Chapter 2 Ingredients of Change: Nonlinear Models

2.1 Exponential Functions and Models

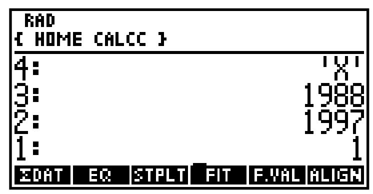
As we consider models that are not linear, it is very important that you be able to use scatter plots, numerical changes in output data, and the underlying shape of the basic functions to be able to identify which model best fits a particular set of data. Finding the model is only a means to an end -- being able to use mathematics to describe the changes that occur in real-world situations.

2.1.1 ENTERING EVENLY-SPACED INPUT VALUES (optional) When an input list consists of many evenly-spaced values, there is a calculator command that will generate the list so that you do not have to type in the values in one by one. To use this sequence command, enter the following on the stack, one by one in this order: *the formula, the variable, the first value, the last value, the increment*.

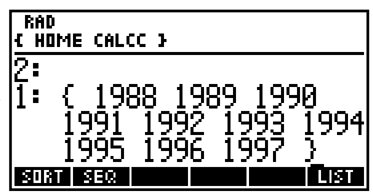
When entering years that differ by 1, the formula is the same as the variable and the increment is 1. Any letter can be used for the variable -- we choose to use X.

Generate the list of years beginning with 1988, ending with 1997, and differing by 1 by first pressing ['] [α] X [ENTER] [ENTER] 1988 [ENTER] 1997 [ENTER] 1 [ENTER] .

(Note that there is also an X in level 5 of the stack that does not appear in the picture on the right.)

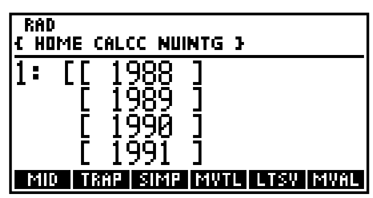


To generate the sequence, press [PRG] [LIST] [NXT] [PROC] [NXT] [SEQ] . (Instead of all these keystrokes, you could instead press and hold down [α] while typing S E Q. Release [α] and press [ENTER] .)



Press [VAR] (maybe [NXT]) [NUINT] [NXT] [LTSV] .

Program LTSV converts the <u>l</u>ist of dates <u>t</u>o a <u>s</u>ingle-<u>v</u>ariable matrix so that output can be included to form the data matrix ΣDAT.



2.1.2 FINDING PERCENTAGE CHANGE When the input values are evenly spaced, use program DIFF to compute percentage change in the output values. If the data are perfectly exponential (i.e., every data point falls on an exponential model), the percentage change in the output values is constant. If the percentage change is "close" to constant, this is an indication that an exponential model *may* be appropriate.

Suppose the population of a small town between the years 1988 and 1997 is as follows:

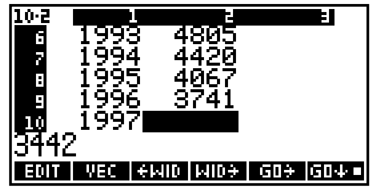
Year	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997

Population	7290	6707	6170	5677	5223	4805	4420	4067	3741	3442

Clear any old data, and enter the above data in ΣDAT. If you have entered the years into the first column of ΣDAT with the sequence command as described in Section 2.1.1 of this *Guide*, enter the second column of ΣDAT as described below.

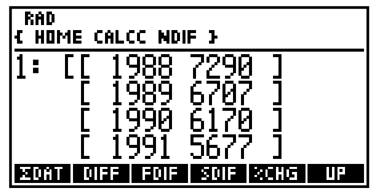
<table>
<tr>
<td>With the one-column matrix of dates on the stack, press [▽] so that you can edit ΣDAT.

Have [GO↓] chosen, and press [▷] to move the cursor to the first position in the second column. Type in the corresponding population values, pressing [ENTER] after each one.</td>
<td>



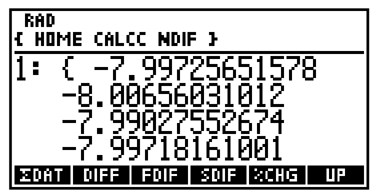
</td>
</tr>
<tr>
<td>Press [VAR] [↱] [▷] ['] (HOME) [CALCC] [NXT] [NDIF] to enter the directory containing program DIFF.

Store the data in ΣDAT with [↰] [ΣDAT] .</td>
<td>



</td>
</tr>
<tr>
<td>Run program DIFF. Remember that ΣDAT must be on level 1 of the stack before pressing [DIFF] , so press [ΣDAT] and then press [DIFF] .

Observe the percentage change after pressing [%CHG] .
See the last 6 values by pressing [▽] several times. [ENTER] returns to the stack.</td>
<td>



An exponential model may be a good fit.</td>
</tr>
</table>

- You should always construct a scatter plot of the data either before or after using program DIFF. For the data in this example, the scatter plot confirms that an exponential model certainly seems appropriate! (It is helpful to press [ΣDAT] to place the data on the stack before you leave this directory because the stat plot program is in a different directory with a different ΣDAT!)

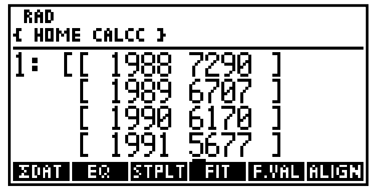
- If you would rather view values to a fixed number of decimal places rather than keep up with the scientific notation format, press [↱] [CST] (MODES) . Press [CHOOSE] from the NUMBER FORMAT location. Press [▽] to choose Fixed and press [OK] . Press [▷], type the number of decimal places you want shown, say 5, and press [OK] . Press [OK] .

2.1.3 FINDING AN EXPONENTIAL MODEL Use your calculator to find an exponential model that fits the data. The exponential model found by the HP-48's built-in routine in the statistics mode uses a base of e. In this text, the model used is of the form $y = ab^x$. Program EXPFT fits an exponential model with base b and is found in the HP-48 Appendix.

Using the instructions below, construct a scatter plot of the data. Notice that the data curves rather than falling in a straight line pattern. An exponential model certainly seems appropriate!

Press UP to get back into the CALCC directory.

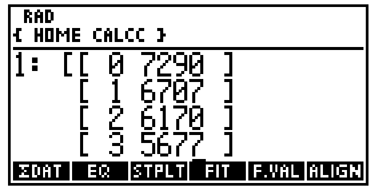
It is very important to align large numbers (like years) whenever you find an exponential model. The model the calculator finds may not even be correct if you don't!



We therefore align the data. Other alignments are possible, but we choose to align so that $x = 0$ in 1988.

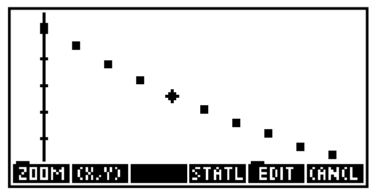
Enter 1988 on the stack and press ALIGN .

Either press ENTER several times to place copies of the data on the stack or store it in ΣDAT.



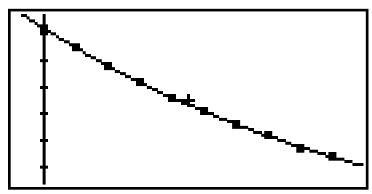
Before running program EXPFT, have ΣDAT on level 1 of the stack.

Press FIT EXPFT . You first see a scatter plot of the data.



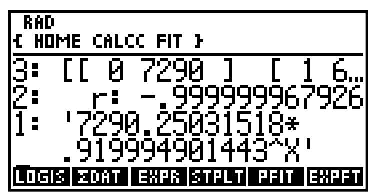
Press ON and the best-fitting exponential model is overdrawn on the scatter plot.

The exponential model is stored in EQ.



After the model is drawn on the scatter plot, the program automatically returns to the stack and puts *r* on level 2 and the model on level 1. (Again, ignore the *r*.)

If you need to recall the model, press EQ .



2.1.4 FINDING A LOGISTIC MODEL

Use your calculator to find a logistic model of the form $y = \dfrac{L}{1 + Ae^{-Bx}}$. The logistic model that you obtain may be slightly different from a logistic model found with another calculator. Logistic models in *Calculus Concepts* were found using a TI-83. Refer to the following discussion for the comparable HP logistic model that best fits the data given in Example 2, Section 2.1 of the text.

(Program LOGIS is given in the HP Appendix.) As with the exponential model $y = ab^x$, large input values must be aligned before fitting a logistic model to data.

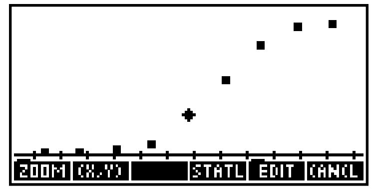
Clear any old data, and enter the following data in ΣDAT:

Aligned end of month	1	2	3	4	5	6	7	8	9
Total number of swimsuits sold	4	12	25	58	230	439	648	748	769

Before running program LOGISTIC, *have* ΣDAT *on level 1 of the stack.*

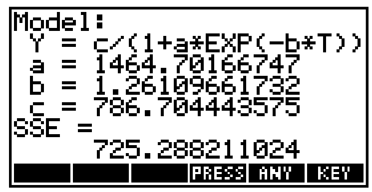
Construct a scatter plot of the data. A logistic model may be appropriate. (Put the data back on the stack.)

Run program LOGIS by pressing [LOGIS] to enter the logistic directory and then press [LOGIS] in that directory to run the program.



Program LOGIS finds a "best-fit" logistic model rather than being a logistic model with a user-input limiting value *L* such that no data value is ever greater than *L.*

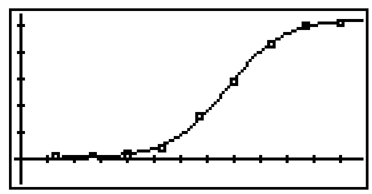
While the program is running you will see several different messages displayed which you should ignore.)



- Note that the variables in the model found by the calculator are called by different names than the variables in the logistic model in your text. The program calls the limiting value c, not L, and the input variable is called T, not X.

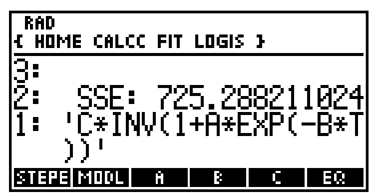
There is a message on the menu. So, press any key to continue.

The program now draws a graph of the logistic model it found on a scatter plot of the data.



Press [ON] after the graph finishes drawing.

The general equation of the logistic model is stored in EQ. (Ignore the SSE value that appears at the end of the program.)

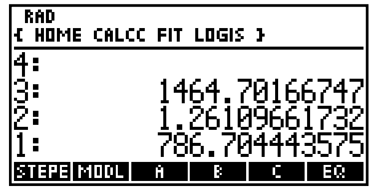


- Provided the input values are evenly spaced, program DIFF might be helpful when you are trying to determine if a logistic model is appropriate for certain data. If the first differences (in list L3 after running program DIFF) *begin small*, *peak in the middle*, and *end small,* this is an indication that a logistic model may provide a good fit to the data. Such is true for this data set because the first differences are 8, 13, 33, 172, 209, 209, 100, and 21.

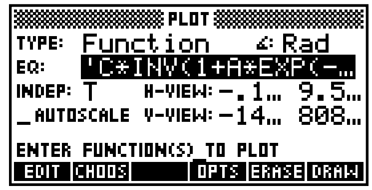
2.1.5 RECALLING MODEL PARAMETERS

Rounding of model parameters can often lead to incorrect or misleading results. You may find that you need to use the full values of model parameters after you have found a model. It would be tedious to copy all these digits into another location of your calculator. You don't have to!

The values of C, A, and B in the logistic model are stored in the LOGIS directory. To recall any of these values, simply press the corresponding menu key.



If you wish to use PLOT, SOLVE , and so forth, the general equation is automatically transferred to those applications because it is stored in EQ. Your calculator will use the correct values of A, B, and C when you do something with this equation as long as you do not store any new values in them.



2.1.6 RANDOM NUMBERS Imagine all the real numbers between 0 and 1, including the 0 but not the 1, written on identical slips of paper and placed in a hat. Close your eyes and draw one slip of paper from the hat. You have just chosen a number “at random”. Your calculator doesn’t offer you a choice of all real numbers between 0 and 1, but it allows you to choose, *with an equal chance of obtaining each one*, any of 10^{12} different numbers between 0 and 1 with its random number generator called rand.

First, “seed” the random number generator. (This is like mixing up all the slips of paper in the hat.)

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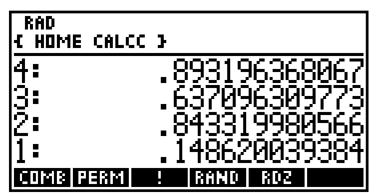
Pick some whole number and enter it on the stack as the “seed”. (Everyone needs to have a different seed, or the choice will not be random.)

With your seed in level 1 of the stack, press [MTH] [NXT] [PROB] [RDZ] .

On the menu, directly to the left of RDZ, is the random number generator RAND.

Press [RAND] several times. Your list of random numbers should be different from the one on the right.



If you want to choose, at random, a whole number between 1 and *N*, IP(N *rand + 1) does the job. (IP is the integer portion of a number.)

Follow the directions shown on the right to generate a random integer between 1 and 10.

Repeat the process to generate more random numbers between 1 and 10.

Press [RAND] .

Press [1] [0] [×] .

Press [1] [+].

Press and hold [α] .

Type I P and press [ENTER] .

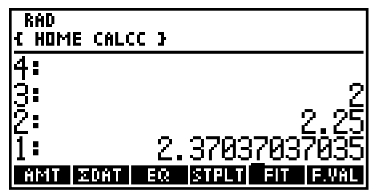
2.2 Exponential Models in Finance

You are probably familiar with the compound interest formulas. This section introduces you to some new methods of using your calculator with familiar formulas.

2.2.1 FINDING FUTURE VALUE One way to find formula outputs on the HP-48 is to create a user-defined formula. This is actually a program that takes a value from level 1 of the stack, substitutes it in a formula, and evaluates the formula.

Formulas can be entered in any directory; just don't forget where you put them. This example places the formula in CALCC. Press [↱] ['] (HOME) and [CALCC].

Suppose you need to evaluate the amount formula $\left(1+\frac{1}{n}\right)^n$ for several different values of n. (Remember that to type a lower-case letter, you should press [α] [↰] before the letter key.)

Enter << →n '(1 + INV(n)) ^ n ' EVAL >>. (Watch, as you type, that you move outside the ')' with [▷] after typing the n in INV(n).) Press [ENTER]. Enter, with ['], the name of this formula, say AMT. Press [STO]. (You can recall the formula with [↱] [AMT]. Using [▽] allows you to edit what you typed.)	Press [↰] [–] for << >>. Press [↱] [0] for →. Insert spaces where needed with [SPC]. Press [↰] [÷] for ( ). Press [1/x] for INV( ).
Once your formula is stored, simply enter a value of n on the stack and press [AMT]. Repeat the procedure for different values of n. The screen to the right shows results for $n = 1, 2,$ and 3.	

- You could have achieved the same results as illustrated above by storing the formula ' (1 + INV(X)) ^ X ' in EQ and finding the outputs using program F.val.
- You could also use the SOLVR application to find outputs. This method is especially useful when there is more than one input variable.

2.2.3 FINDING PRESENT VALUE The present value of an investment is easily found with the calculator's solver or by any of the other means that were discussed to find input values. For instance, suppose you want to solve for the present value P the equation $9438.40 = P\left(1+\frac{0.075}{12}\right)^{60}$.

Refer to 1.1.7 of this *Guide* for instructions on using the HP-48's SOLVR. Enter the equation on the right in EQ.	Type ' 9438.40 = P* (1 + .075 / 12) ^ 60 ' and press [ENTER] [↰] [EQ].

Solve for P with [↰] [P] to obtain the present value $6494.49. (Refer to 1.2.3 of this *Guide* for more detailed instructions on finding x-intercepts.)	If you prefer, you could find the x-intercept of ' 9438.4 – X*(1+.075/12)^60 ' to find the present value.

- If your VAR menu appears cluttered, recall that you can delete variables that are no longer used (such as A, P, R, and T from the above example) by entering them on level 1 of the stack inside { } and pressing [↰] [EEX] (PURGE) .

2.3 Polynomial Functions and Models

You will in this section learn how to fit models to data that have the familiar shape of a parabola or a cubic. Using your calculator to find these models involves basically the same procedure as when using it to find linear and exponential models.

2.3.1 FINDING SECOND DIFFERENCES When the input values are evenly spaced, use program DIFF to compute second differences in the output values. If the data are perfectly quadratic (*i.e.*, every data point falls on a quadratic model), the second differences in the output values are constant. If the second differences are "close" to constant, this is an indication that a quadratic model *may* be appropriate.

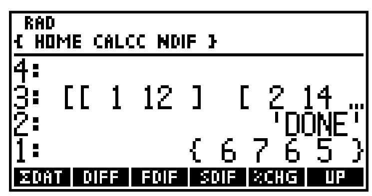
Clear any old data, and enter the roofing job data in ΣDAT:

Months after January	1	2	3	4	5	6
Number of jobs	12	14	22	37	58	84

Run program DIFF and observe the second differences in list SDIF.

The second differences are close to constant, so a quadratic model may be a reasonably good fit.

Construct a scatter plot of the data. A quadratic model seems appropriate!

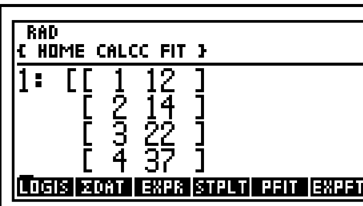


2.3.2 FINDING A QUADRATIC MODEL Use your calculator to obtain a quadratic model that fits the data. The HP-48 quadratic model is found using program PFIT with $n = 2$ and is of the form $y = ax^2 + bx + c$.

Before running program PFIT,
have ΣDAT on level 2 of the stack and the value of n on level 1 of the stack.

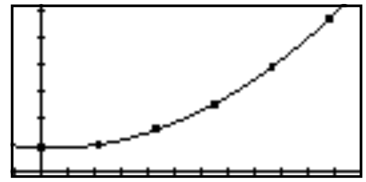
Check to be sure that the data shown to the right is what is in level 1 of the stack. (Remember, to see all the data, press [▽] when ΣDAT is on the stack.)

Find program PFIT and store ΣDAT in the FIT directory.



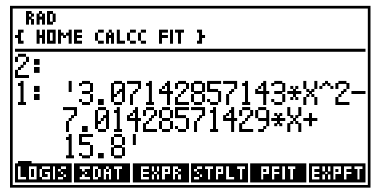
Press ΣDAT to place the data on the stack. Enter 2 to tell the program you want to fit a quadratic model. Press PFIT .

A scatter plot of the data appears. ON overdraws the graph of the quadratic model.



As before, you are automatically returned to the stack after the graph is drawn.

The right-hand side of the quadratic model is in level 1 of the stack and in EQ.

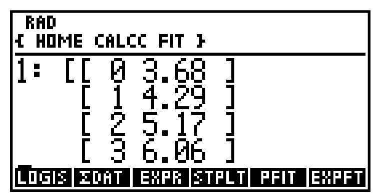


2.3.3 FINDING A CUBIC MODEL Whenever a scatter plot of the data shows a single change in concavity, a cubic or logistic model is appropriate. If a limiting value is apparent, use the logistic model. Otherwise, a cubic model should be considered. When appropriate, use your calculator to obtain the cubic model that best fits data. Program PFIT with $n = 3$ finds the cubic model of the form $y = ax^3 + bx^2 + cx + d$ that best fits the data.

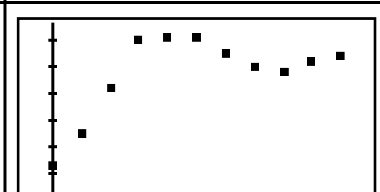
Clear any old data, and enter the average price in dollars per 1000 cubic feet of natural gas for residential use in the U.S. from 1980 through 1990 in ΣDAT:

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Price	3.68	4.29	5.17	6.06	6.12	6.12	5.83	5.54	5.47	5.64	5.77

First, use program ALIGN to align the data so that x represents the number of years since 1980.



Draw a scatter plot of these data with STPLT.

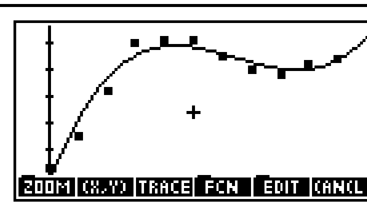


Notice that a concavity change is evident, but there do not appear to be any limiting values. Thus, a cubic model may fit the data.

As before, remember to have ΣDAT on level 2 of the stack and the value of n on level 1 before using program PFIT.

Press ΣDAT 3 PFIT .

Press ON after viewing the scatter plot to see the plot of the model.



You are automatically returned to the stack after the graph is drawn.

The right-hand side of the cubic model is in level 1 of the stack and in EQ.