Chapter 10
Analyzing Multivariable Change: Optimization

Section 10.1 Multivariable Critical Points

1. a. A relative maximum occurs when a table value is greater than all 8 values surrounding it.
   b. A relative minimum occurs when a table value is less than all 8 values surrounding it.
   c. If a table value appears to be a maximum in one direction but a minimum in another direction, then the value corresponds to a saddle point.

   d, e. If all the edges of a table are terminal edges, then the absolute maximum and minimum are simply the largest and smallest values in the table. If all the edges are not terminal edges, then you must know whether any critical points exist outside the table in order to determine whether absolute extrema exist. If no critical points exist outside the table, then in determining absolute extrema, you must consider relative extrema, output values on terminal edges, and the behavior of the function beyond the edges of the table. It is often helpful to sketch contour curves on a table when determining critical points and absolute extrema.

2. a. A relative maximum occurs when a point lies within concentric simple closed contour curves and the values of the contour curves are increasing as the curves get closer to the point.
   b. A relative minimum occurs when a point lies within concentric simple closed contour curves and the values of the contour curves are decreasing as the curves get closer to the point.
   c. A saddle point occurs when the contour curves near a point are all curved away from the point and some of the contours decrease as you move away from the point, while others increase. A contour curve that passes exactly through a saddle point forms an X.

   d, e. In a contour graph whose edges are all terminal edges, the absolute extrema are the largest and smallest output values among the relative extrema and the edges of the graph. If there are non-terminal edges, then you must consider the behavior of the graph beyond the edges. In some cases, there may be no absolute maximum or absolute minimum.

3. The point is a relative maximum point because the values of the contour curves decrease in all directions away from the point.

4. The point is a saddle point because it is the maximum along a line beginning near the lower left corner, ending near the upper right corner, and passing through the point at \( x = 5, y = 3 \); and it is a minimum for a line in the opposite diagonal direction. Also the point lies at the intersection of the 50-contour curves. **Note: the wrong point was identified in the answer key in the back of the text.**
Chapter 10: Analyzing Multivariable Change: Optimization

Calculus Concepts

5. a.

\[ R(g, h) \]

Relative maximum

Saddle point

b. Relative maximum point: \((g, h, R) = (2, 3, 95)\); Saddle point: \((g, h) = (6, 3, 30)\)

6. a.

\[ T(p, f) \]

Saddle point

Relative minimum

b. Relative minimum point: \((p, f, T) = (5, 3, 30)\); Saddle point: \((p, f, T) = (5, 7, 70)\)

7. The point is a saddle point because it is a maximum contour level along a cross-section extending from the \(x=4, y=0\) corner diagonally back through the point and is a minimum contour level along a cross-section extending from the \(x=0, y=0\) corner diagonally through the point.

8. a. This point is a relative maximum point.
   
   b. This point is a saddle point.
   
   c. This point is none of these.

9. Relative maximum point: (May, 1995, $1.45 per pound)
   
   Relative maximum point: (May, 1998, $0.88 per pound)

10. There are two relative maximum points: (1995, July, 52 cents) and (1998, July, 53 cents).
    The points corresponding to June and July in 1997 suggest a saddle point, because the entries are relative maximum values in the 1997 row, and relative minimum values in the June and July columns.
11. a. Yes; The table gives monthly averages, so it doesn’t make sense to extend the columns. However, the choice of January as the first column is not mandatory. The best way to visualize this table is to wrap it around a cylinder so that the January and December columns are adjacent columns and there are no left or right edges on the table. The top and bottom rows are terminal edges.

b. | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
---|---|---|---|---|---|---|---|---|---|---|---|
North Pole | — | — | 3.1 | 6.9 | 8.9 | 7.9 | 6.9 | 4.9 | 6.9 | — | — | — |
80° | — | 0.5 | 0.77 | 3.4 | 6.9 | 8.8 | 7.8 | 6.8 | 5.8 | 4.3 | 2.2 | 0.81 | 0.34 |
70° | — | 0.52 | 0.56 | 3.4 | 6.9 | 8.8 | 7.8 | 6.8 | 5.8 | 4.3 | 2.2 | 0.81 | 0.34 |
60° | 6.9 | 1.53 | 3.4 | 5.7 | 7.6 | 8.8 | 7.8 | 6.8 | 5.8 | 4.3 | 2.2 | 0.81 | 0.34 |
50° | 1.66 | 2.8 | 4.7 | 5.7 | 8.4 | 9.1 | 8.8 | 7.4 | 7.1 | 3.4 | 1.97 | 1.35 |
40° | 3.0 | 4.2 | 5.9 | 5.7 | 8.8 | 9.3 | 9.0 | 9.1 | 5.9 | 4.8 | 3.4 | 2.6 |
30° | 4.4 | 5.6 | 6.9 | 5.7 | 9.0 | 9.2 | 9.1 | 8.4 | 7.4 | 6.1 | 4.7 | 4.1 |
20° | 4.9 | 6.1 | 7.8 | 8.5 | 8.8 | 8.9 | 8.8 | 8.6 | 8.0 | 7.4 | 6.0 | 5.5 |
10° | 6.1 | 7.1 | 7.7 | 8.3 | 8.4 | 8.3 | 8.4 | 8.3 | 8.2 | 7.9 | 7.3 | 6.8 |
Equator | 6.1 | 8.5 | 8.6 | 8.3 | 7.8 | 7.5 | 7.6 | 8.0 | 8.4 | 8.2 | 7.9 |
10° | 8.9 | 8.9 | 8.8 | 8.4 | 8.7 | 6.9 | 6.4 | 6.5 | 7.2 | 8.1 | 8.6 | 8.8 |
20° | 9.4 | 8.6 | 8.4 | 6.9 | 8.5 | 5.7 | 5.4 | 5.4 | 6.4 | 8.6 | 9.2 | 9.5 |
30° | 9.6 | 8.8 | 7.4 | 5.8 | 4.4 | 3.8 | 4.1 | 5.2 | 6.7 | 8.2 | 9.3 | 9.8 |
40° | 9.6 | 9.3 | 8.5 | 6.9 | 3.1 | 2.5 | 2.7 | 3.9 | 5.6 | 7.6 | 9.1 | 9.9 |
50° | 9.6 | 7.6 | 5.4 | 3.3 | 1.8 | 1.25 | 1.49 | 2.4 | 4.5 | 6.6 | 8.7 | 9.7 |
60° | 8.7 | 6.8 | 4.1 | 2.0 | 0.72 | 0.31 | 0.47 | 1.36 | 3.1 | 5.8 | 8.2 | 9.3 |
70° | 5.7 | 2.8 | 0.84 | — | — | — | — | 0.38 | 1.78 | 4.7 | 9.1 |
80° | 4.2 | 1.7 | 0.14 | 0.09 | — | — | — | — | 0.62 | 3.2 | 7.0 | 9.3 |
South Pole | 8.1 | 4.6 | 0.60 | — | — | — | — | — | — | 2.9 | 7.8 | 9.4 |

c. There are three relative maximum points: (June, North Pole, 8.9 kW-h/m²), (June, 40° North, 9.3 kW-h/m²), and (December, 40° South, 9.9 kW-h/m²).

It is difficult to estimate relative minima points accurately because of the dashes in the table, which we can interpret to mean radiation levels of essentially zero. Thus we conclude that the regions of the underlying function represented by the dashes in the table are those in which the minimum radiation level occurs. There are two such regions: one at and near the North Pole between March and October and one at and near the South Pole between April and September. If there are two specific relative minima of the underlying function, then we estimate that they occur at the end of December at the North Pole and in the middle of June at the South Pole.

There are four points that can be considered saddle points: (April, 10° North, 8.5 kW-h/m²) (August, 10° North, 8.4 kW-h/m²), (June, 70° North, 8.5 kW-h/m²), and (December, 70° South, 9.1 kW-h/m²). (Answers may vary.)

d. The greatest radiation level shown in the table is 9.9 kW-h/m² which occurs in December at 40° South. The smallest radiation level shown is 0.06 kW-h/m² which occurs in November at 80° North. If we consider the dashes to be zeros, then the smallest radiation level is zero and occurs many times in the table.
e. The absolute maximum value is 9.9 kW-h/m², and the absolute minimum is approximately zero. Because the table cannot extend in any direction, these answers do correspond to those in part d.

f. The largest and smallest values in the table will be the absolute maximum and minimum, respectively, if the table cannot be extended in any direction. That is, either the edges are terminal edges or the table “wraps around,” as in this case.

12. a.

<table>
<thead>
<tr>
<th>Δγ (°C)</th>
<th>-40</th>
<th>-30</th>
<th>-20</th>
<th>-10</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
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<td>0</td>
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<td>0</td>
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<td>1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

b. The absolute maximum percentage is 59%. This occurs at the point (0, 0, 59). This means that the most likely situation to occur is that precipitation and temperature are at their normal levels. This situation occurs 59% of the time.

13. a. The expected corn yield is 100% of the annual average yield. That is, there is no expected increase or decrease in yield from the average.

b. The expected corn yield is 40% of the annual average yield.
c. The maximum percentage yield is 109%. This maximum occurs twice, at the points (40%, 0°C, 109%) and (20%, -1°C, 109%). This means that a yield of 109% above normal can be expected when temperatures are average (a change of 0°C) and there is 40% more precipitation than normal or when temperatures are 1°C below normal and precipitation is 20% above normal.

14. a. Answers will vary based on your latitude.

b. Relative maximum: 17.3 mm per day in June at a latitude between 40° and 42°

c. The table wraps around as in Activity 11; however, the rows can extend above the top row for degrees of latitude greater than 50.

d. The greatest amount of extraterrestrial radiation is approximately 17.3 mm per day, which occurs in June for latitudes between 40° and 42°. We cannot accurately estimate the least amount of radiation from the table, although we suspect that it is near zero and occurs in December at the North Pole.

15. a. Because there are no values smaller than all 8 surrounding values in the table, there are no relative minimum points. A relative maximum point occurs at an average daily weight gain of 1.01 kilograms per day for a 91-kilogram pig at an air temperature of 21.1°C. When we consider the edges of the table in our search for absolute extrema, we find the absolute maximum weight gain to be 1.09 kilograms per day for a 156-kilogram pig and air temperature of 15.6°C. The absolute minimum point corresponds to a weight loss of 1.15 kilograms per day for a 156-kilogram pig and temperature of 37.8°C.

b. 15.6°C = 60.08°F, 21.1°C = 69.98°F, 37.8°C = 100.04°F

c. The greatest average daily weight gain for pigs weighing between 45 kg and 156 kg and air temperatures between 4.4°C and 37.8°C is approximately 1.09 kilograms per day for a
156-kilogram pig at an air temperature of about 60°F. The greatest average daily weight loss is 1.15 kilograms per day for a 156-kilogram pig at an air temperature of about 100°C. These answers indicate that the heaviest pigs can gain or lose weight more quickly than lighter pigs, depending on the temperature.

16. a.  
<table>
<thead>
<tr>
<th>Storage time (months)</th>
<th>Blanching temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>43 48 54 53 45</td>
</tr>
<tr>
<td>0.5</td>
<td>38 44 48 47 49</td>
</tr>
<tr>
<td>1</td>
<td>36 41 45 44 43</td>
</tr>
<tr>
<td>1.5</td>
<td>54 49 49 44 40</td>
</tr>
<tr>
<td>2</td>
<td>33 38 44 44 34</td>
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<tr>
<td>2.5</td>
<td>33 38 43 43 33</td>
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<tr>
<td>3</td>
<td>34 40 45 46 36</td>
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<tr>
<td>3.5</td>
<td>35 40 47 49 38</td>
</tr>
<tr>
<td>4</td>
<td>39 40 47 50 50</td>
</tr>
</tbody>
</table>

b. Saddle point: \(\approx (63°C, 2.5 \text{ months, } 43\%)\)

c. Absolute maximum: 54% occurring at 0 months storage time and a temperature of 59°C.

Absolute minimum: 33% occurring at 2 months and a temperature of 35°C and at 2.5 months and temperatures of 35°C and 83°C.

d. One possible answer: Thick applesauce is desirable because water can be added to it, increasing the amount that can be processed.

e. One possible answer: In order to use the least added water, manufacturers should store apples for 2.5 months.

17. a.  
<table>
<thead>
<tr>
<th>Storage time (months)</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.1 2.8 2.6 2.6 2.8</td>
</tr>
<tr>
<td>1</td>
<td>2.5 3.1 2.8 2.8 2.6</td>
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<td>2</td>
<td>3.5 3.2 3.1 2.9 3.2</td>
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<tr>
<td>3</td>
<td>3.4 3.2 3.1 2.9 3.2</td>
</tr>
<tr>
<td>4</td>
<td>3.3 3.2 2.9 2.7 3.0</td>
</tr>
</tbody>
</table>

b. Saddle point: \(\approx (71°C, 2.5 \text{ months, } 2.9 \text{ cm})\)

c. Absolute maximum: 3.5 cm at 35°C and 2 months

Absolute minimum: 2.6 cm at 59°C and 71°C and 0 months

d. One possible answer: The saddle point is important because it shows the optimal storage time and temperature to keep applesauce from getting too thick or too thin.

18. a. The only relative extreme point occurs at (0.5 miles north, 0.9 miles west, 800.2 feet above sea level)

b. The saddle point occurs near (0.5 miles north, 0.5 miles east, 799.9 feet above sea level)

c. By sketching some contours, we get a general idea of where a saddle point may occur. We pinpoint the saddle point by finding the table value that is smaller than those on either side of it in a row and larger than the values on either side of it in a column.
19. a. 

b. Both points correspond to maximum temperatures.

20. a. Maximum volume

b. The maximum occurs at approximately 4.3 grams of leavening and a baking time between 29 and 30 minutes.

c. The maximum volume index appears to be approximately 114. This probably means that the maximum volume possible is 114% of the volume of the cake batter.

21. a,b,c.
b. Land subsidence in the Santa Clara Valley is at a maximum of approximately 5.25 feet north-west of Sunnyvale and north-east of San Jose.

Section 10.2 Multivariable Optimization

1. Find the partial derivatives of $R$, and set them equal to zero.
   
   \[ R_k = 6k - 2m - 20 = 0 \]
   
   \[ R_m = -2k + 6m - 4 = 0 \]

   Solving for $k$ and $m$ gives $k = 4$ and $m = 2$.

   \[ R_{kk} = 6, \quad R_{mm} = 6, \quad R_{km} = R_{mk} = -2 \]

   \[ D(4, 2) = \begin{vmatrix} 6 & -2 \\ -2 & 6 \end{vmatrix} = 36 - 4 = 32 > 0 \]

   \[ R_{kk}(4, 2) = 6 > 0 \]

   Because $D > 0$ and $R_{kk} > 0$, we know the critical point corresponds to a minimum. We conclude that a relative minimum of $R(4, 2) = 16$ is located at $k = 4$ and $m = 2$.

2. Find the partial derivatives of $H$, and set them equal to zero.
   
   \[ H_r = s + 2r = 0 \]
   
   \[ H_s = r + 4s = 0 \]

   Solving for $r$ and $s$ gives $r = 0$ and $s = 0$.

   \[ H_{rr} = 2, \quad H_{ss} = 4, \quad H_{rs} = H_{sr} = 1 \]

   \[ D(0, 0) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7 > 0 \]

   \[ H_{rr}(0, 0) = 2 > 0 \]

   Because $D > 0$ and $H_{rr} > 0$, we know the critical point corresponds to a minimum. We conclude that a relative minimum of $H(0, 0) = 0$ is located at $r = 0$ and $s = 0$.

3. Find the partial derivatives of $G$, and set them equal to zero.
   
   \[ G_t = pe^t = 0 \]
\[ G_p = e^t - 3 = 0 \]

To solve for \( t \) and \( p \), note that in order for \( pe^t = 0 \), either \( p = 0 \) or \( e^t = 0 \). Because \( e^t \) can never be zero, we conclude that \( p = 0 \). The second equation gives \( e^t = 3 \) or \( t = \ln 3 \approx 1.099 \).

\[ G_{tt} = pe^t, \quad G_{pp} = 0, \quad G_{tp} = G_{pt} = e^t \]

At \( p = 0 \) and \( t = \ln 3 \),

\[ G_{tt} = 0, \quad G_{pp} = 0, \quad G_{tp} = G_{pt} = e^{\ln 3} = 3 \]

\[ D(\ln 3, 0) = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0 \]

Because \( D < 0 \), we know the point is a saddle point. The output at that point is \( G(\ln 3, 0) = 0 \). A saddle point is located at \((\ln 3, 0)\).

4. Find the partial derivatives of \( f \) and set them equal to zero.

\[ f_a = 2a - 4 = 0 \]
\[ f_b = 2b - 2 = 0 \]

Solving for \( a \) and \( b \) gives \( a = 2 \) and \( b = 1 \).

\[ f_{aa} = 2, \quad f_{bb} = 2, \quad f_{ab} = f_{ba} = 0 \]

\[ D(2, 1) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \]

\[ f_{aa}(2, 1) = 2 > 0 \]

Because \( D > 0 \) and \( f_{aa} > 0 \), we know the critical point corresponds to a minimum.

A relative minimum of \( f(2, 1) = -17 \) is located at \( a = 2 \) and \( b = 1 \).

5. Find the partial derivatives of \( h \), and set them equal to zero.

\[ h_w = 1.2w - 4.7z = 0 \]
\[ h_z = 3.0z^2 - 4.7w = 0 \]

To solve this system of equations, solve the first equation for \( w \):

\[ w = \frac{4.7z}{1.2} \]

Substitute this expression into the second equation:

\[ 3.0z^2 - 4.7\left(\frac{4.7z}{1.2}\right) = 0 \]

\[ z\left(3.0z - \frac{4.7^2}{1.2}\right) = 0 \]

Solving for \( z \) gives two solutions: \( z = 0 \) and \( z \approx 4.720 \). The two critical points are \( w = 0, z = 0, h = 0 \) and \( w = 18.487, z \approx 4.720, \) and \( h \approx -68.353 \)

The second partials are

\[ h_{ww} = 1.2, \quad h_{zz} = 6.0z \]
\[ h_{zw} = h_{wz} = -4.7 \]
\[ D(0, 0) = \begin{bmatrix} 1.2 & -4.7 \\ -4.7 & 0 \end{bmatrix} = -20.89 < 0, \text{ thus } (0, 0, 0) \text{ is a saddle point.} \]
\[ D(18.487, 4.720) = \begin{bmatrix} 1.2 & -4.7 \\ -4.7 & 28.32 \end{bmatrix} \approx 11.89 > 0 \text{ and } h_{ww} > 0, \text{ thus } (18.487, 4.720, -68.353) \text{ is a relative minimum.} \]

6. Find the partial derivatives of \( R \), and set them equal to zero.
\[ R_s = 3.3s^2 - 5.2s + 0.9 = 0 \]
\[ R_t = -6.2t + 5.3 = 0 \]
Solving for \( s \) and \( t \) gives \( s = 0.1979 \) and \( t = 0.8548 \) or \( s = 1.3778 \) and \( t = 0.8548 \).
\[ R_{ss} = 6.6s - 5.2, \ R_{tt} = -6.2, \ R_{st} = R_{ts} = 0 \]
\[ D(0.1979, 0.8548) = \begin{bmatrix} 1.2021 & 0 \\ 0 & -6.2 \end{bmatrix} < 0 \]
\[ D(1.3778, 0.8548) = \begin{bmatrix} 3.8935 & 0 \\ 0 & -6.2 \end{bmatrix} < 0 \]
Because \( D < 0 \) in both cases, we know that both points are saddle points.
Saddle points are located at about \((0.1979, 0.8548, 8.0583)\) and at about \((1.3778, 0.8548, 7.1549)\).

7. Solving \( f_x = 6x - 3x^2 = 3x(2-x) = 0 \) \text{ and } \( f_y = 24y - 24y^2 = 24y(1-y) = 0 \) \text{ yields } x = 0, x = 2, y = 0, y = 1. Thus we consider 4 points: \((0, 0)\), \((0, 1)\), \((2, 0)\), \((2, 1)\). The second partials are \( f_{xx} = 6 - 6x \), \( f_{yy} = 24 - 48y \), and \( f_{xy} = f_{yx} = 0 \).
For \((0, 0)\), \( D = \begin{bmatrix} 6 & 0 \\ 0 & 24 \end{bmatrix} = 144 > 0 \) \text{ and } \( f_{xx} > 0 \) \text{ thus } \((0, 0, 60)\) \text{ is a relative minimum.}
For \((0, 1)\), \( D = \begin{bmatrix} 6 & 0 \\ 0 & -24 \end{bmatrix} = -144 < 0, \text{ thus } \((0, 1, 64)\) \text{ is a saddle point.}
For \((2, 0)\), \( D = \begin{bmatrix} -6 & 0 \\ 0 & 24 \end{bmatrix} = -144 < 0, \text{ thus } \((2, 0, 64)\) \text{ is a saddle point.}
For \((2, 1)\), \( D = \begin{bmatrix} -6 & 0 \\ 0 & -24 \end{bmatrix} = 144 > 0 \text{ and } f_{xx} < 0, \text{ thus } \((2, 1, 68)\) \text{ is a relative maximum.}

8. Find the partial derivatives of \( g \), and set them equal to zero.
\[ g_x = 4y - 4x^3 = 0 \]
\[ g_y = 4x - 4y^3 = 0 \]
Solving for \( x \) and \( y \) gives 3 solutions:
\[ x = 0 \text{ and } y = 0 \]
\[ x = 1 \text{ and } y = 1 \]
\[ x = -1 \text{ and } y = -1 \]
$g_{xx} = -12x^2, \ g_{yy} = -12y^2, \ g_{xy} = g_{yx} = 4$

$D(0,0) = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$

$D(1,1) = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 128 > 0$

$D(-1,-1) = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 128 > 0$

Because $D(0,0) < 0$, we know that (0, 0, 0) is a saddle point. Because the other two determinants are positive and $g_{xx}$ is negative, we know that $(1, 1, 2)$ and $(-1, -1, 2)$ are relative maximum points.

9. a. Solving $R_b = 14 - 6b - p = 0$ and $R_p = -b - 4p + 12 = 0$ yields $b \approx 1.91$ and $p \approx 2.52$.

The manager should try to buy ground beef at $1.91$ a pound and sausage at $2.52$ a pound.

b. We verify that these inputs give a maximum revenue by finding the determinant of the second partials matrix: $R_{bb} = -6, \ R_{pp} = -4, \ R_{bp} = R_{pb} = -1$

$D = \begin{vmatrix} -6 & -1 \\ -1 & -4 \end{vmatrix} = 23 > 0$ and $R_{bb} < 0$

Thus we have found the prices that result in maximum quarterly revenue.

c. $R(1.91, 2.52) \approx$ $28.5$ thousand

10. a. Solving $P_b = -p - 4b + 120 = 0$ and $P_p = 144 - 6p - b = 0$ yields $b \approx 25.04$ and $p \approx 19.83$.

The nursery should charge $25.04$ per load for bark mulch and $19.83$ per load for pine straw.

b. We verify that these inputs give a maximum revenue by finding the determinant of the second partials matrix: $P_{bb} = -4, \ P_{pp} = -6, \ P_{bp} = P_{pb} = -1$

$D = \begin{vmatrix} -6 & -1 \\ -1 & -4 \end{vmatrix} = 23 > 0$ and $P_{bb} < 0$

Thus we have found the prices that result in maximum profit.

c. $P(19.83, 25.04) \approx$ $2965$

11. a. Find the partial derivatives of $E$, and set them equal to zero.

$E_T = 299.7038 - 10.4420T - 0.4023H = 0$

$E_H = 23.1412 - 0.1874H - 0.4023T = 0$
Solving for \( T \) and \( H \) gives:
\[
T = 26.1032^\circ \text{C} \quad \text{and} \quad H = 67.4488\%.
\]
\( E(26.1032, 67.4488) \approx 500.343 \) eggs The critical point is approximately \((26.1, 67.4, 500)\).

b. When exposed to approximately \( 26.1^\circ \text{C} \) and \( 67.4\% \) relative humidity, a \( C. \) grandis female will lay approximately \( 500 \) eggs in 30 days.
\[
E_{TT} = -10.442, \quad E_{RR} = -1.874, \quad E_{TR} = E_{RT} = -0.4023
\]
\[
D = \begin{bmatrix}
-10.442 & -0.4023 \\
-0.4023 & -1.874
\end{bmatrix}
\]

Because \( E_{TT} < 0 \) and \( D > 0 \), the critical point is a maximum.

12. a. Find the partial derivatives of \( D \), and set them equal to zero.
\[
D_H = -0.838 + 0.0168H - 0.0181L = 0 \quad D_L = 2.4297 - 0.1452L - 0.0181H = 0
\]
Solving for \( H \) and \( L \) gives
\[
H \approx 59.8688 \quad \text{and} \quad L \approx 9.2705.
\]
\( D(59.8688, 9.2705) \approx 11.8463 \)
The critical point is approximately \((59.9, 9.3, 11.8)\).

b. When the relative humidity is approximately \( 59.9\% \) and the eggs are exposed to approximately \( 9.3 \) hours of light each day, a \( C. \) grandis egg will develop into an adult in approximately \( 11.8 \) days.
\[
D_{HH} = 0.0168, \quad D_{LL} = -0.1452
\]
\[
D_{HL} = D_{LH} = -0.0181
\]
\[
D = \begin{bmatrix}
0.0168 & -0.0181 \\
-0.0181 & -0.1452
\end{bmatrix}
\]

The critical point is a saddle point.

13. a. Find the partial derivatives of \( R \), and set them equal to zero.
\[
R_P = -1.544 + 9.810P - 3T = 0
\]
\[
R_T = -1.625 - 14.106T - 3P = 0
\]
Solving for \( P \) and \( T \) gives \( P = -0.1307 \) and \( T = -0.0874 \).
\( R(-0.1307, -0.0874) \approx 32.8 \)
The critical point is approximately \((-0.13, -0.09, 32.8)\).

b. The second partials are
\[
R_{PP} = -9.810, \quad R_{TT} = -14.106, \quad R_{PT} = R_{TP} = -3
\]
\[
D = \begin{bmatrix}
-9.810 & -3 \\
-3 & -14.106
\end{bmatrix}
\]

Because \( D > 0 \) and \( R_{PP} < 0 \), the critical point is a maximum.

To maximize the rate, the \( \text{pH} \) is about \( 5.5 + 1.5(-0.13) = 5.3 \) and the temperature is about \( 60 + 8(-0.09) \approx 59.3^\circ \text{C} \).

14. Find the partial derivatives of \( F \), and set them equal to zero.
\[
F_P = -1.29 - 0.892p - 1.249t = 0
\]
\[
F_t = -3.052 - 7.522t - 1.249p = 0
\]
Solving for \( P \) and \( T \) gives \( P \approx 0.5518 \) and \( T \approx -0.4974 \).
\[ F(0.5518, -0.4974) \approx 25.95 \]
The critical point is approximately \((0.55, -0.50, 25.95)\).

The second partials are
\[
\begin{align*}
F_{pp} &= -0.892, \quad F_{tt} = -7.522 \\
F_{pt} &= F_{tp} = -1.249 \\
D &= \begin{vmatrix}
-0.892 & -1.249 \\
-1.249 & -7.522 \\
\end{vmatrix} = 5.15 > 0
\end{align*}
\]
Because \( F_{pp} < 0 \) and \( D > 0 \), the critical point is a maximum.

To maximize the number of fatty acids per 100 grams of water, the pH is about
\[ 5.5 + 1.5(0.5518) = 6.3 \]
and the temperature is about \[ 60 + 8(-0.4974) = 56.0 \]°C. This is verified above.

15. a. From the graph, the maximum appears to be about 2.35 mg when pH is about 9 and the
    temperature is about 65 °C.

b. Find the partial derivatives of \( P \), and set them equal to zero.
\[
\begin{align*}
P_x &= -0.26 - 0.46x - 0.25y = 0 \\
P_y &= -0.34 - 0.32y - 0.25x = 0
\end{align*}
\]
Solving for \( x \) and \( y \) gives \( x \approx 0.0213 \) and \( y \approx -1.0791 \).
The second partials are \( P_{xx} = -0.46 \), \( P_{yy} = -0.32 \), and \( P_{xy} = P_{yx} = -0.25 \)
\[
D = \begin{vmatrix}
-0.46 & -0.25 \\
-0.25 & -0.32 \\
\end{vmatrix} = 0.21 > 0
\]
Because \( P_{xx} < 0 \) and \( D > 0 \), the critical point is a maximum.

A pH of about \( 9 + 0.0213 \approx 9.02 \) and a temperature of about
\[ 70 + 5(-1.0791) = 64.6 \]°F will result in the maximum production of
\( P(0.0213, -1.0791) \approx 2.32 \) mg.

16. a. From the graph, the maximum appears to be about 2.3 mg when the pH is about 8.3 and the
    time is about 3.5 hours.

b. Find the partial derivatives of \( P \), and set them equal to zero.
\[
\begin{align*}
P_x &= -0.26 - 0.46x - 0.07y = 0 \\
P_y &= 0.04 - 0.08y - 0.07x = 0
\end{align*}
\]
Solving for \( x \) and \( y \) gives \( x \approx -0.7398 \) and \( y \approx 1.1473 \).
\[
P_{xx} = -0.46, \quad P_{yy} = -0.08
\]
The second partials are \( P_{xy} = P_{yx} = -0.07 \)
\[
D = \begin{vmatrix}
-0.46 & -0.07 \\
-0.07 & -0.08 \\
\end{vmatrix} \approx 0.03 > 0
\]
Because \( P_{xx} < 0 \) and \( D > 0 \), the critical point is a maximum.

The maximum amount of peptides is realized when the pH is about
\[ 9 - 0.7398 \approx 8.26 \]
and the processing time is about \( 2.5 + 1.1473 \approx 3.65 \) hours.
17. Find the partial derivatives of \( L \), and set them equal to zero.

\[
L_w = 1.13 - 5.83s = 0
\]
\[
L_s = 1.04 - 5.83w = 0
\]

Solving for \( w \) and \( s \) gives \( w \approx 0.18 \) and \( s \approx 0.19 \). The second partials are
\[
L_{ww} = 0, \ L_{ss} = 0, \ L_{sw} = L_{ws} = -5.83
\]
\[
D = \begin{vmatrix}
0 & -5.83 \\
-5.83 & 0
\end{vmatrix} \approx -34 < 0
\]

Because \( D < 0 \), the critical point is a saddle point. The corresponding proportions are
- whey protein: \( w \approx 0.18 \)
- skim milk powder: \( s \approx 0.19 \)
- sodium caseinate: \( c = 1 - 0.18 - 0.19 = 0.63 \)

18. a. Find the partial derivatives of \( P \), and set them equal to zero.

\[
P_T = -9.6544 + 0.14736T = 0
\]
\[
P_r = 1.9836 - 0.05926r = 0
\]

Solving for \( T \) and \( r \) gives \( T \approx 65.5157 \) and \( r \approx 33.3042 \). The critical point is about \((65.5157, 33.3042, 23.756)\).

b. The second partials are
\[
P_{TT} = 0.14736, \ P_{rr} = -0.05926
\]
\[
P_{Tr} = P_{rt} = 0
\]
\[
D = \begin{vmatrix}
0.14736 & 0 \\
0 & -0.05926
\end{vmatrix} \approx -0.0087 < 0
\]

Because \( D < 0 \), the critical point is a saddle point. Thus the model does not have a relative maximum or minimum.

19. a. Find the partial derivatives of \( E \), and set them equal to zero.

\[
E_e = -30.372e^2 + 42.694e - 13.972 = 0
\]
\[
E_n = -5n + 2.497 = 0
\]

Solving for \( n \) gives \( n \approx 0.4994 \) and solving for \( e \) gives \( e \approx 0.5185 \) or \( e \approx 0.8872 \). Thus there are two critical points:

\[
A = (0.5185, 0.4994, 799.91)
\]
\[
B = (0.8872, 0.4994, 800.16)
\]

b. We use the contour graph to conclude that Point \( A \) is a saddle point and point \( B \) corresponds to a relative maximum.

c. When \( e \approx 0.5185 \) and \( n \approx 0.4994 \),
\[
D = -59.98 \text{ so point } A \text{ is confirmed to be a saddle point. When } e \approx 0.8872 \text{ and } n \approx 0.4994,
\]
\[
D = 59.98 \text{ and } e = -11.20 \text{ so point } B \text{ is confirmed to correspond to relative maximum.}
20. Find the partial derivatives of \( R \), and set them equal to zero.

\[
R_c = -768.772c - 59.128r + 10,299.325 = 0
\]

\[
R_r = -59.128c - 104.392r + 5935.497 = 0
\]

Solving for \( c \) and \( r \) gives \( c \approx 9.4351 \) and \( r \approx 51.5137 \). \( R(0.5185, 0.4994) \approx 64,594 \)

The critical point is

\( (9.4351, 51.5137, 64,594) \)

The second partials are

\[
R_{cc} = -768.772, \quad R_{rr} = -104.392
\]

\[
R_{cr} = R_{rc} = -59.128
\]

\[
D = \begin{vmatrix}
-768.772 & -59.128 \\
-59.128 & -104.392
\end{vmatrix} = 76,758 > 0
\]

Because \( D > 0 \) and \( R_{cc} < 0 \), we know the critical point is a maximum. For maximum revenue, about 9.4 thousand tons of 40% cheese and 51.5 thousand tons of regular cheese should be sold.

21. a. Find the partial derivatives of \( A \), and set them equal to zero.

\[
A_g = 4.26 - 0.1g = 0
\]

\[
A_m = 5.69 - 0.28m - 0.07h = 0
\]

\[
A_s = 0.67 - 0.06s = 0
\]

\[
A_h = 2.48 - 0.1h - 0.07m = 0
\]

Solving this system, we get

\( g \approx 42.6\% \), \( m \approx 17.1169\% \),

\( s \approx 11.1667\% \), \( h \approx 12.8182 \) days

\( A(42.6, 7.1169, 11.1667, 12.8182) = 7.29 \)

b. One method is to evaluate points close to the critical point. By doing so, it is possible to conjecture that the point is a relative maximum.

22. a. Find the partial derivatives of \( V \), and set them equal to zero.

\[
V_g = 1000(-1.757 + 0.282g + 0.186s) = 0
\]

\[
V_s = 1000(-3.436 + 3.145s + 1.217l - 1.066sl + 0.186g) = 0
\]

\[
V_l = 1000(0.375 + 1.217s - 0.533s^2) = 0
\]

Solving \( V_l = 0 \) for \( s \) gives two solutions \( s \approx 2.5583, -0.2750 \).

We discard the negative solution. Substituting \( s \approx 2.5583 \) in \( V_g = 0 \) yields \( g \approx 4.5431 \).

Substituting these two values into \( V_l = 0 \), gives \( l \approx 3.6274 \).

\( V(4.5431, 2.5583, 3.6274) = 4268 \)

The measure of thickness at this critical point is about 4268.

b. Testing other points near the point found in part \( a \) indicates that it is minimum.
23. **One possible answer:** In order to locate a critical point of a three-dimensional function, determine both first partial derivatives, set each of them equal to zero and solve. Any input \((a, b)\) for which both first partial derivatives are zero will yield a critical point.

To determine the type of critical point found, write the four second partial derivatives of the function. Next evaluate the product of the non-mixed second partials minus the product of the mixed partials at \((a, b)\)—this is known as the \(D(a, b)\), the determinant of the second partials matrix. If \(D(a, b)\) is negative, then \((a, b)\) yields a saddle point. If \(D(a, b)\) is positive, then evaluate one of the non-mixed second partials at \((a, b)\). If that second partial is exactly zero, then the determinant test fails and estimation graphically or numerically may help.

24. **One possible answer:** If \(D > 0\), then the point is either a relative maximum or a relative minimum. If \(f_{xx} < 0\), then the cross section with respect to \(x\) is concave down, so the point is a relative maximum in the \(x\) direction. Also, when \(f_{xx} < 0\) and \(D > 0\), then \(f_{xy}\) must also be less than zero. Thus there is also a relative maximum in the \(y\) direction. Because \(D > 0\) guarantees a maximum or minimum, this critical point must be a relative maximum. Similarly, if \(f_{xx} > 0\), then the point is a relative minimum. If \(D < 0\), then the point is a saddle point regardless of whether \(f_{xx}\) and \(f_{yy}\) are positive or negative.

### Section 10.3 Optimization Under Constraints

1. a. The optimal point is \((45, 45, 2025)\). The optimal value is 2025.
   
   b. The point is a constrained maximum.
   
   c. \(f_a = b, \ f_b = a, \ g_a = 1, \ g_b = 1\)
   
   We have the following system of equations:
   
   \[
   b = \lambda(1)
   \]
   
   \[
   a = \lambda(1)
   \]
   
   \[
   a + b = 90
   \]

   Solving this system, we get \(a = 45, \ b = 45, \) and \(\lambda = 45\). Thus \(f(45, 45) = 2025\) is a constrained optimal value.

2. a. The optimal points are \((-1, 1, 2)\) and \((1, 1, 2)\). The optimal value of \(f\) subject to the constraint is 2.
   
   b. The point is a constrained maximum.
   
   c. \(f_k = 4kh, \ f_h = 2k^2, \ g_k = 2k, \ g_h = 1\)
   
   We have the following system of equations:
   
   \[
   4kh = \lambda(2k)
   \]
   
   \[
   2k^2 = \lambda(1)
   \]
   
   \[
   k^2 + h = 2
   \]

   Solving this system, we get \(k = -1, \ h = 1\) and \(\lambda = 2\) or \(k = 1, \ h = 1, \) and \(\lambda = 2\). This verifies part \(a\).
3. a. The constrained minimum is approximately 7, which occurs when \( G \approx 26\% \) and \( M \approx 15\% \).

4. a. The approximate relative maximum point is \((42.5\%, 17.5\%, 7.5)\). The maximum measure of adhesiveness is about 7.5, which occurs when there is 42.5\% glucose and maltose and 17.5\% moisture.

b. The maximum is a little more than 6 and occurs when \( G \approx 38.5\% \) and \( M = 16\% \): \((39, 16, 6.25)\).

5. \( f_r = 4r + p \), \( f_h = r - 2p + 1 \), 
\( g_k = 2 \), \( g_h = 3 \)

We have the following system of equations.

\[
\begin{align*}
4r + p &= \lambda(2) \\
r - 2p + 1 &= \lambda(3) \\
2r + 3p &= 1
\end{align*}
\]

Solving this system we get \( r = \frac{1}{16} \) and \( p = \frac{3}{8} \). \( f \left( \frac{1}{16}, \frac{3}{8} \right) = \frac{7}{32} \)

The constrained optimal point is \( \left( \frac{1}{16}, \frac{3}{8}, \frac{7}{32} \right) \).
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Calculus Concepts

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a. A contour graph confirms that the point is a constrained minimum, because the contour to which the constraint line is tangent is the smallest-valued contour that the constraint line touches.

b. We evaluate $f(r, p)$ for values of $r$ near $\frac{1}{16} = 0.0625$. We choose 0.6 and 0.65 (these are arbitrarily chosen). We find the corresponding $p$-values that lie on the constraint curve $2r + 3p = 1$, but substituting each $r$-value and solving for $p$. Thus we have the input pairs $(0.065, 0.29)$ and $(0.06, 0.2933)$. Evaluating $f$ at these inputs gives the outputs 0.2321 and 0.2332. Both of these values are greater than $f\left(\frac{-1}{16}, \frac{3}{8}\right) = \frac{7}{32} = 0.21875$. These calculations suggest that the point is the location of a constrained minimum.

6. $f_x = 80, f_y = 10y, g_x = 2, g_y = 2$

We have the following system of equations.

\[
80 = \lambda(2)
\]
\[
10y = \lambda(2)
\]
\[
2x + 2y = 1.4
\]

Solving this system we get $x = -7.3, y = 8$, and $\lambda = 40. f(-7.3, 8) = -264$

The constrained optimal point is $(-7.3, 8, -264)$.

a. A contour graph like the one above confirms that the point is a constrained minimum, because the contour to which the constraint line is tangent is the smallest-valued contour that the constraint line touches.

b. Evaluating $f(x, y)$ for values of $x$ near $-7.3$ and $y$ near 8 gives values greater than $-264$, which suggest the point is a constrained minimum.
7. We have the following system of equations (the first two are the same as in Example 2).
1.13 - 5.83s = \lambda
1.04 - 5.83w = \lambda
w + s = 0.9
Solving this system gives \( s \approx 0.4577 \) and \( w \approx 0.4423 \). These values correspond to an output \( P(0.4577, 0.4423) \approx 10.45 \), so the minimum percentage loss is about 10.45%. The approximation in Example 2 was 10.46%.

8. We have the following system of equations (the first two are the same as in Example 1).
\[ 28.86L^{-0.4}K^{0.4} = 8\lambda \]
\[ 119.24L^{0.6}K^{-0.6} = \lambda \]
\[ 8L + K = 99 \]
Solving for \( L \) and \( K \), we get \( L \approx 7.425 \) and \( K \approx 39.6 \).
\[ f(7.425, 39.6) \approx 698 \text{ mattresses} \]
Maximum production is 698 mattresses. That this is a maximum can be verified by examining the contour graph on page 720.

9. a. \( g(r, n) = 12r + 6n = 504 \)
\[ A_r = 0.2rn, \ A_n = 0.1r^2, \ g_r = 12, \ g_n = 6 \]
We have the following system of equations.
\[ 0.2rn = \lambda(12) \]
\[ 0.1r^2 = \lambda(6) \]
\[ 12r + 6n = 504 \]
Solving this system, we get \( r = 28, \ n = 28 \), and \( \lambda = 13.07 \) (or \( r = 0, \ n = 0, \) and \( \lambda = 0 \) which gives 0 responses.) The club should allocate (28 ads)($12 per ad) = $336 for 28 radio ads and (28 ads)($6 per ad) = $168 for 28 newspaper ads.

b. \( A(28, 28) \approx 2195 \text{ responses} \)

c. The Lagrange multiplier is \( \lambda = 13.07 \) responses per dollar. The change in the number of responses can be approximated as
\[ \Delta A \approx (13.07 \text{ responses per dollar})(26) \approx 340 \text{ additional responses.} \]

10. a. We solve the system of equations: \( A_G = 4.26 - 0.1G = 0 \) and \( A_M = 4.85 - 0.28M = 0 \) to obtain \( G = 42.6\% \), \( M = 17.3\% \), and an absolute maximum adhesiveness of \( A(42.6, 17.3) = 7.3 \). Our estimate in Activity 4 part a is 7.5, slightly higher than the actual maximum.

b. \( g(G, M) = G + M = 55 \)
Using the partial derivatives from part a and \( g_G = 1 \) and \( g_M = 1 \), we have the following system of equations:
\[ 4.26 - 0.1G = \lambda(1) \]
\[ 4.85 - 0.28M = \lambda(1) \]
\[ G + M = 55 \]
Solving this system, we obtain
\( G = 39\% \), \( M = 16\% \), and \( \lambda = 0.38 \).
The maximum measure of adhesive possible is about \( A(39, 16) \approx 6.37 \).
c. Figure 10.3.4 verifies that the point is a maximum since the contour to which the constraint line is tangent is the largest-valued contour that the constraint line touches.
d. The estimate in Activity 4 part b (38.5, 16, 6.25) is close to the solution of (39, 16, 6.37) found in part b of this activity.

11. a. We solve the system of equations: 
\[ C_G = -3.76 + 0.08G + 0.06M = 0 \]
\[ C_M = -4.71 + 0.16M + 0.06G = 0 \]
to obtain \( G = 34.7\% \), \( M = 16.4\% \), and an absolute minimum cohesiveness of honey of \( C(34.7, 16.4) = 2.5 \). Our estimate in Activity 3 part a was 2.5, agreeing to one decimal place with the actual minimum.
b. \( g(G, M) = G + M = 40 \)
Using the partial derivatives from part a and \( g_G = 1 \) and \( g_M = 1 \), we have the following system of equations:
\[ -3.76 + 0.08G + 0.06M = \lambda(1) \]
\[ -4.71 + 0.16M + 0.06G = \lambda(1) \]
\[ G + M = 40 \]
Solving this system, we get \( G \approx 25.4 \), \( M \approx 14.6 \), and \( \lambda \approx -0.85 \).
The minimum measure of cohesiveness possible is about \( C(25.4, 14.6) = 7.2 \).
c. Figure 10.3.3 verifies that the point is a minimum since the contour to which the constraint line is tangent is the smallest-valued contour that the constraint line touches.
d. The estimate in Activity 3 part b of 7.5 is slightly higher than 7.2, the actual constrained minimum found in part b of this activity.

12. \( C_l = w - \frac{3.2}{l^2} \), \( C_w = t - \frac{4.8}{w^2} \), \( g_l = 1 \), and \( g_w = 2 \)
We have the following system of equations:
\[ w - \frac{3.2}{l^2} = \lambda(1) \]
\[ l - \frac{4.8}{w^2} = \lambda(2) \]
\[ l + 2w = 5 \]
This system has more than one solution, but only one gives positive length and width. We get \( l = 1.2966 \), \( w = 1.8517 \), and \( \lambda = -0.0517 \). The length is about 1.3 ft and the width is about 1.9 ft. The minimum cost is \( C(1.2966, 1.8517) = $7.46 \).

13. a. Worker expenditure:
\[ \frac{(7.50 / \text{hour})(100L \text{ hours})}{1000} \]
\[ = 0.75L \text{ thousand dollars} \]
The constraint is
\[ g(L, K) = 0.75L + K \]
The partial derivatives are
\[ f_L = 3.16389L^{-0.7}K^{0.5} \]
\[ f_K = 5.27315L^{0.3}K^{-0.5} \]
\[ g_L = 7.5 \text{, } g_K = 1 \]
We solve the following system of equations
\[3.16389L^{-0.7}K^{0.5} = 0.75\lambda\]
\[5.27315L^{0.3}K^{-0.5} = \lambda\]
\[0.75L + K = 15\]
to obtain \(L = 7.5, K = 9.375, \lambda \approx 3.152\).
Maximum production will be achieved by using 750 labor-hours and $9375 in capital expenditures.

b. To verify that the value in part \(a\) is a maximum, evaluate the production at close points on the constraint curve, or examine the constraint curve graphed on a contour graph of the production function.

c. The marginal productivity of money is \(\lambda \approx 3.152\) radios per thousand dollars. An increase in the budget of $1000 will result in an increase in output of about 3 radios.

14. a. From Activity 10, \(\lambda = 0.38\), so \(\frac{dA}{dk} = 0.38\) adhesiveness unit per percentage point.

b. The maximum adhesiveness measure should increase by about 0.38.

c. Find the partial derivatives of \(A\), and set them equal to zero.
\[A_G = 4.26 - 0.1G = 0\]
\[A_M = 4.85 - 0.28M = 0\]
Solving for \(G\) and \(M\) gives \(G = 42.6\) and \(M \approx 17.3214\).
The second partials are
\[A_{GG} = -0.1, A_{MM} = -0.28, A_{GM} = A_{MG} = 0\]
\[D = \begin{vmatrix} -0.1 & 0 \\ 0 & -0.28 \end{vmatrix} = 0.028 > 0\]
Because \(A_{GG} < 0\) and \(D > 0\), the critical point is a maximum.
\(A(42.6, 17.3214) = 7.26\)
The relative maximum when there are no constraints is approximately 7.3, which is obtained when the percentage of glucose and maltose is 42.6% and the percentage of moisture is approximately 17.3%.

15. a. From Activity 11, \(\lambda = -0.85\), so \(\frac{dC}{dk} = -0.85\) cohesiveness unit per percentage point.

b. The minimum cohesiveness measure should decrease by about \((0.85\text{ unit per percentage point}) / (2\text{ percentage points}) = 1.7\).

c. Find the partial derivatives of \(C\), and set them equal to zero.
\[C_G = -3.76 + 0.08G + 0.06M = 0\]
\[C_M = -4.71 + 0.16M + 0.06G = 0\]
Solving for \(G\) and \(M\) gives \(G = 34.6739\) and \(M = 16.4348\).
The second partials are
\[C_{GG} = 0.08, C_{MM} = 0.16, C_{GM} = C_{MG} = 0.06\]
\[ D = \begin{bmatrix} 0.08 & 0.06 \\ 0.06 & 0.16 \end{bmatrix} = 0.0092 > 0 \]

Because \( C_{GG} > 0 \) and \( D > 0 \), the critical point is a minimum.
\( C(34.6739, 16.4348) \approx 3.08 \)
The relative minimum when there are no constraints is approximately 3.1 which is obtained when the percentage of glucose and maltose is approximately 34.7% and the percentage of moisture is approximately 16.4%.

16. a. From Activity 12, \( \lambda \approx -0.0517 \), so \( \frac{dC}{dk} \approx -0.0517 \) dollar per foot.

b. \( \Delta M \approx (-0.0517)(0.5) = -0.03 \)
\( M \approx 7.46 - 0.03 = 7.43 \)

17. a. From Activity 13, \( \lambda \approx 3.152 \), so \( \frac{dP}{dc} \approx 3.152 \) radios/thousand dollars.

b. \( \Delta P \approx (3.152 \text{ radios per thousand dollars})(1.5 \text{ thousand dollars}) \)
\( \approx 4.7 \) radios

c. \( \Delta P \approx (3.152 \text{ radios per thousand dollars})(-1 \text{ thousand dollars}) \)
\( \approx -3.2 \) radios
\( P(7.5, 9.375) = 30 \) radios
We estimate the maximum production to be about 30 – 3 = 27 radios.

18. a. \( A(l, w) = lw \) square feet where \( l \) is the length in feet and \( w \) is the width in feet

b. \( g(w, l) = 2w + 2l - 6 = 200 \) or
\( g(w, l) = 2w + 2l = 206 \)

c. \( A_l = w, A_w = l, g_l = 2, g_w = 2 \)

We solve the following system of equations:
\begin{align*}
w &= 2\lambda \\
l &= 2\lambda \\
2W + 2l &= 206
\end{align*}
which yields \( w = l = 51.5 \) feet and \( A(51.5, 51.5) = 2652.25 \) square feet.

19. a. \( S(r, h) = 2\pi rh + \pi r^2 + \pi \left(r + \frac{9}{8}\right)^2 \)

square inches when the radius is \( r \) inches and the height is \( h \) inches

b. \( V(r, h) = \pi r^2 h = 808.5 \) cubic inches

c. We solve the equations
\begin{align*}
2\pi h + 2\pi r + 2\pi \left(r + \frac{9}{8}\right) &= \lambda 2\pi rh \\
2\pi r &= \lambda \pi r^2 \\
\pi r^2 h &= 808.5
\end{align*}
by isolating \( \lambda \) in the second equation and \( h \) in the third equation to obtain
20. a. \( R(s, p) = (50 + s)p \) dollars where \( s \) is the number of students in excess of 50 and \( p \) is the price per student

b. \( p = 1200 - 10s \) or 
\( g(s, p) = p + 10s = 1200 \)

c. \( R_s = p, \ R_p = 50 + s, \ g_s = 10, \ g_p = 1 \)

We solve the following system of equations:
\[
\begin{align*}
\lambda &= \frac{2\pi r}{\pi r^2} = \frac{2}{r} \quad \text{and} \quad h = \frac{808.5}{\pi r^2}.
\end{align*}
\]
Substituting these expressions into the first equation gives
\[
\frac{1617}{r^2} + 4\pi r + \frac{9\pi}{4} = \frac{3234}{r^2}
\]
Solving for \( r \) gives \( r \approx 4.87 \) inches, \( h = \frac{808.5}{\pi(4.87^2)} = 10.86 \) inches, and 
\( S(r, h) \approx 519.5 \) square inches.

21. The condition \( \frac{f_x}{g_s} = \frac{f_x}{g_y} \) is equivalent to guaranteeing that the slope of the extreme-contour curve is the same as the slope of the constraint curve at their point of intersection.

22. It is important that the estimated point is actually on both the constraint curve and the extreme-contour curve.

Section 10.4 Least-Squares Optimization

1. a. \( f(a, b) = (7 - a - b)^2 + (11 - 6a - b)^2 + (19 - 12a - b)^2 \)

b. \( \frac{\partial f}{\partial a} = 2(7 - a - b)(-1) + 2(11 - 6a - b)(-6) + 2(19 - 12a - b)(-12) = 362a + 38b - 602 \)
\( \frac{\partial f}{\partial b} = 2(7 - a - b)(-1) + 2(11 - 6a - b)(-1) + 2(19 - 12a - b)(-1) = 38a + 6b - 74 \)
The second partials are \( f_{aa} = 362, \ f_{bb} = 6, \ f_{ab} = f_{ba} = 38 \)
\( D = \begin{vmatrix} 362 & 38 \\ 38 & 6 \end{vmatrix} = 728 \)
c. Set $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$, and solve the resulting system of equations.

The solution is $a = 1.0989$ and $b = 5.3736$ corresponding to an output of $f(1.0989, 5.3736) \approx 1.4066$. This is a minimum because $D > 0$ and $f_{aa} > 0$.

d. The linear model that best fits the data is $y = 1.099x + 5.374$.

2. a. $f(a, b) = (5 - 2a - b)^2 + (7 - 3a - b)^2 + (11 - 6a - b)^2 + (15 - 8a - b)^2$

b. $\frac{\partial f}{\partial a} = 2(5 - 2a - b)(-2) + 2(7 - 3a - b)(-3) + 2(11 - 6a - b)(-6) + 2(15 - 8a - b)(-8)
= 226a + 38b - 434$

$\frac{\partial f}{\partial b} = 2(5 - 2a - b)(-1) + 2(7 - 3a - b)(-1) + 2(11 - 6a - b)(-1) + 2(15 - 8a - b)(-1)
= 38a + 8b - 76$

The second partials are $f_{aa} = 226$, $f_{bb} = 8$, $f_{ab} = f_{ba} = 38$.

$D = \begin{vmatrix} 226 & 38 \\ 38 & 8 \end{vmatrix} = 364$

c. Set $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$, and solve the resulting system of equations.

The solution is $a = \frac{146}{91} = 1.6044$ and $b = \frac{171}{91} = 1.8791$, corresponding to an output of $f(1.6044, 1.8791) \approx 0.4396$. This is a minimum because $D > 0$ and $f_{aa} > 0$.

d. The linear model that best fits the data is $y = 1.604x + 1.879$.

3. a. $f(a, b) = (3 - b)^2 + (2 - 10a - b)^2 + (1 - 20a - b)^2$

b. $\frac{\partial f}{\partial a} = 2(2 - 10a - b)(-10) + 2(1 - 20a - b)(-20) = 1000a + 60b - 80$

$\frac{\partial f}{\partial a} = 2(3 - b)(-1) + 2(2 - 10a - b)(-1) + 2(1 - 20a - b)(-2) = 60a + 6b - 12$

The second partials are $f_{aa} = 1000$, $f_{bb} = 6$, $f_{ab} = f_{ba} = 60$.

$D = \begin{vmatrix} 1000 & 60 \\ 60 & 6 \end{vmatrix} = 2400$

Set $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$, and solve the resulting system of equations.

The solution is $a = -0.1$ and $b = 3$, corresponding to an output of $f(-0.1, 3) = 0$.

This is a minimum because $D > 0$ and $f_{aa} > 0$.

Because the minimum SSE is zero, we know that the line of best fit is a perfect line—that is, all of the data points lie on the line.

c. The linear model that best fits the data is $y = -0.1x + 3$ percent, where $x$ is the number of years since 1970.

d. Answers will vary depending on the year.
4. a. \[ f(a, b) = (46 - b)^2 + (60 - 10a - b)^2 + (72 - 20a - b)^2 \]

b. \[ \frac{\partial f}{\partial a} = 2(60 - 10a - b)(-10) + 2(72 - 20a - b)(-20) = 1000a + 60b - 4080 \]
\[ \frac{\partial f}{\partial b} = 2(46 - b)(-1) + 2(60 - 10a - b)(-1) + 2(72 - 20a - b)(-1) = 60a + 6b - 356 \]
The second partials are \( f_{aa} = 1000 \), \( f_{bb} = 6 \), \( f_{ab} = f_{ba} = 60 \).

\[ D = \begin{vmatrix} 1000 & 60 \\ 60 & 6 \end{vmatrix} = 2400 \]

Set \( \frac{\partial f}{\partial a} = 0 \) and \( \frac{\partial f}{\partial b} = 0 \), and solve the resulting system of equations.

The solution is \( a = 1.3 \) and \( b = \frac{139}{3} = 46.3333 \), corresponding to an output of \( f(1.3, 46.3333) \approx 0.6667 \). This is a minimum because \( D > 0 \) and \( f_{aa} > 0 \).

c. The linear model that best fits the data is \( y = 1.3x + 46.333 \) percent, where \( x \) is the number of years since 1970.

d. When \( x = 30 \), \( y = 85.3 \) percent.

e. Using technology we find the linear model that best fits the data is \( y = 1.07x + 49.4 \) percent, where \( x \) is the number of years since 1970.

When \( x = 30 \), \( y = 81.5 \) percent

f. Both estimates were over-estimates. The estimate in part \( d \) was 7.3 percentage points too high. The estimate in part \( e \) was 3.5 percentage points too high.

g. Discussion will vary.

5. a. \[
\begin{array}{c|c|c}
\hline
x & y & \hline
0 & 3 & 0 \\
1 & 6 & 3 \\
2 & 9 & 6 \\
3 & 12 & 9 \\
4 & 15 & 12 \\
5 & 18 & 15 \\
\hline
\end{array}
\]

b. Using technology, \( y = 1.176x + 1.880 \) dollars to make \( x \) cases of ball bearings. The vertical intercept is the fixed cost per case. The slope is the cost to produce one case.
c. \begin{array}{|c|c|c|c|c|}
\hline
x & y(x) & Data value & Deviation: \text{data} - y(x) & Squared deviations \\
\hline
1 & 3.0561 & 3.10 & 0.044 & 0.00192 \\
2 & 4.2324 & 4.25 & 0.018 & 0.00031 \\
6 & 8.9375 & 8.95 & 0.013 & 0.00016 \\
9 & 12.4663 & 12.29 & -0.176 & 0.03108 \\
14 & 18.3477 & 18.45 & 0.102 & 0.01047 \\
\hline
\end{array}

\text{Sum of squared deviations} \approx 0.044

d. To find the best-fitting line, first construct the function \( f \) with inputs \( a \) and \( b \), which represents the sum of the squared errors of the data points from the line \( y = ax + b \). Find the partial derivatives of \( f \) with respect to \( a \) and \( b \). Simplify the partials, and find the point \((a, b)\) where the partials are simultaneously zero. These are the coefficients of the model given in part \( b \). The function \( f \) evaluated at \((a, b)\) gives the value of \( \text{SSE} \) shown in part \( c \).

6. a. \( f(a, b) = (5.50 - a - b)^2 + (3.25 - 2a - b)^2 + (1.35 - 3a - b)^2 \)

b. \( \frac{\partial f}{\partial a} = 2(5.50 - a - b)(-1) + 2(3.25 - 2a - b)(-2) + 2(1.35 - 3a - b)(-3) = 28a + 12b - 32.1 \)

\( \frac{\partial f}{\partial b} = 2(5.50 - a - b)(-1) + 2(3.25 - 2a - b)(-1) + 2(1.35 - 3a - b)(-1) = 12a + 6b - 20.2 \)

The second partials are \( f_{aa} = 28 \), \( f_{bb} = 6 \), \( f_{ab} = f_{ba} = 12 \).

\[ D = \begin{vmatrix}
28 & 12 \\
12 & 6
\end{vmatrix} = 24 \]

Set \( \frac{\partial f}{\partial a} = 0 \) and \( \frac{\partial f}{\partial b} = 0 \), and solve the resulting system of equations.

The solution is \( a = 2.075 \) and \( b = 7.517 \), corresponding to an output of \( f(2.075, 7.517) = 0.0204 \). This is a minimum because \( D > 0 \) and \( f_{aa} > 0 \).

C. The linear model that best fits the data is \( y = 2.075x + 7.517 \) inches, where \( x \) is the month.

7. \( f(a, b) = (55 - b)^2 + (5 - 5a - b)^2 + (4.8 - 8a - b)^2 + (4.6 - 10a - b)^2 \)

\( \frac{\partial f}{\partial a} = 2(55 - 5a - b)(-5) + 2(4.8 - 8a - b)(-8) + 2(4.6 - 10a - b)(-10) = 378a + 46b - 218.8 \)

\( \frac{\partial f}{\partial b} = 2(55 - 5a - b)(-1) + 2(4.8 - 8a - b)(-1) + 2(4.6 - 10a - b)(-1) = 46a + 8b - 39.8 \)

Set \( \frac{\partial f}{\partial a} = 0 \) and \( \frac{\partial f}{\partial b} = 0 \), and solve the resulting system of equations. The solution is \( a \approx -0.0885 \) and \( b \approx 5.4841 \).

The linear model that best fits the data is \( y = -0.0885x + 5.4841 \) million experiments, where \( x \) is the number of years since 1970.
8. \( f(a,b) = (24 - 30a - b)^2 + (10 - 38a - b)^2 + (5 - 45a - b)^2 + (0.5 - 50a - b)^2 \)

\[
\begin{align*}
\frac{\partial f}{\partial a} &= 2(24 - 30a - b)(-30) + 2(10 - 38a - b)(-38) + 2(5 - 45a - b)(-45) + \\
&
\quad 2(0.5 - 50a - b)(-50) = 13,738a + 326b - 2700 \quad (1) \\
\frac{\partial f}{\partial b} &= 2(24 - 30a - b)(-1) + 2(10 - 38a - b)(-1) + 2(5 - 45a - b)(-1) + 2(0.5 - 50a - b)(-1) \\
&= 326a + 8b - 78 \quad (2)
\end{align*}
\]

Set \( \frac{\partial f}{\partial a} = 0 \) and \( \frac{\partial f}{\partial b} = 0 \), and solve the resulting system of equations.

The solution is \( a \approx -1.145 \) and \( b \approx 56.533 \).

The linear model that best fits the data is \( y = -1.145x + 56.533 \) day where \( x \) is the temperature in °F.

9. a. The plot of the data points appears to be concave up.

b. Plot the data points \((0, \ln 1.1)\), \((80, \ln 2.0)\), \((125, \ln 4.0)\), and \((163, \ln 8.0)\). The plot of the data points also appears to be concave up, but less so than the plot in part a.

c. Using the least-squares technique, we begin with the function describing the sum of the squared errors:

\[
\begin{align*}
f(a,b) &= (\ln 1.1 - b)^2 + (\ln 2.0 - 80a - b)^2 + (\ln 4.0 - 125a - b)^2 + (\ln 8.0 - 163a - b)^2 \quad (3)
\end{align*}
\]

Next we find the first partial derivatives and set them equal to zero:

\[
\begin{align*}
\frac{\partial f}{\partial a} &= 2(\ln 2.0 - 80a - b)(-80) + 2(\ln 4.0 - 125a - b)(-125) + 2(\ln 8.0 - 163a - b)(-163) \\
&= 97,188a + 736b - (160\ln 2 + 250\ln 4 + 326\ln 8) = 0
\end{align*}
\]
The solution to this system of linear equations is \( a = 0.012 \) and \( b = -0.037 \).

\[
y = 0.012x - 0.037 \text{ whose output is the natural log of the population in billions } x \text{ years after 1850}
\]

d. The plot of the data points appears to be concave up.

b. Plot the data points (1, ln 11), (2, ln 7), (3, ln 5), and (5, ln 2).

The plot of these data points appears to be linear.

c. We begin with the function describing the sum of the squared errors:

\[
f(a,b) = (\ln 11 - a - b)^2 + (\ln 7 - 2a - b)^2 + (\ln 5 - 3a - b)^2 + (\ln 3 - 4a - b)^2 + (\ln 2 - 5a - b)^2
\]

Next we find the first partial derivatives and set them equal to zero:

\[
\frac{\partial f}{\partial a} = 2(\ln 1.1 - b)(-1) + 2(\ln 2.0 - 80a - b)(-1) + 2(\ln 4.0 - 125a - b)(-1) + 2(\ln 8.0 - 163a - b)(-1)
\]

\[
= 736a + 8b - 2(\ln 1.1 + \ln 2 + \ln 4 + \ln 8) = 0
\]

Using technology, we get

\[
y = 0.964(1.012^x) \text{ billion people } x \text{ years after 1850. This confirms our result in part } d \text{ because}
\]

\[
0.964 \approx e^{-0.037} \text{ and } 1.012 \approx e^{0.012}.
\]
\[
\frac{\partial f}{\partial a} = 2(\ln 11 - a - b)(-1) + 2(\ln 7 - 2a - b)(-2) + 2(\ln 5 - 3a - b)(-3) + \\
2(\ln 3 - 4a - b)(-4) + 2(\ln 2 - 5a - b)(-5) \\
= 110a + 30b - 2(\ln 11 + 2\ln 7 + 3\ln 5 + 4\ln 3 + 5\ln 2) = 0 \\
\frac{\partial f}{\partial b} = 2(\ln 11 - a - b)(-1) + 2(\ln 7 - 2a - b)(-1) + 2(\ln 5 - 3a - b)(-1) + \\
2(\ln 3 - 4a - b)(-1) + 2(\ln 2 - 5a - b)(-1) \\
= 30a + 10b - 2(\ln 11 + \ln 7 + \ln 5 + \ln 3 + \ln 2) = 0 \\
\]

The solution to this system of linear equations is \( a \approx -0.426 \) and \( b \approx 2.826 \).

\( y = -0.426x + 2.826 \), the natural log of infants born in the \( x \)th generation of a family.

d. \( y = e^{(-0.426x + 2.826)} \) infants born in the \( x \)th generation of a family.

e. Using technology, we get \( y = 16.787(0.6533^x) \) infants born in the \( x \)th generation of a family. This confirms our result in part d because \( 16.787 \approx 2.826 \) and \( 0.6533 \approx e^{-0.426} \).

11. \textit{One possible answer:} A single large outlier will not have as profound an influence on the overall fit of the line if absolute errors are used as it would if squared errors are used. However, algebraically simplifying (and solving) the sum of absolute error expressions is much more complicated than simplifying the sum of squared algebraic expressions.
Chapter 10 Concept Review

1. a. Highest ozone level:
   Approximately 450 thousandths of a centimeter at 90°N in mid-March.

   Lowest ozone level:
   Approximately 250 thousandths of a centimeter at or just north of 0° (the equator) between October and March.

2. a. | Blanching temperature (°C) | Blanching time (minutes) |
     |-------------------------------|--------------------------|
     | 50                            | 2, 4.2, 6.8, 4.2         |
     | 60                            | 4.5, 4.7, 7.1, 8.6       |
     | 70                            | 8.2, 11.4, 10, 8.6       |
     | 80                            | 7.3, 6.9, 3.9           |

   b. The maximum crispness is approximately 11.4, occurring for a blanching time of 15 minutes at 70°C.

   c. Find the partial derivatives of $C$ and set them equal to zero.
      
      $C_x = -10.4x + 299.7 - 0.4y = 0$
      $C_y = -0.2y + 23.1 - 0.4x = 0$

      Solving for $x$ and $y$ gives $x = 26.4$ minutes and $y = 62.7$ °C.
      The crispness index is $C(26.4, 62.7) = 10$.

   d. $C_{xx} = -10.4, C_{yy} = -0.2, C_{xy} = C_{yx} = -0.4$

      $D = \begin{vmatrix} -10.4 & -0.4 \\ -0.4 & -0.2 \end{vmatrix} = 1.92 > 0$ and $C_{xx} < 0$, so the point is a maximum. Because the point corresponds to the only critical value and the function decreases in every direction away from the maximum, it is an absolute maximum.

3. a. The maximum profit is approximately $202,500 for about 52,000 shirts and 9000 hats sold.
b.  \( 4s + 1.25h = 150 \)

c. The constrained maximum revenue is approximately $185,000 for about 34,000 shirts and 10,000 hats sold.

4. a. 

\[ -104.4s + 5935.5 - 59.1h = 4\lambda \]
\[ -59.1s + 10,299.3 - 768.6h = 1.25\lambda \]
\[ 4s + 1.25h = 150 \]

b. Solving this system of linear equations, we get \( s \approx 34.3611, h = 10.0446, \) and \( \lambda \approx 438.64. \) The corresponding profit is about \( P(34.3611, 10.0446) \approx $186,599. \)

c. \( \lambda \approx $438.64 \) of profit per thousand dollars spent, which means that for each additional thousand dollars budgeted, profit will increase by approximately $439.

d. The change in budget is 2 thousand dollars, so expect profit to increase by about \( ($438.64 \text{ of profit per thousand dollars budgeted})(2 \text{ thousand}) = $877. \)

5. a. 

\[ f(a, b) = (29.9 - 10a - b)^2 + (33.4 - 15a - b)^2 + (37.5 - 20a - b)^2 \]

b. 

\[ \frac{\partial f}{\partial a} = 2(29.9 - 10a - b)(-10) + 2(33.4 - 15a - b)(-15) + 2(37.5 - 20a - b)(-20) \]
\[ = 1450a + 90b - 3100 \]

\[ \frac{\partial f}{\partial b} = 2(29.9 - 10a - b)(-1) + 2(33.4 - 15a - b)(-1) + 2(37.5 - 20a - b)(-1) \]
\[ = 90a + 6b - 2016 \]

Set \( \frac{\partial f}{\partial a} = 0 \) and \( \frac{\partial f}{\partial b} = 0, \) and solve the resulting system of equations. The solution is \( a = 0.76 \) and \( b = 22.2, \) corresponding to \( f(0.76, 22.2) = 0.06. \)

\[ f_{aa} = 1450, \quad f_{bb} = 6, \quad f_{ab} = f_{ba} = 90 \]

\[ D = \begin{vmatrix} 1450 & 90 \\ 90 & 6 \end{vmatrix} = 600 \]

This is a minimum because \( D > 0 \) and \( f_{aa} > 0. \)

c. The linear model that best fits the data is \( y = 0.76x + 22.2 \) kilograms, where \( x \) is the body temperature in °C. The sum of the squared deviations from this line is 0.06, and this is the smallest possible sum.