHA] E [STO] to store the model in location E.

Store the calculator's derivative in some location, say C, and your derivative formula, $\frac{dg}{dt} = g'(t)$, in some other location, say D.



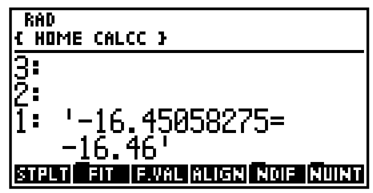
(*Hint:* Instead of trying to retype the model, simply refer to it as E -- the location in which it is stored. You must use the menu key, not the alphabetic key .)

Since the *g(t)* model represents average fuel consumption per car where *t* = 80 in 1980, it makes sense to use only whole number values of *t* = *x* greater than or equal to 80 when checking the derivative formula. We want to check to see if C ≈ D.

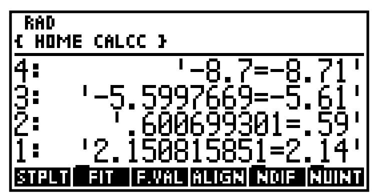
Enter 'C=D' on the stack and store it as EQ by pressing [↰] [EQ]. (Access the equal sign with [↰] 0.)

Enter 80 on the stack and press [NXT] [F.VAL].

Press [NXT] as many times as necessary until you see numerical results. The values of the calculator's numeric derivative and your derivative formula are very close to the same. (Recall that we are using a rounded derivative in D.)



Repeat the process for at least 4 other values of *x*. If <u>all</u> the values of the HP's derivative (C) and your derivative (D) are very close, your formula for the derivative is *probably* correct.



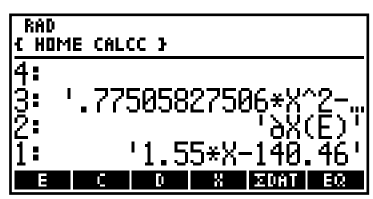
- If the two sides of the equation are *not* very close to the same, you have either incorrectly entered a function or made a mistake in your derivative formula.

4.4.2 GRAPHICALLY CHECKING SLOPE FORMULAS

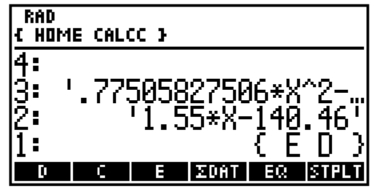
Another method of checking your answer for a slope formula (derivative) is to draw the graph of the calculator's numerical derivative and draw the graph of your derivative. If the graphs appear identical *in the same viewing window*, your derivative is probably correct.

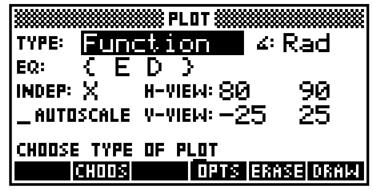
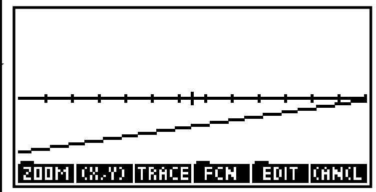
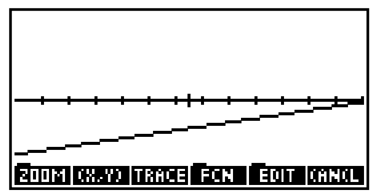
We again use the fuel consumption functions from Section 4.4.1 of this *Guide*.

Have the quadratic model stored in location E, the calculator's derivative in some location, say C, and your derivative formula, $\frac{dg}{dt} = g'(t)$, in some other location, say D.



<table>
<tr><td>Enter a list containing the function you are taking the derivative of and your derivative formula. Be sure you put the function first. Store this list in EQ.

(First, press the menu keys to see that you are using the correct functions in your list.)</td><td></td></tr>
<tr><td>Go to the PLOT application.

Set an appropriate viewing window such as x between 80 and 90 and y between $^-25$ and 25 or draw the graph with AUTOSCALE .</td><td></td></tr>
<tr><td>Draw the graph of the function and your derivative with ERASE DRAW . Do not be concerned that the view does not include the function. We are only interested in the graph of the derivative.</td><td></td></tr>
<tr><td>Press FCN NXT F' to have the HP plot the derivative of the first function in the list, the quadratic model, and your derivative. Since you see only one graph, your derivative formula is probably correct.</td><td></td></tr>
<tr><td colspan="2">If your derivative is such that it cannot be seen in the viewing window in which you see the calculator's derivative or vice versa, you will see only one graph and think that your slope formula is correct. This is certainly possible with the above method. It is better to perform a numerical check on the derivatives than to incorrectly use the graphical checking process.</td></tr>
</table>

- When trying to determine an appropriate viewing window, read the problem again; it will likely indicate the values for the horizontal view. Also use your knowledge of the general shape of the function being graphed to know what you should see.

4.4.3 SYMBOLICALLY FINDING SLOPE FORMULAS The HP-48 offers another way of checking your slope formulas - it will find some of them for you! However, your calculator can only find the formula form of certain derivatives, not all of them. Thus, it is still very important that you learn the slope (derivative) formulas!

The symbolic mode of the HP-48 also offers you an opportunity to experience how the sum, difference, and product formulas as well as the chain rule, power rule, etc. apply to certain functions. We illustrate these ideas by having the calculator show the process of finding the formula form of the derivative of $f(x) = 5x^3 - x^2 + 3$.

First, press ↰ CST (MODES) MISC and check to see that the symbolic mode is active. If it is, SYM▫ should have a small white box after the "M". If not, press SYM and the box will appear. Press VAR .

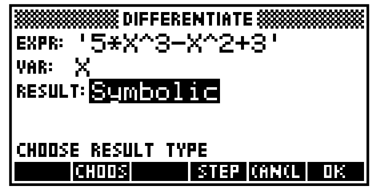
To use the symbolic mode of the HP-48, the input variable cannot appear on the menu in your directory or any directory above your directory. Thus, if you are using X *as the input variable, you must purge* X *in the* CALCC *directory and any directory you must pass through to get to your current directory.*

[To purge a variable, put its' name on the stack, in ticks, and press [↰] [EEX] (PURGE) .]

Press [↱] [9] (SYMBOLIC) and use [▽] [OK] to choose Differentiate.

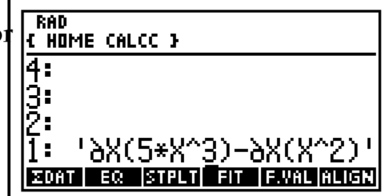
Enter ' 5*X^3–X^2+3 ' in the expression location. Press [OK].

Enter X in the variable location. Press [OK]. Press [+/–] if SYMBOLIC is not in the RESULT location.



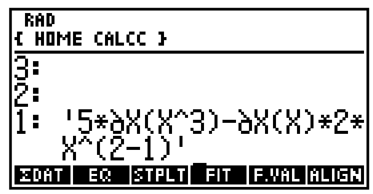
Press [STEP] [EVAL]. Remember that the HP's notation for $\frac{dy}{dx}$ is ∂X(y).

The step shown to the right shows the Sum Rule being applied.



After viewing what is on the screen, press [EVAL] again.

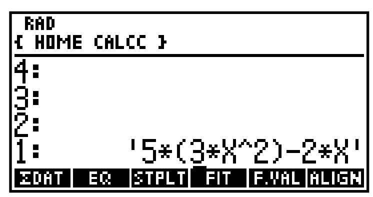
The step to the right shows the Constant Multiplier Rule being applied to the first term and the Simple Power Rule being applied to the second term.



As you continue to press [EVAL], you are lead through a series of steps showing how the formulas for taking the derivative of a constant, the sum rule, and the power rule apply to obtain the derivative of this function. Carefully examine each step that you see to understand the process.

When the expression on the stack no longer changes as you press [EVAL], the steps are finished.

If you only wish to view the symbolic form to check your use of the derivative formulas, press [OK] instead of [STEP] after you have entered the function and chosen SYMBOLIC in the SYMBOLIC menu.



The expression that appears is the derivative, but it has not been simplified. Press [↰] [9] (SYMBOLIC) [COLCT].

(For more complicated expressions, keep pressing [COLCT] until there are no more changes.

Warning: Some expressions never simplify to a recognizable form.)