C.2 Separation of Variables

Use separation of variables to solve differential equations. • Use differential equations to model and solve real-life problems.

Separation of Variables

The simplest type of differential equation is one of the form $y' = f(x)$. You know that this type of equation can be solved by integration to obtain

$$y = \int f(x)\, dx.$$

In this section, you will learn how to use integration to solve another important family of differential equations—those in which the variables can be separated. This technique is called **separation of variables.**

Separation of Variables

If f and g are continuous functions, then the differential equation

$$\frac{dy}{dx} = f(x)g(y)$$

has a general solution of

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx + C.$$

TECHNOLOGY

You can use a symbolic integration utility to solve a separable variables differential equation. Use a symbolic integration utility to solve the differential equation

$$y' = \frac{x}{y^2 + 1}.$$

Essentially, the technique of separation of variables is just what its name implies. For a differential equation involving x and y, you separate the x variables to one side and the y variables to the other. After separating variables, integrate each side to obtain the general solution. Here is an example.

EXAMPLE 1 Solving a Differential Equation

Find the general solution of $\dfrac{dy}{dx} = \dfrac{x}{y^2 + 1}$.

Solution Begin by separating variables, then integrate each side.

$$\frac{dy}{dx} = \frac{x}{y^2 + 1} \qquad \text{Differential equation}$$

$$(y^2 + 1)\, dy = x\, dx \qquad \text{Separate variables.}$$

$$\int (y^2 + 1)\, dy = \int x\, dx \qquad \text{Integrate each side.}$$

$$\frac{y^3}{3} + y = \frac{x^2}{2} + C \qquad \text{General solution}$$

EXAMPLE 2 Solving a Differential Equation

Find the general solution of

$$\frac{dy}{dx} = \frac{x}{y}.$$

Solution Begin by separating variables, then integrate each side.

$$\frac{dy}{dx} = \frac{x}{y} \qquad \text{Differential equation}$$

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate each side.}$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1 \qquad \text{Find antiderivatives.}$$

$$y^2 = x^2 + C \qquad \text{Multiply each side by 2.}$$

So, the general solution is $y^2 = x^2 + C$. Note that C_1 is used as a temporary constant of integration in anticipation of multiplying each side of the equation by 2 to produce the constant C.

STUDY TIP

After finding the general solution of a differential equation, you should use the techniques demonstrated in Section C.1 to check the solution. For instance, in Example 2 you can check the solution by differentiating the equation $y^2 = x^2 + C$ to obtain $2yy' = 2x$ or $y' = x/y$.

EXAMPLE 3 Solving a Differential Equation

Find the general solution of

$$e^y \frac{dy}{dx} = 2x.$$

Use a graphing utility to graph several solutions.

Solution Begin by separating variables, then integrate each side.

$$e^y \frac{dy}{dx} = 2x \qquad \text{Differential equation}$$

$$e^y\,dy = 2x\,dx \qquad \text{Separate variables.}$$

$$\int e^y\,dy = \int 2x\,dx \qquad \text{Integrate each side.}$$

$$e^y = x^2 + C \qquad \text{Find antiderivatives.}$$

By taking the natural logarithm of each side, you can write the general solution as

$$y = \ln(x^2 + C). \qquad \text{General solution}$$

The graphs of the particular solutions given by $C = 0, C = 5, C = 10$, and $C = 15$ are shown in Figure A.10.

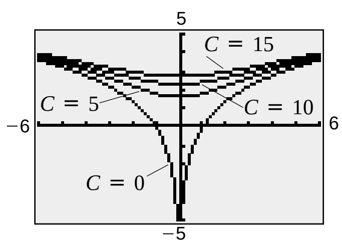

FIGURE A.10

EXAMPLE 4 Finding a Particular Solution

Solve the differential equation $xe^{x^2} + yy' = 0$ subject to the initial condition $y = 1$ when $x = 0$.

Solution

$$xe^{x^2} + yy' = 0 \qquad \text{Differential equation}$$

$$y\frac{dy}{dx} = -xe^{x^2} \qquad \text{Subtract } xe^{x^2} \text{ from each side.}$$

$$y\,dy = -xe^{x^2}\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int -xe^{x^2}\,dx \qquad \text{Integrate each side.}$$

$$\frac{y^2}{2} = -\frac{1}{2}e^{x^2} + C_1 \qquad \text{Find antiderivatives.}$$

$$y^2 = -e^{x^2} + C \qquad \text{Multiply each side by 2.}$$

To find the particular solution, substitute the initial condition values to obtain

$$(1)^2 = -e^{(0)^2} + C.$$

This implies that $1 = -1 + C$, or $C = 2$. So, the particular solution that satisfies the initial condition is

$$y^2 = -e^{x^2} + 2. \qquad \text{Particular solution}$$

EXAMPLE 5 Solving a Differential Equation

Example 3 in Section C.1 uses the differential equation

$$\frac{dx}{dt} = k(10 - x)$$

to model the sales of a new product. Solve this differential equation.

Solution

$$\frac{dx}{dt} = k(10 - x) \qquad \text{Differential equation}$$

$$\frac{1}{10 - x}\,dx = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{10 - x}\,dx = \int k\,dt \qquad \text{Integrate each side.}$$

$$-\ln(10 - x) = kt + C_1 \qquad \text{Find antiderivatives.}$$

$$\ln(10 - x) = -kt - C_1 \qquad \text{Multiply each side by } -1.$$

$$10 - x = e^{-kt - C_1} \qquad \text{Exponentiate each side.}$$

$$x = 10 - Ce^{-kt} \qquad \text{Solve for } x.$$

STUDY TIP

In Example 5, the context of the original model indicates that $(10 - x)$ is positive. So, when you integrate $1/(10 - x)$, you can write $-\ln(10 - x)$, rather than $-\ln|10 - x|$.

Also note in Example 5 that the solution agrees with the one that was given in Example 3 in Section C.1.

Applications

EXAMPLE 6 Modeling National Income

Let y represent the national income, let a represent the income spent on necessities, and let b represent the percent of the remaining income spent on luxuries. A commonly used economic model that relates these three quantities is

$$\frac{dy}{dt} = k(1 - b)(y - a)$$

where t is the time in years. Assume that b is 75%, and solve the resulting differential equation.

Corporate profits in the United States are closely monitored by New York City's Wall Street executives. Corporate profits, however, represent only about 11.9% of the national income. In 1999, the national income was more than \$7 trillion. Of this, about 71% was employee compensation.

Solution Because b is 75%, it follows that $(1 - b)$ is 0.25. So, you can solve the differential equation as follows.

$$\frac{dy}{dt} = k(0.25)(y - a) \qquad \text{Differential equation}$$

$$\frac{1}{y - a}\,dy = 0.25k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - a}\,dy = \int 0.25k\,dt \qquad \text{Integrate each side.}$$

$$\ln(y - a) = 0.25kt + C_1 \qquad \text{Find antiderivatives, given } y - a > 0.$$

$$y - a = Ce^{0.25kt} \qquad \text{Exponentiate each side.}$$

$$y = a + Ce^{0.25kt} \qquad \text{Add } a \text{ to each side.}$$

The graph of this solution is shown in Figure A.11. In the figure, note that the national income is spent in three ways.

$$(\text{National income}) = (\text{necessities}) + (\text{luxuries}) + (\text{capital investment})$$

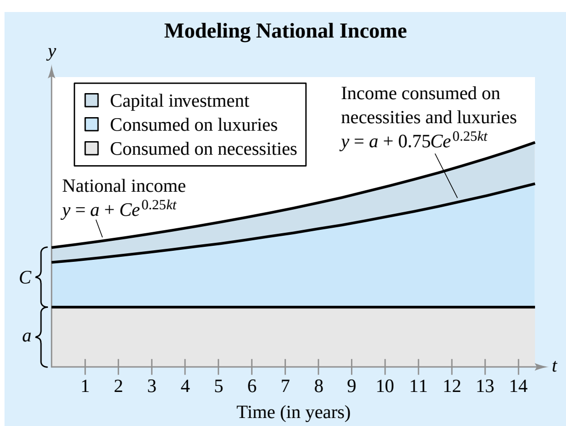

FIGURE A.11

EXAMPLE 7 Using Graphical Information

Find the equation of the graph that has the characteristics listed below.

1. At each point (x, y) on the graph, the slope is $-x/2y$.
2. The graph passes through the point $(2, 1)$.

Solution Using the information about the slope of the graph, you can write the differential equation

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

Using the point on the graph, you can determine the initial condition $y = 1$ when $x = 2$.

$$\frac{dy}{dx} = -\frac{x}{2y} \qquad \text{Differential equation}$$

$$2y\,dy = -x\,dx \qquad \text{Separate variables.}$$

$$\int 2y\,dy = \int -x\,dx \qquad \text{Integrate each side.}$$

$$y^2 = -\frac{x^2}{2} + C_1 \qquad \text{Find antiderivatives.}$$

$$2y^2 = -x^2 + C \qquad \text{Multiply each side by 2.}$$

$$x^2 + 2y^2 = C \qquad \text{Simplify.}$$

Applying the initial condition yields

$$(2)^2 + 2(1)^2 = C$$

which implies that $C = 6$. So, the equation that satisfies the two given conditions is

$$x^2 + 2y^2 = 6. \qquad \text{Particular solution}$$

As shown in Figure A.12, the graph of this equation is an ellipse.

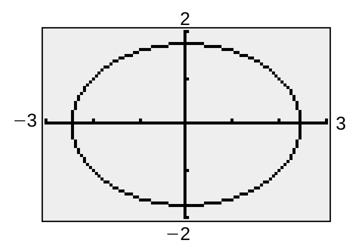

FIGURE A.12

TAKE ANOTHER LOOK

Classifying Differential Equations

In which of the differential equations can the variables be separated?

a. $\dfrac{dy}{dx} = \dfrac{3x}{y}$

b. $\dfrac{dy}{dx} = \dfrac{3x}{y} + 1$

c. $x^2\dfrac{dy}{dx} = \dfrac{3x}{y}$

d. $\dfrac{dy}{dx} = \dfrac{3x + y}{y}$

WARM-UP C.2

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–6, find the indefinite integral and check your result by differentiating.

1. $\int x^{3/2}\,dx$

2. $\int (t^3 - t^{1/3})\,dt$

3. $\int \frac{2}{x-5}\,dx$

4. $\int \frac{y}{2y^2+1}\,dy$

5. $\int e^{2y}\,dy$

6. $\int xe^{1-x^2}\,dx$

In Exercises 7–10, solve the equation for C or k.

7. $(3)^2 - 6(3) = 1 + C$

8. $(-1)^2 + (-2)^2 = C$

9. $10 = 2e^{2k}$

10. $(6)^2 - 3(6) = e^{-k}$

EXERCISES C.2

In Exercises 1–6, decide whether the variables in the differential equation can be separated.

1. $\frac{dy}{dx} = \frac{x}{y+3}$

2. $\frac{dy}{dx} = \frac{x+1}{x}$

3. $\frac{dy}{dx} = \frac{1}{x} + 1$

4. $\frac{dy}{dx} = \frac{x}{x+y}$

5. $\frac{dy}{dx} = x - y$

6. $x\frac{dy}{dx} = \frac{1}{y}$

In Exercises 7–26, use separation of variables to find the general solution of the differential equation.

7. $\frac{dy}{dx} = 2x$

8. $\frac{dy}{dx} = \frac{1}{x}$

9. $3y^2\frac{dy}{dx} = 1$

10. $\frac{dy}{dx} = x^2y$

11. $(y+1)\frac{dy}{dx} = 2x$

12. $(1+y)\frac{dy}{dx} - 4x = 0$

13. $y' - xy = 0$

14. $y' - y = 5$

15. $\frac{dy}{dt} = \frac{e^t}{4y}$

16. $e^y\frac{dy}{dt} = 3t^2 + 1$

17. $\frac{dy}{dx} = \sqrt{1-y}$

18. $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$

19. $(2+x)y' = 2y$

20. $y' = (2x-1)(y+3)$

21. $xy' = y$

22. $y' - y(x+1) = 0$

23. $y' = \frac{x}{y} - \frac{x}{1+y}$

24. $\frac{dy}{dx} = \frac{x^2+2}{3y^2}$

25. $e^x(y'+1) = 1$

26. $yy' - 2xe^x = 0$

In Exercises 27–32, use the initial condition to find the particular solution of the differential equation.

	Differential Equation	*Initial Condition*
27.	$yy' - e^x = 0$	$y = 4$ when $x = 0$
28.	$\sqrt{x} + \sqrt{y}\,y' = 0$	$y = 4$ when $x = 1$
29.	$x(y+4) + y' = 0$	$y = -5$ when $x = 0$
30.	$\frac{dy}{dx} = x^2(1+y)$	$y = 3$ when $x = 0$
31.	$dP - 6P\,dt = 0$	$P = 5$ when $t = 0$
32.	$dT + k(T-70)\,dt = 0$	$T = 140$ when $t = 0$

In Exercises 33 and 34, find an equation for the graph that passes through the point and has the specified slope. Then graph the equation.

33. Point: $(-1, 1)$

Slope: $y' = \dfrac{6x}{5y}$

34. Point: $(8, 2)$

Slope: $y' = \dfrac{2y}{3x}$

Velocity In Exercises 35 and 36, solve the differential equation to find velocity v as a function of time t if $v = 0$ when $t = 0$. The differential equation models the motion of two people on a toboggan after consideration of the force of gravity, friction, and air resistance.

35. $12.5 \dfrac{dv}{dt} = 43.2 - 1.25v$

36. $12.5 \dfrac{dv}{dt} = 43.2 - 1.75v$

Chemistry: Newton's Law of Cooling In Exercises 37–39, use Newton's Law of Cooling, which states that the rate of change in the temperature T of an object is proportional to the difference between the temperature T of the object and the temperature T_0 of the surrounding environment. This is described by the differential equation $dT/dt = k(T - T_0)$.

37. A steel ingot whose temperature is 1500°F is placed in a room whose temperature is a constant 90°F. One hour later, the temperature of the ingot is 1120°F. What is the ingot's temperature 5 hours after it is placed in the room?

38. A room is kept at a constant temperature of 70°F. An object placed in the room cools from 350°F to 150°F in 45 minutes. How long will it take for the object to cool to a temperature of 80°F?

39. Food at a temperature of 70°F is placed in a freezer that is set at 0°F. After 1 hour, the temperature of the food is 48°F.

(a) Find the temperature of the food after it has been in the freezer 6 hours.

(b) How long will it take the food to cool to a temperature of 10°F?

40. ***Biology: Cell Growth*** The rate of growth of a spherical cell with volume V is proportional to its surface area S. For a sphere, the surface area and volume are related by $S = kV^{2/3}$. So, a model for the cell's growth is

$$\frac{dV}{dt} = kV^{2/3}.$$

Solve this differential equation.

41. ***Learning Theory*** The management of a factory has found that a worker can produce at most 30 units per day. The number of units N per day produced by a new employee will increase at a rate proportional to the difference between 30 and N. This is described by the differential equation

$$\frac{dN}{dt} = k(30 - N)$$

where t is the time in days. Solve this differential equation.

42. ***Sales*** The rate of increase in sales S (in thousands of units) of a product is proportional to the current level of sales and inversely proportional to the square of the time t. This is described by the differential equation

$$\frac{dS}{dt} = \frac{kS}{t^2}$$

where t is the time in years. The saturation point for the market is 50,000 units. That is, the limit of S as $t \to \infty$ is 50. After 1 year, 10,000 units have been sold. Find S as a function of the time t.

43. ***Economics: Pareto's Law*** According to the economist Vilfredo Pareto (1848–1923), the rate of decrease of the number of people y in a stable economy having an income of at least x dollars is directly proportional to the number of such people and inversely proportional to their income x. This is modeled by the differential equation

$$\frac{dy}{dx} = -k\frac{y}{x}.$$

Solve this differential equation.

44. ***Economics: Pareto's Law*** In 1998, 8.6 million people in the United States earned more than \$75,000 and 50.3 million people earned more than \$25,000 (see figure). Assume that Pareto's Law holds and use the result of Exercise 43 to determine the number of people (in millions) who earned (a) more than \$20,000 and (b) more than \$100,000. *(Source: U.S. Census Bureau)*