8 COMPLEX VECTOR SPACES

Charles Hermite

1822 – 1901

Charles Hermite was born on Christmas Eve in Dieuze, France—the sixth of seven children. His father was a cloth merchant who had studied engineering. Although much of his early education occurred at home, he went to Paris to study when he was eighteen.

At the age of 20, Hermite published a paper titled *Considerations on the Algebraic Solution of the Equation of the Fifth Degree*. (Later in his life, Hermite showed how to solve a general fifth-degree equation by means of elliptic modular functions—such equations cannot be solved generally by algebraic means.) That same year, Hermite was admitted to the École Polytechnique, but he was dismissed from the school after one year. While at the Polytechnique, Hermite became acquainted with Joseph Liouville, who in turn introduced Hermite to Carl Jacobi. Hermite's correspondence with Jacobi reveals Hermite's early understanding of abstract mathematics.

In 1848, Hermite gained a position at the Polytechnique (the same institution that had dismissed him five years earlier). With this appointment, Hermite's career finally began to take shape. In 1856 he was appointed to the Academy of Sciences, in 1869 to the École Normale, and finally in 1870 to a professorship at the Sorbonne, a position he held until his retirement in 1897.

Hermite is best known for his work with elliptic functions, for his proof that e is a transcendental number, and for his introduction of what are today called Hermite polynomials. Hermite was the first to use the term *orthogonal matrices*, and was the first to prove that if a matrix is equal to its own conjugate transpose, then its eigenvalues must be real.

8.1 COMPLEX NUMBERS

Thus far in the text, the scalar quantities we have used have been *real numbers*. In this chapter, we expand the set of scalars to include **complex numbers.**

In algebra we often need to solve quadratic equations such as $x^2 - 3x + 2 = 0$. The general quadratic equation is $ax^2 + bx + c = 0$ and its solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the quantity in the radical $b^2 - 4ac$ is called the **discriminant.** If $b^2 - 4ac \geq 0$, then the solutions are ordinary real numbers. But what can we conclude about the solutions of a quadratic equation whose discriminant is negative? For example, the equation

$x^2 + 4 = 0$ has a discriminant of $b^2 - 4ac = -16$. From our experience with ordinary algebra, it is clear that there is no real number whose square is -16. However, by writing

$$\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16}\sqrt{-1} = 4\sqrt{-1}$$

we see that the essence of the problem is that there is no real number whose square is -1. To solve the problem, mathematicians invented the **imaginary unit *i*,** which has the property that $i^2 = -1$. In terms of this imaginary unit, we can write

$$\sqrt{-16} = 4\sqrt{-1} = 4i.$$

We define the imaginary unit i as follows.

Definition of Imaginary Unit *i*

The number i is called the **imaginary unit** and is defined by

$$i = \sqrt{-1}$$

where $i^2 = -1$.

REMARK: When working with products involving square roots of negative numbers, be sure to convert to a multiple of i *before* multiplying. For instance, consider the following.

$$\sqrt{-1}\sqrt{-1} = i \cdot i = i^2 = -1 \qquad \textbf{Correct}$$

$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \qquad \textbf{Incorrect}$$

With this single addition to the real number system, we can develop the system of **complex numbers.**

Definition of a Complex Number

If a and b are real numbers, then the number

$$a + bi$$

is a **complex number,** where a is the **real part** and bi is the **imaginary part** of the number. The form $a + bi$ is the **standard form** of a complex number.

Some examples of complex numbers written in standard form are $2 = 2 + 0i$, $4 + 3i$, and $-6i = 0 - 6i$. The set of real numbers is a subset of the set of complex numbers. To see this, note that every real number a can be written as a complex number using $b = 0$. That is, for every real number,

$$a = a + 0i.$$

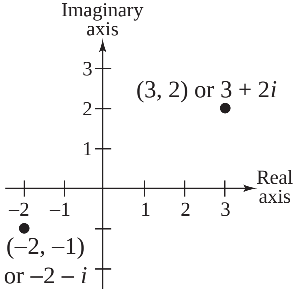
A complex number is uniquely determined by its real and imaginary parts. Thus, we say that two complex numbers are equal if and only if their real and imaginary parts are equal. That is, if $a + bi$ and $c + di$ are two complex numbers written in standard form, then

$$a + bi = c + di$$

if and only if $a = c$ and $b = d$.

The Complex Plane

Figure 8.1

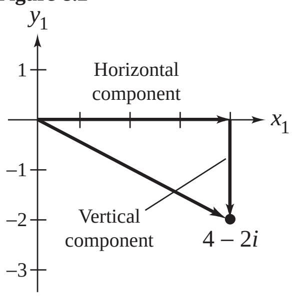


The Complex Plane

Since a complex number is uniquely determined by its real and imaginary parts, it is natural to associate the number $a + bi$ with the ordered pair (a, b). With this association, we can graphically represent complex numbers as points in a coordinate plane that we call the **complex plane.** This plane is an adaptation of the rectangular coordinate plane. Specifically, we call the horizontal axis the **real axis** and the vertical axis the **imaginary axis.** For instance, Figure 8.1 shows the graph of two complex numbers, $3 + 2i$ and $-2 - i$. The number $3 + 2i$ is associated with the point $(3, 2)$ and the number $-2 - i$ is associated with the point $(-2, -1)$.

Another way to represent the complex number $a + bi$ is as a vector whose *horizontal* component is a and *vertical* component is b. (See Figure 8.2.) (Note that the use of the letter i to represent the imaginary unit is unrelated to the use of $\mathbf{i}$ to represent a unit vector.)

Figure 8.2



Vector Representation of a Complex Number

Addition and Scalar Multiplication of Complex Numbers

Since a complex number consists of a real part added to a multiple of i, we define the operations of addition and multiplication in a manner consistent with the rules for operating with real numbers. For instance, to add (or subtract) two complex numbers, we add (or subtract) the real and imaginary parts separately.

Definition of Addition and Subtraction of Complex Numbers

The **sum** and **difference** of $a + bi$ and $c + di$ are defined as follows.

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad \textbf{Sum}$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i \qquad \textbf{Difference}$$

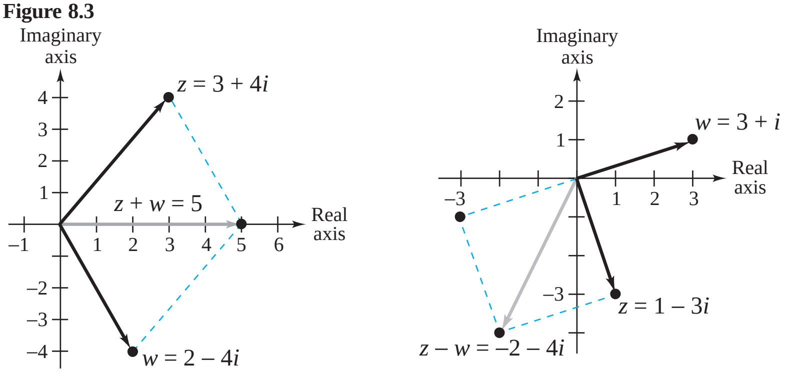
EXAMPLE 1 ***Adding and Subtracting Complex Numbers***

(a) $(2 - 4i) + (3 + 4i) = (2 + 3) + (-4 + 4)i = 5$
(b) $(1 - 3i) - (3 + i) = (1 - 3) + (-3 - 1)i = -2 - 4i$

REMARK: Note in part (a) of Example 1 that the sum of two complex numbers can be a real number.

Using the vector representation of complex numbers, we can add or subtract two complex numbers geometrically using the parallelogram rule for vector addition, as shown in Figure 8.3.

Figure 8.3



Addition of Complex Numbers

Subtraction of Complex Numbers

Many of the properties of addition of real numbers are valid for complex numbers as well. For instance, addition of complex numbers is both *associative* and *commutative*. Moreover, to find the sum of three or more complex numbers, we extend the definition of addition in the natural way. For example,

$$\begin{aligned}(2 + i) + (3 - 2i) + (-2 + 4i) &= (2 + 3 - 2) + (1 - 2 + 4)i \\ &= 3 + 3i.\end{aligned}$$

To multiply a complex number by a *real* scalar, we use the following definition.

Definition of Scalar Multiplication

If c is a real number and $a + bi$ is a complex number, then the **scalar multiple** of c and $a + bi$ is defined to be

$$c(a + bi) = ca + cbi.$$

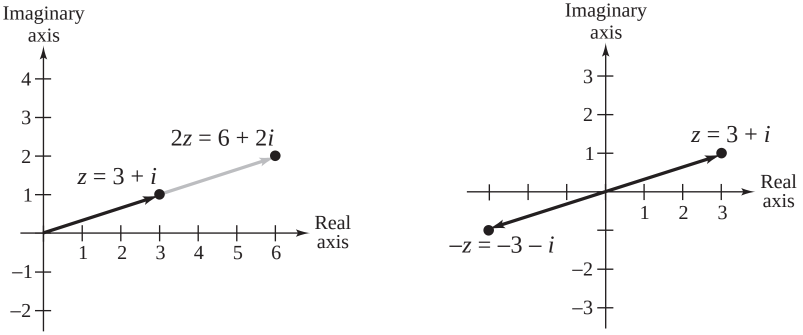
Geometrically, multiplication of a complex number by a real scalar corresponds to the multiplication of a vector by a scalar, as shown in Figure 8.4.

EXAMPLE 2 ***Operations with Complex Numbers***

(a) $$\begin{aligned}3(2 + 7i) + 4(8 - i) &= 6 + 21i + 32 - 4i \\ &= 38 + 17i\end{aligned}$$

(b) $$\begin{aligned}-4(1 + i) + 2(3 - i) - 3(1 - 4i) &= -4 - 4i + 6 - 2i - 3 + 12i \\ &= -1 + 6i\end{aligned}$$

Figure 8.4



Multiplication of a Complex Number by a Real Number

With addition and scalar multiplication, the set of complex numbers forms a vector space of dimension two (where the scalars are the real numbers). You are asked to verify this in Exercise 55.

Multiplication of Complex Numbers

The operations of addition, subtraction, and multiplication by a real number have exact counterparts with the corresponding vector operations. By contrast, there is no direct counterpart for the multiplication of two complex numbers.

Definition of Multiplication of Complex Numbers

The **product** of the complex numbers $a + bi$ and $c + di$ is defined to be

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Rather than try to memorize this definition of the product of two complex numbers, you should simply apply the distributive law as follows.

$$\begin{aligned}
(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive law} \\
&= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive law} \\
&= ac + (ad)i + (bc)i + (bd)(-1) && \text{Use } i^2 = -1 \\
&= ac - bd + (ad)i + (bc)i && \text{Commutative law} \\
&= (ac - bd) + (ad + bc)i && \text{Distributive law}
\end{aligned}$$

This is demonstrated in the following example.

EXAMPLE 3 ***Multiplying Complex Numbers***

(a) $(-2)(1 - 3i) = -2 + 6i$

(b) $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2 = 8 + 6i - 4i - 3(-1)$

$= 8 + 3 + 6i - 4i = 11 + 2i$

TECHNOLOGY NOTE

Many computers and graphing calculators are capable of calculating with complex numbers. For example, on the TI-85 or the HP 48G, you express a complex number $a + bi$ as an ordered pair (a, b). Try verifying the result of Example 3(b) by multiplying $(2, -1)$ and $(4, 3)$. You should obtain the ordered pair $(11, 2)$.

EXAMPLE 4 ***Complex Zeros of a Polynomial***

Use the Quadratic Formula to find the zeros of the polynomial

$$p(x) = x^2 - 6x + 13$$

and verify that $p(x) = 0$ for each zero.

Solution Using the Quadratic Formula, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 52}}{2} \\ &= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i. \end{aligned}$$

Substituting these values of x into the polynomial $p(x)$, we have

$$\begin{aligned} p(3 + 2i) &= (3 + 2i)^2 - 6(3 + 2i) + 13 \\ &= 9 + 12i - 4 - 18 - 12i + 13 = 0 \end{aligned}$$

and

$$\begin{aligned} p(3 - 2i) &= (3 - 2i)^2 - 6(3 - 2i) + 13 \\ &= 9 - 12i - 4 - 18 + 12i + 13 = 0. \end{aligned}$$

In Example 4, the two complex numbers $3 + 2i$ and $3 - 2i$ are **complex conjugates** of each other (together they are a **conjugate pair**). A well known result from algebra states that the complex zeros of a polynomial with real coefficients must occur in conjugate pairs. (See Review Exercise 72.) We will say more about complex conjugates in Section 8.2.

Complex Matrices

Now that we are able to add, subtract, and multiply complex numbers, we can apply these operations to matrices whose entries are complex numbers. We call such a matrix **complex.**

Definition of a Complex Matrix

A matrix whose entries are complex numbers is called a **complex matrix.**

All of the ordinary operations for matrices work for complex matrices, as demonstrated in the next two examples.

EXAMPLE 5 ***Operations with Complex Matrices***

Let A and B be the complex matrices given by

$$A = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix}$$

and determine the following matrices.

(a) $3A$ (b) $(2-i)B$ (c) $A+B$ (d) BA

Solution

(a) $$3A = 3\begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} = \begin{bmatrix} 3i & 3+3i \\ 6-9i & 12 \end{bmatrix}$$

(b) $$(2-i)B = (2-i)\begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} = \begin{bmatrix} 2+4i & 0 \\ 1+2i & 4+3i \end{bmatrix}$$

(c) $$A+B = \begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} + \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix} = \begin{bmatrix} 3i & 1+i \\ 2-2i & 5+2i \end{bmatrix}$$

(d) $$BA = \begin{bmatrix} 2i & 0 \\ i & 1+2i \end{bmatrix}\begin{bmatrix} i & 1+i \\ 2-3i & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2+2i \\ 7+i & 3+9i \end{bmatrix}$$

EXAMPLE 6 ***Finding the Determinant of a Complex Matrix***

Find the determinant of the matrix

$$A = \begin{bmatrix} 2-4i & 2 \\ 3 & 5-3i \end{bmatrix}.$$

Solution

$$\begin{aligned} \det(A) = \begin{vmatrix} 2-4i & 2 \\ 3 & 5-3i \end{vmatrix} &= (2-4i)(5-3i) - (2)(3) \\ &= 10 - 20i - 6i - 12 - 6 \\ &= -8 - 26i. \end{aligned}$$

TECHNOLOGY NOTE

Many computers and graphing calculators are capable of performing matrix operations on complex matrices. Try verifying the determinant calculation of the matrix from Example 6. You should obtain the same answer, $(-8, -26)$.

SECTION 8.1 ❑ EXERCISES

In Exercises 1–6, determine the value of the given expression.

1. $\sqrt{-2}\sqrt{-3}$

2. $\sqrt{8}\sqrt{-8}$

3. $\sqrt{-4}\sqrt{-4}$

4. i^3

5. i^4

6. $(-i)^7$

In Exercises 7–12, plot the given complex number.

7. $z = 6 - 2i$

8. $z = 3i$

9. $z = -5 + 5i$

10. $z = 7$

11. $z = 1 + 5i$

12. $z = 1 - 5i$

In Exercises 13 and 14, use vectors to illustrate graphically the given operation.

13. $-u$ and $2u$, where $u = 3 - i$

14. $3u$ and $-\frac{3}{2}u$, where $u = 2 + i$

In Exercises 15–18, determine x so that the complex numbers in each pair are equal.

15. $x + 3i, 6 + 3i$

16. $(2x - 8) + (x - 1)i, 2 + 4i$

17. $(x^2 + 6) + (2x)i, 15 + 6i$

18. $(-x + 4) + (x + 1)i, x + 3i$

In Exercises 19–26, find the sum and difference of the given complex numbers. Use vectors to illustrate your answers graphically.

19. $2 + 6i, 3 - 3i$

20. $1 + \sqrt{2}i, 2 - \sqrt{2}i$

21. $5 + i, 5 - i$

22. $i, 3 + i$

23. $6, -2i$

24. $12 - 7i, 3 + 4i$

25. $2 + i, 2 + i$

26. $2 + i, 2 - i$

In Exercises 27–36, find the product.

27. $(5 - 5i)(1 + 3i)$

28. $(3 + i)\left(\frac{2}{3} + i\right)$

29. $\left(\sqrt{7} - i\right)\left(\sqrt{7} + i\right)$

30. $\left(4 + \sqrt{2}i\right)\left(4 - \sqrt{2}i\right)$

31. $(a + bi)^2$

32. $(a + bi)(a - bi)$

33. $(1 + i)^3$

34. $(2 - i)(2 + 2i)(4 + i)$

35. $(a + bi)^3$

36. $(1 + i)^2(1 - i)^2$

In Exercises 37–44, determine the zeros of the given polynomial.

37. $2x^2 + 2x + 5$

38. $x^2 + x + 1$

39. $x^2 - 5x + 6$

40. $x^2 - 4x + 5$

41. $x^3 - 3x^2 + 4x - 2$

42. $x^3 - 2x^2 - 11x + 52$

43. $x^4 - 16$

44. $x^4 + 10x^2 + 9$

In Exercises 45–54, perform the indicated matrix operation using the following complex matrices.

$$A = \begin{bmatrix} 1+i & 1 \\ 2-2i & -3i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1-i & 3i \\ -3 & -i \end{bmatrix}$$

45. $A + B$

46. $B - A$

47. $2A$

48. $\frac{1}{2}B$

49. $2iA$

50. $\frac{1}{4}iB$

51. $\det(A + B)$

52. $\det(B)$

53. $5AB$

54. BA

55. Prove that the set of complex numbers, with the operations of addition and scalar multiplication (with *real* scalars), is a vector space of dimension two.

56. (a) Evaluate i^n for $n = 1, 2, 3, 4,$and 5.

(b) Calculate i^{57} and i^{1995}.

(c) Find a general formula for i^n for any positive integer n.

57. Let $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

(a) Calculate A^n for $n = 1, 2, 3, 4,$ and 5.

(b) Calculate A^{57} and A^{1995}.

(c) Find a general formula for A^n for any positive integer n.

58. Prove that if the product of two complex numbers is zero, then at least one of the numbers must be zero.