Chapter 3 Describing Change: Rates

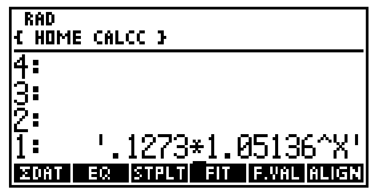
3.1 Average Rates of Change

As you calculate average and other rates of change, remember that each numerical answer should be accompanied by units telling how the quantity is measured. You should also be able to interpret each numerical answer. It is only through their interpretations that the results of your calculations will be useful in real-world situations.

3.1.1 FINDING AVERAGE RATES OF CHANGE Finding an average rate of change using a model is just a matter of evaluating the model at two different values of the input variable and dividing by the difference in those input values. Consider this example.

The population density of Nevada from 1950 through 1990 can be approximated by the model $P(t) = 0.1273(1.05136)^t$ people per square mile where t is the number of years since 1900. You are asked to calculate the average rates of change between from 1950 through 1980 and between 1980 and 1990.

<table>
<tr>
<td>Store ' 0.1273* (1.05136)^X ' in EQ in the directory containing program F.val.

(X must be the name of the input variable whenever using program F.val.)</td>
<td>
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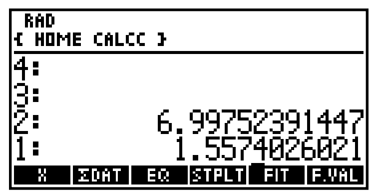
<td>The average rate of change of the population density between 1950 and 1980 is $\frac{P(80) - P(50)}{80 - 50}$.

Evaluate the function at each of these values by placing each value on the stack and pressing F.VAL .</td>
<td>
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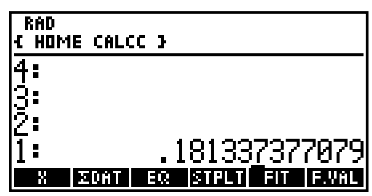
<td>Press − to find the difference in the outputs.

Next, enter 80, then 50, and press − .

Press ÷ to find the average rate of change.

Repeat the procedure to find the average rate of change of the population between 1980 and 1990.</td>
<td>
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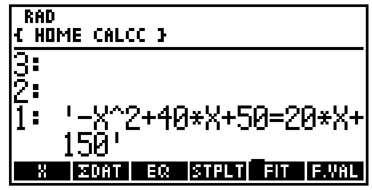
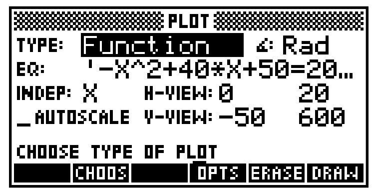
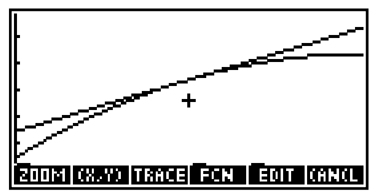
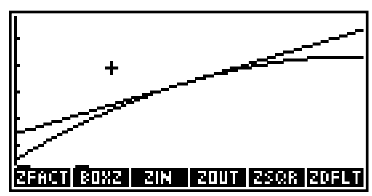
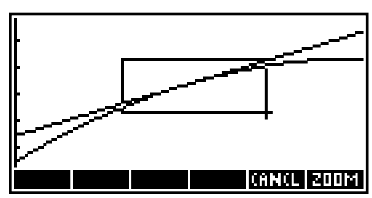
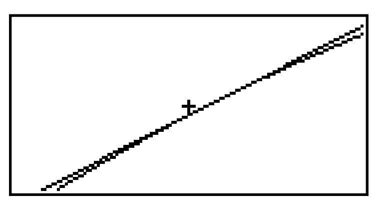
Recall that rate of change units are output units per input units. We see that on average, the population density increased by about 0.18 person per square mile per year between 1950 and 1980 and by approximately 0.45 person per square mile per year between 1980 and 1990.

3.3 Tangent Lines

We first examine the principle of local linearity which says that if you are close enough, the tangent line and the curve are indistinguishable. We then use the calculator to draw tangent lines.

3.3.2 MAGNIFYING A PORTION OF A GRAPH

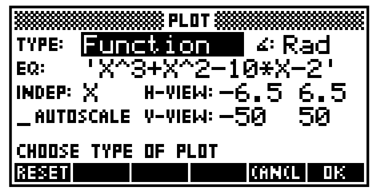
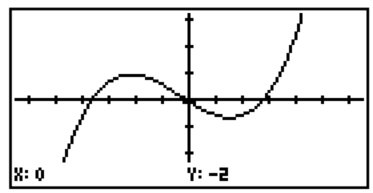
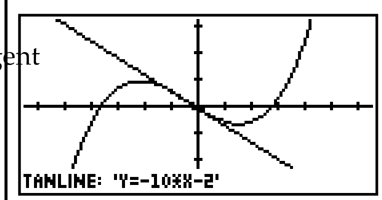
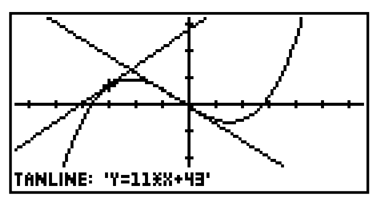
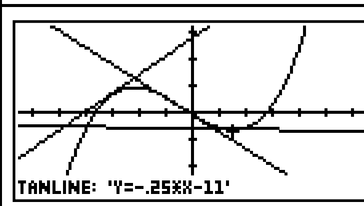
The ZOOM menu of your calculator allows you to magnify any portion of the graph of a function. Suppose we are investigating the graph of $y = {}^{-}x^2 + 40x + 50$ and the tangent line, $y = 20x + 150$, to the graph of this function at $x = 10$.

Enter $^{-}x^2 + 40x + 50$ and $20x + 150$ on the stack. Press [↰] [0] (=) to set the two equations equal so that they will both graph at the same time when we graph. Store this in EQ. Note that instead of joining the two equations with =, you could have put them in a list separated by a space. This procedure works for drawing the graph of any number of equations: { ' -X^2 +40*X +50 ' '20*X+150 ' }	
Press [↱] [8] (PLOT) , and choose FUNCTION as the TYPE: (the current EQ should then appear in the second line). Change the variable to X if necessary. Set the horizontal and vertical views shown to the right.	
Graph the function and the line tangent to it at $x = 10$ with [ERASE] [DRAW] . We now want to "box in" the point of tangency and magnify that portion of the graph.	
Using the arrow keys, move the cursor to a position that would be at the upper left-hand corner of a box that you would draw enclosing the point of tangency. Press [ZOOM] [BOXZ] .	
Using [▷] and [▽], move the cursor so that you form a box enclosing the point of tangency. (Your box will probably not look exactly like the one shown to the right.) Press [ZOOM] .	
It is easy to see that the graph of the function and the graph of the tangent line are almost the same close to the point of tangency.	

- You should verify that the function and its tangent have close output values near the point of tangency by tracing the graphs near the point of tangency. Recall that you jump from one function to the other with [▲] or [▼] as you trace.

3.3.3 DRAWING A TANGENT LINE

When there is a graph on the screen, the [FCN] menu of your calculator contains the instruction to draw a tangent line to a curve at a point. To illustrate the process, we draw some tangent lines on the graph of $f(x) = x^3 + x^2 - 10x - 2$. We also investigate what the calculator does when you ask it to draw a tangent line where the line cannot be drawn.

Store $x^3 + x^2 - 10x - 2$ in EQ. Also set the horizontal and vertical views shown to the right.	
Draw the graph of the function. Press [TRACE] and use [(X,Y)] to be sure that the cursor is currently at a position where $x = 0$ (that is, at the y-intercept).	
Press any white key to return the menu to the bottom of the screen. Press [FCN] [NXT] [TANL] to draw the tangent line at the point (0, 2). Notice that the tangent line cuts through the curve at $x = 0$. It appears that (0, $^-2$) is an inflection point. Note that your HP also prints the equation of the tangent line at the bottom of the screen.	
Press any white key to return the menu to the bottom of the screen. Press [PICT] to return to the original menu so that you can activate trace to go to a specific point, and so forth. Repeat the above procedure to draw the tangent line at $x = {}^-3$.	
Go through the process again to draw the tangent line at $x = 1.5$. The tangent line is almost, but not quite, horizontal at $x = 1.5$. Return to the stack and notice that the equations of the three tangent lines appear on the stack in the order in which they were found.	

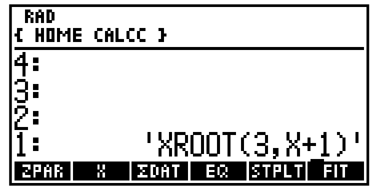
Let us now look at some special cases:

1. What happens if the tangent line is vertical? We consider the function $f(x) = (x + 1)^{1/3}$ which has a vertical tangent at $x = {}^-1$.
2. How does the calculator respond when the tangent line cannot be drawn at a point? We illustrate what happens with $g(x) = |x| - 1$, a function that has a sharp point at (0, $^-1$).

3. Does the calculator draw the tangent line at the joining point(s) of a piecewise continuous function? We consider two situations:
   a. *h(x)*, a piecewise continuous function that is continuous at all points and
   b. *m(x)*, a piecewise continuous function that is not continuous at $x = 1$.

1. Enter $(x+1)^{1/3}$ in EQ. Access XROOT with [↱] [√x].

Note that if you enter ' (X+1)^(1/3) ' that the HP will only draw the positive portion of this function.

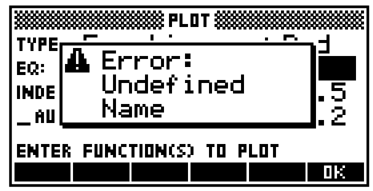


Draw the graph of the function by going to the plot application and choosing the default views:

H-VIEW = ⁻6.5 6.5 V-VIEW = ⁻3.1 3.2

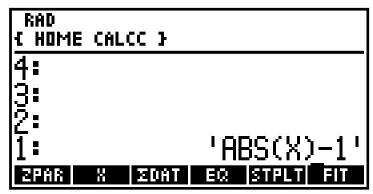
Draw the graph and trace to where $x = {}^-1$. Use [TANL] to draw the tangent line at this point.

A vertical tangent line at $x = {}^-1$ does not draw (as it should), and the message on the right appears.



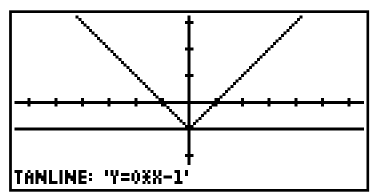
The tangent line at this point is vertical, and the calculator does not draw the tangent line.

2. Enter $|x| - 1$ in EQ. The absolute value symbol is obtained with [MTH] [REAL] [NXT] [ABS] or by typing ABS. Press [VAR] and store this in EQ.



Draw the graph in the default view you used before (H-VIEW = ⁻6.5 6.5 V-VIEW = ⁻3.1 3.2).

Trace to where $x = 0$. Use [TANL] to draw the tangent line at this point.

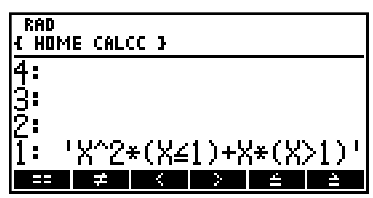


This is not correct! There is a sharp point at (0, ⁻1), and the limiting positions of secant lines from the left and the right of that point are different. A tangent line cannot be drawn at (0, ⁻1) because the instantaneous rate of change at that point does not exist.

3a. Next, enter, as indicated, the function

$$h(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x & \text{when } x > 1 \end{cases}$$

[Recall that the inequality symbols are accessed with [PRG] [TEST]].



h(x) is continuous for all values of *x*.

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Draw the graph of the function, and, using the same procedure as before and the default view, draw the tangent line to this function at $x = 1$.

The calculator is correct -- the tangent line can not be drawn because secant lines drawn with points on the right and left of $x = 1$ do not approach the same slope.

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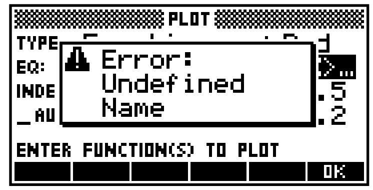

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3b. Edit EQ to enter, as indicated, the function

$$m(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x+1 & \text{when } x > 1 \end{cases}$$

([↰] [+/–] puts you in the stack edit mode while [▽] puts you in the equation editor mode.)

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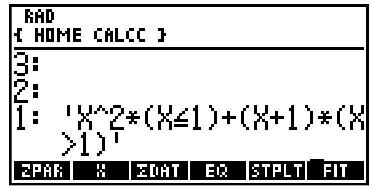

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Choose [OPTS] and "dot" mode by unchecking CONNECT as indicated to the right. Press [OK] .

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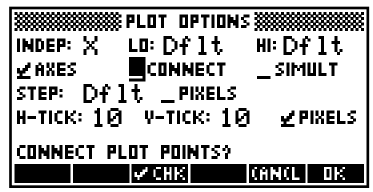

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Draw the graph in the default view.

Trace to where $x = 1$.

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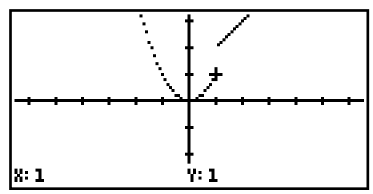

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Attempt to draw the tangent line at this point with the [TANL] instruction.

The calculator is correct. Since *m(x)* is not continuous at $x = 1$, the instantaneous rate of change does not exist at that point. The tangent line cannot be drawn at (1, 1).

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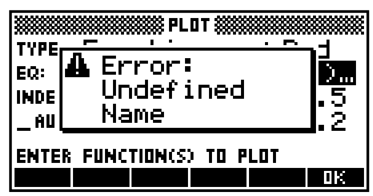

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Caution: Be certain that the instantaneous rate of change exists at a point before using your calculator to draw a tangent line at that point. Because of the way your calculator computes instantaneous rates of change, it may draw a tangent line at a point on a curve where the tangent line does not exist. If you receive an error message, be certain you understand why that message is the result of your action.

3.5 Percentage Change and Percentage Rates of Change

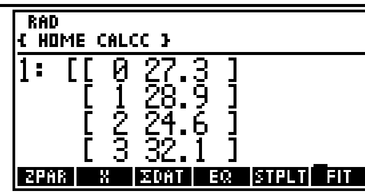
The calculations in this section involve no new calculator techniques. When calculating percentage change or percentage rates of change, you have the option of using a program or the stack.

3.5.1 CALCULATING PERCENTAGE CHANGE

Recall that program DIFF stores percentage changes (also called percentage differences) in a list you find when you press [%CHG]. Consider the following data giving quarterly earnings for a business:

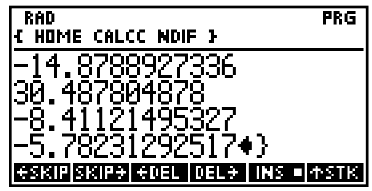
Quarter ending	Mar 1994	June 1994	Sept 1994	Dec 1994	Mar 1995	June 1995
Earnings (millions)	27.3	28.9	24.6	32.1	29.4	27.7

Align the input data so that x is the number of quarters since March, 1993, and input x in the first column of ΣDAT and earnings (in millions) in the second column of ΣDAT. Go to the NDIF directory and store the data as the current ΣDAT.



Run program DIFF and view the percentage change with [%CHG].

Notice that the percentage change from the end of September 1994 through December 1994 is about 30.5%. Also, from the end of March 1995 through June 1995, the percentage change is approximately ⁻5.8%.



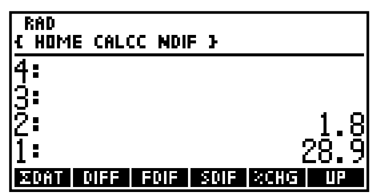
- You may find it easier to use the stack to calculate percentage change than to have the program do it for you.

3.5.2 CALCULATING PERCENTAGE RATE OF CHANGE

Consider again the quarterly earnings for a business. Suppose you are told or otherwise find that the rate of change at the end of the June 1994 is 1.8 million dollars per quarter. Evaluate the percentage rate of change at the end of June 1994.

Divide the rate of change at the end of June 1994 by the earnings, in millions, at the end of June 1994 and multiply by 100 to obtain the percentage rate of change at that point.

With your stack as it appears to the right, press [÷], enter 100 and press [×].



The percentage rate of change in earnings at the end of June 1994 was approximately 6.2% per quarter.