

# Chapter 10 Analyzing Multivariable Change: Optimization



## 10.2 Multivariable Optimization

As you might expect, multivariable optimization techniques on your calculator are very similar to those that were discussed in Chapter 5. The basic difference is that the algebra required to get the expression that comes from solving a system of equations in several unknowns down to one equation in one unknown is sometimes fairly difficult. However, once your equation is of that form, all the optimization procedures are the same as previously discussed.

**10.2.1 FINDING CRITICAL POINTS** To find critical points for a multivariable function (points at which maxima, minima, or saddle points occur), find the point or points at which the partial derivatives with respect to each of the input variables are zero. Once you use derivative formulas to find the partial derivative with respect to each input variable, set these partial derivatives formulas each equal to 0, and use algebraic methods to obtain one equation in one unknown input. Then, you can use the your calculator's solve routine to obtain the solution to that equation.

We illustrate these ideas with a model for the total daily intake of organic matter required by a beef cow grazing the Northern Great Plains rangeland:

$$I(s, m) = 8.61967 - 1.244s + 0.0897s^2 - 0.20988m + 0.035947m^2 + 0.214915sm \text{ kg per day}$$

Find the two partial derivatives and set each of them equal to 0 to obtain

$$\frac{\partial I}{\partial s} = -1.244 + 0.1794s + 0.214915m = 0 \quad \text{and} \quad \frac{\partial I}{\partial m} = -0.20988 + 0.071894m + 0.214915s = 0$$

Solve one of the equations for one of the variables, say  $\frac{\partial I}{\partial s}$  for  $s$ , to obtain

$$s = \frac{1.244 - 0.214915m}{0.1794}$$

(Remember that to solve for a quantity means that it must be by itself on one side of the equation and the other side of the equation cannot contain that letter at all.) Now, let your calculator do the rest of the work.

**Warning:** Everything that you do in your calculator depends on the partial derivative formulas that you have found using derivative rules. Double check your analytic work using one of the methods suggested in Chapter 9 before you use any of the methods illustrated below.

On the stack, enter the expression for  $s$ , using the *same* letters that are in the problem situation, in EQ.

Remember that any numerator, denominator, or power that consists of more than one number or letter should be enclosed in parentheses.

Store this expression to the name S.

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RAD          ALG
{ HOME CALC }
2:
1: '(1.244-.214915*M)/.1794'
'S'
MODE EQ  STAT  FIT  F-VAR  ALIGN

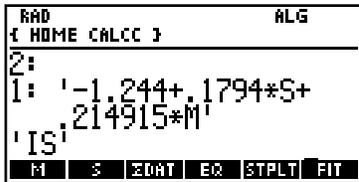
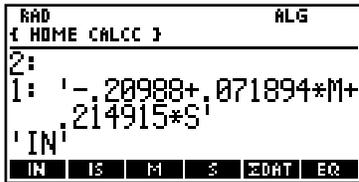
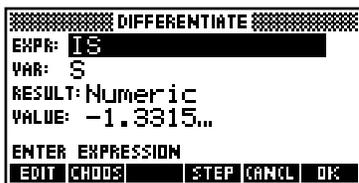
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<p>Type the expression for the <i>other</i> partial derivative, <math>\frac{\partial I}{\partial m}</math>, exactly as you have it written on your paper, on the stack. (The expression you type should be equal to zero.)</p> <p>Store this expression in EQ.</p>	
<p>Press <math>\rightarrow</math> <math>\boxed{7}</math> (SOLVE), and have the cursor in the M location.</p> <p>Press <math>\boxed{\text{SOLVE}}</math>. (Note: A guess for M is not really necessary because EQ is a linear equation and therefore has only one root.)</p>	
<p>Press <math>\boxed{\text{NXT}}</math> <math>\boxed{\text{OK}}</math> to return to the stack. To find the value of S simply press <math>\boxed{\text{S}}</math> and <math>\boxed{\text{EVAL}}</math>.</p>	
<p>Now, enter <math>I(s, m)</math> on the stack and then press <math>\boxed{\text{EVAL}}</math> <math>\boxed{\text{EVAL}}</math>. (You already have the values you found for <math>s</math> and <math>m</math> stored in those variables.)</p> <p>What all this has done is locate a critical point of <math>I(s, m)</math> at <math>s \approx -1.33</math> and <math>m \approx 6.90</math> with <math>I(s, m) \approx 8.72</math>.</p>	
<p>Drop the value of <math>I(s, m)</math> from level one of the stack. Store the value of S, -1.33159744075, in S for use in the next section.</p>	

- It is very important to note that because the HP-48 accepts strings of letters as variable names, whenever you perform an operation on two separate variables, you must put the indicated symbol for that multiplication. For instance, we typed the product of S and M in the above evaluation as S\*M. This is necessary; if it were not done, the calculator would think we meant a single variable called “SM”.

**10.2.2 THE DETERMINANT TEST** Now that you have found one or more critical points, the next step is to classify those point(s) as points at which a maximum, a minimum, or a saddle point occurs. The Determinant Test will often give you the answer. Because this test uses derivatives, the calculator can help you.

For the beef cattle example in Section 10.2.1 with critical point  $s \approx -1.33$  and  $m \approx 6.90$ , you now need to calculate some second partial derivatives evaluated at this point. Recall that we have  $\frac{\partial I}{\partial m} = I_m$  in EQ. However, S does not contain  $I_s = \frac{\partial I}{\partial s}$ .

<p>Enter <math>\frac{\partial I}{\partial s} = -1.244 + 0.1794s + 0.214915m</math> and store it some name that you can remember, say IS.</p>	
<p>You should still have <math>\frac{\partial I}{\partial m}</math> stored in EQ. Press EQ to return it to the stack. (If not, enter <math>I_m</math> on the stack.) Store <math>I_m</math> to the name IN.</p> <p>(An error results if you try to store to the name IM because it is reserved for use by the calculator's memory.)</p>	
<p>Return to the home screen, and recall S and M to be certain that the critical point values are still stored in those locations. If not, store the critical values in those locations. <b>This is a very important step!</b></p> <p>(You will have better accuracy with the Determinant Test if you use several decimal places for the values.)</p>	
<p>Now, if we take the derivative of <math>IS = I_s</math> with respect to <math>s</math>, we will have <math>I_{ss}</math>, the second partial derivative of <math>I(s, m)</math>. Have the SYMBOLIC screen as shown to the right and press <math>\text{ⓄK}</math>. Find <math>I_{ss} \approx 0.1794</math>.</p> <p>(Be certain the value you type after the variable matches the input that is varying.)</p>	
<p>If we take the derivative of <math>IS = I_s</math> with respect to <math>m</math>, we will have <math>I_{sm}</math>, the partial derivative of <math>I(s, m)</math> with respect to <math>s</math> and then <math>m</math>. Have the SYMBOLIC screen as shown to the right and press <math>\text{ⓄK}</math>. Find <math>I_{sm} \approx 0.2149</math>.</p>	
<p>If we take the derivative of <math>IN = I_m</math> with respect to <math>s</math>, we will have <math>I_{ms}</math>, the partial derivative of <math>I(s, m)</math> with respect to <math>m</math> and then <math>s</math>. Have the SYMBOLIC screen as shown to the right and press <math>\text{ⓄK}</math>. Find <math>I_{ms} \approx 0.2149</math>.</p>	

<p>Now, if we take the derivative of <math>IN = I_m</math> with respect to <math>m</math>, we will have <math>I_{mm}</math>, the second partial derivative of <math>I(s, m)</math>. Find <math>I_{mm} \approx 0.0719</math>.</p>	
<p>The values of all the partial derivatives are now on the stack.</p>	
<p>Now, find the value of <math>D \approx (0.1794)(0.0719) - (0.2149)^2</math> by pressing <math>\leftarrow</math> <math>\blacktriangle</math> (STACK) 3 <b>ROLLD</b> <math>\times</math> 3 <b>ROLLD</b> <math>\times</math> <math>\blacktriangleright</math> (SWAP) <math>\ominus</math>.</p> <p>Since <math>D &lt; 0</math>, the Determinant Test tells us that the point <math>(s, m, I) \approx (-1.33, 6.90, 8.72)</math> a saddle point.</p>	

- The values of the second partial derivatives were not very difficult to determine without the calculator in this example. However, with a more complicated function, we strongly suggest using the above methods to provide a check on your analytic work to avoid making simple mistakes.

## 10.3 Optimization Under Constraints

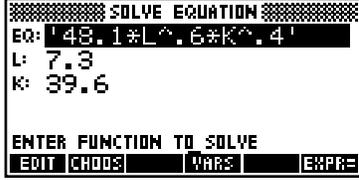
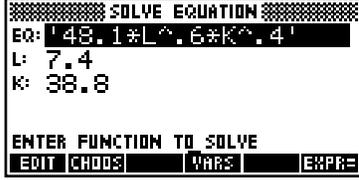
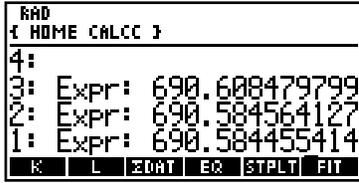
Optimization techniques on your calculator when a constraint is involved are exactly the same as those discussed in Sections 10.2.1 and Section 10.2.2 except that there is one more equation involved in the analytic process.

### 10.3.1 CLASSIFYING OPTIMAL POINTS UNDER CONSTRAINED OPTIMIZATION

Use the methods indicated in Section 10.2.1 to find the critical point  $L = 7.35$ ,  $K = 39.2$ . We illustrate the procedure for determining if this is the point at which a maximum or minimum occurs for the Cobb-Douglas Production function  $f(L, K) = 48.1L^{0.6}K^{0.4}$  subject to the constraint  $g(L, K) = 8L + K = 98$ .

<p>Find <math>f(L, K)</math> at <math>L = 7.35</math>, <math>K = 39.2</math> to be 690.60479799. One way to do this is to use the SOLVE application and press <b>EXPR=</b>.</p> <p>We now use the methods of Section 10.2.2 to test close points that are on the constraint.</p>	
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You need to remember that whatever close points you test, they must be near the critical point and *they must be on the constraint*  $g(L, K)$ . Be wary of rounding during the following procedure. Rounding of intermediate calculations and/or critical points can give a false result when the “close” point is very close to the optimal point.

<p>Choose a new value of <math>L</math> that is less than <math>L</math>, say 7.3. Determine the value of <math>K</math> so that <math>8L + K = 98</math>:</p> $8(7.3) + K = 98, \text{ so } K = 98 - 58.4 = 39.6.$ <p>Now find <math>f(L, K)</math> at <math>L = 7.3, K = 39.6</math>.</p>	
<p>Choose another nearby value of <math>L</math>, but this time choose one that is more than <math>L</math>, say 7.4. Determine the value of <math>K</math> so that <math>8L + K = 98</math>:</p> $8(7.4) + K = 98, \text{ so } K = 98 - 59.2 = 38.8.$ <p>Now find <math>f(L, K)</math> at <math>L = 7.4, K = 38.8</math>.</p>	
<p>Press <b>NXT</b> <b>OK</b> to return to the stack and observe the outputs.</p>	

Since  $f(7.35, 39.2) \approx 690.608$  is greater than both  $f(7.3, 39.6) \approx 690.585$  and  $f(7.4, 38.8) \approx 690.584$ , we conclude that  $L = 7.35$  thousand worker hours,  $K = \$39.2$  thousand is the point at which the maximum value of  $f(L, K)$  occurs. Rounding this answer to make sense in the context of the problem, we find that 691 mattresses is a maximum production level.

- When you test close points, it is best to use all the decimal places provided by the calculator for the optimal values when classifying optimal points since rounding will, in many cases, give inaccurate or misleading results in the classification.