Businesses seek to maximize their profits while operating under budget, supply, labor, and space constraints. Determining which combination of variables will result in the maximum profit may be done through the use of linear programming. Although many factors affect a business’s profitability, linear programming can help a business owner determine the “ideal” conditions for business success.

4.1 Graphing Linear Inequalities
- Graph linear inequalities
- Set up and solve systems of linear inequalities

4.2 Solving Linear Programming Problems Graphically
- Determine the feasible region of a linear programming problem
- Solve linear programming problems in two variables

4.3 Solving Standard Maximization Problems with the Simplex Method
- Apply the simplex method to solve multivariable standard maximization problems

4.4 Solving Standard Minimization Problems with the Dual
- Find the solution to a standard minimization problem by solving the dual problem with the simplex method

4.5 Solving General Linear Programming Problems with the Simplex Method
- Solve general linear programming problems with the simplex method
Many students work multiple part-time jobs to finance their education. Often the jobs pay different wages and offer varying hours. Suppose that a student earns $10.50 per hour delivering pizza and $8.00 per hour working in a campus computer lab. If the student has only 30 hours per week to work and must earn $252 during that period, how many hours must he spend at each job in order to meet his earnings goal? In this section, we will explain how linear inequalities may be used to answer this question. We will demonstrate how to graph linear inequalities and show that the solution region of a system of linear inequalities is the intersection of the graphs of the individual inequalities.

Linear Inequalities

In many real-life applications, we are interested in a range of possible solutions instead of a single solution. For example, when you prepare to buy a house, a lender will calculate the maximum amount of money it is willing to lend you; however, the lender doesn’t require you to borrow the maximum amount. You may borrow any amount of money up to the maximum. Recall that in mathematics, we use inequalities to represent the range of possible solutions that meet the given criteria.

INEquality NOTATION

\[ x \leq y \] is the set of all values of \( x \) less than or equal to \( y \).
\[ x \geq y \] is the set of all values of \( x \) greater than or equal to \( y \).
\[ x < y \] is the set of all values of \( x \) less than but not equal to \( y \).
\[ x > y \] is the set of all values of \( x \) greater than but not equal to \( y \).

The inequalities \( x < y \) and \( x > y \) are called strict inequalities because the two variables cannot ever be equal. Although strict inequalities have many useful applications, we will focus on the nonstrict inequalities in this chapter.

An easy way to keep track of the meaning of an inequality is to remember that the inequality sign always points toward the smaller number. Consider these everyday examples of inequalities:

You must be at least 16 years old to get a driver’s license. \((16 \leq a)\) or \((a \geq 16)\).
You must be at least 21 years old to legally buy alcohol. \((21 \leq a)\) or \((a \geq 21)\).
The maximum fine for littering is $200. \((200 \geq f)\) or \((f \leq 200)\).
Your carry-on bag must be no more than 22 inches long. \((22 \geq l)\) or \((l \leq 22)\).
A linear inequality looks like a linear equation with an inequality sign in the place of the equal sign. Recall that linear inequalities may be manipulated algebraically in the same way as linear equations, with one major exception: When we multiply or divide both sides of an inequality by a negative number, we must reverse the direction of the inequality sign. For example, if we multiply both sides of \(-3x + 2y \leq 10\) by \(-1\), we get \(3x - 2y \geq -10\), not \(3x - 2y \leq -10\).

**Graphing Linear Inequalities**

The graph of a linear inequality is a region bordered by a line called a **boundary line**. The **solution region** of a linear inequality is the set of all points (including the boundary line) that satisfy the inequality.

Consider the inequality \(x + 2y \geq 4\) (see Figure 4.1). The boundary line of the solution region is \(x + 2y = 4\), since the points satisfying this linear equation are on the border of the solution region.

We need to find all points \((x, y)\) that satisfy the inequality. We know that all points on the line satisfy the inequality. Which points off the line satisfy the inequality? Let’s pick a few points off the line (see Table 4.1) and test them to see if they satisfy the inequality. In order to satisfy the inequality, \(x + 2y\) must be at least 4.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(x + 2y)</th>
<th>In Solution Region?</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>No</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>7</td>
<td>Yes</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>6</td>
<td>Yes</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>No</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>9</td>
<td>Yes</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>8</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Graphically speaking, what do the four points in the solution region have in common (see Figure 4.2)?

They are all on the same side of the boundary line! In fact, all points on or above this boundary line satisfy the inequality. We represent this notion by shading the region above the boundary line, as shown in Figure 4.3.

Although we checked multiple points in this problem, we need to check only one point off the boundary line in order to determine which region to shade.

The linear inequality graphing process is summarized as follows.
Linear Inequality Graphing Technique

To graph the solution region of the linear inequality $ax + by \leq c$ (or $ax + by \geq c$), do the following:

1. Graph the boundary line $ax + by = c$.
2. Select a point on one side of the line. [If the line doesn’t pass through the origin, $(0, 0)$ is an excellent choice for easy computations.]
3. Substitute the point into the linear inequality and simplify. If the simplified statement is true, the selected point and all other points on the same side of the line are in the solution region. If the simplified statement is false, all points on the opposite side of the line are in the solution region.
4. Shade the solution region.

**EXAMPLE 1**

Graphing the Solution Region of a Linear Inequality

Graph the solution region of the linear inequality $2x + y \leq 4$.

**SOLUTION** As shown in Chapter 1, the $x$-intercept is easily found by dividing the constant term by the coefficient on the $x$ term.

$$x = \frac{4}{2} = 2$$

The point $(2, 0)$ is the $x$-intercept.

The $y$-intercept is found by dividing the constant term by the coefficient on the $y$ term.

$$y = \frac{4}{1} = 4$$

The point $(0, 4)$ is the $y$-intercept. We graph the $x$- and $y$-intercepts and then draw the line through the intercepts, as shown in Figure 4.4.

Next, we will pick the point $(0, 0)$ to plug into the inequality.

$$2(0) + 0 \leq 4$$

$0 \leq 4$

The statement is true, so all points on the same side of the line as the origin are in the solution region. We shade the solution region (see Figure 4.5).
If you choose to convert lines from standard to slope-intercept form before graphing them, the following properties will help you to quickly identify the solution region without having to check a point.

**SOLUTION REGION OF A LINEAR INEQUALITY**

The solution region of a linear inequality \( y \geq mx + b \) contains the line \( y = mx + b \) and the shaded region above the line.

The solution region of a linear inequality \( y \leq mx + b \) contains the line \( y = mx + b \) and the shaded region below the line.

### Graphing Systems of Linear Inequalities

Just as we can graph systems of linear equations, we can graph systems of linear inequalities. The solution region of a system of linear inequalities is the intersection of the solution regions of the individual inequalities. When we graph a solution region by hand, we will typically place arrows on the boundary lines to indicate which side of the lines satisfies the given inequality. Once all of the linear inequality graphs have been drawn, we will shade the region that has arrows pointing into the interior of the region from all sides.

#### Graphing the Solution Region of a System of Linear Inequalities

Graph the solution region of the system of linear inequalities.

\[
3x + 2y \leq 5 \\
x \geq 0 \\
y \geq 0
\]
We first rewrite the linear inequality as a linear inequality in slope-intercept form.

\[ 3x + 2y \leq 5 \]

Subtract 3x from both sides

\[ 2y \leq -3x + 5 \]

Divide both sides by 2

\[ y \leq -\frac{3}{2}x + \frac{5}{2} \]

Write as a decimal (optional)

\[ y \leq -1.5x + 2.5 \]

The boundary line is a line with slope \(-1.5\) and \(y\)-intercept \((0, 2.5)\). Since \(y\) is less than or equal to the expression \(-1.5x + 2.5\), we will shade the region below the line, as shown in Figure 4.6.

The next two inequalities \((x \geq 0, y \geq 0)\) limit the solution region to positive values of \(x\) and \(y\). The line \(x = 0\) is the \(y\) axis. The line \(y = 0\) is the \(x\) axis. Therefore, the solution region of the system of inequalities is the triangular region to the right of the line \(x = 0\), above the line \(y = 0\), and below the line \(y = -1.5x + 2.5\) (see Figure 4.7).

If it is possible to draw a circle around the solution region, the solution region is bounded. If no circle can be drawn that will enclose the entire solution region, the solution region is unbounded. The solution region in Example 2 was bounded. The solution region in Example 3 will be unbounded.

**EXAMPLE 3**

Graphing the Solution Region of a System of Linear Inequalities

Graph the solution region of the system of linear inequalities.

\[ 4x + y \geq 4 \]

\[ -x + y \geq 1 \]

**SOLUTION** The \(x\)-intercept of the boundary line \(4x + y = 4\) is \((1, 0)\), and the \(y\)-intercept is \((0, 4)\). Plugging in the point \((0, 0)\), we get

\[ 4(0) + (0) \geq 4 \]

\[ 0 \geq 4 \]

Since the statement is false, we graph \(4x + y = 4\) and place arrows on the side of the line not containing the origin, as shown in Figure 4.8.
The \( x \)-intercept of the boundary line \(-x + y = 1\) is \((-1, 0)\), and the \( y \)-intercept is \((0, 1)\). Plugging in the point \((0, 0)\), we get
\[
-(0) + (0) \geq 1 \\
0 \geq 1
\]
Since the statement is false, we graph \(-x + y = 1\) and place arrows on the side of the line not containing the origin, as shown in Figure 4.8.

The solution region is not bounded above the line \(y = x + 1\) or above the line \(y = -4x + 4\). Consequently, the shaded solution region is unbounded.

### Example 4

**Graphing a System of Linear Inequalities with an Empty Solution Region**

Graph the solution region of the system of linear inequalities.
\[
-2x + 2y \geq 6 \\
-x + y \leq 1
\]

**SOLUTION** We will graph the boundary lines by first rewriting them in slope-intercept form.

Solving the first inequality for \(y\), we get \(y \geq x + 3\) and draw arrows pointing to the region above the line \(y = x + 3\). Solving the second inequality for \(y\), we get \(y \leq x + 1\) and draw arrows pointing to the region below the line \(y = x + 1\) (see Figure 4.9).

Because the lines have the same slope \((m = 1)\), they are parallel. Consequently, the lines will never intersect. As seen in the graph, the two regions also will never intersect; they will always be separated by the region between the two lines. Therefore, this system of linear inequalities does not have a solution. That is, there is no ordered pair \((x, y)\) that can satisfy both inequalities simultaneously.
The corners of a solution region are called **corner points**. To find the coordinates of each corner point, we solve the system of equations formed by the two intersecting boundary lines that create the corner.

**Finding the Corner Points of a Solution Region**

Graph the solution region of the system of linear inequalities and determine the coordinates of each corner point.

\[
\begin{align*}
2x + y &\leq 6 \\
-x + y &\geq 0 \\
x &\geq 0
\end{align*}
\]

**SOLUTION** We will graph the boundary line \(2x + y = 6\) using its \(x\)- and \(y\)-intercepts. The \(x\)-intercept is \((3, 0)\), since \(\frac{6}{2} = 3\). The function has \(y\)-intercept \((0, 6)\), since \(\frac{6}{1} = 6\). Plugging in the point \((0, 0)\), we get \(2(0) + (0) \leq 6\). The statement is true, so we shade the side of the line containing the point \((0, 0)\). Solving the second inequality for \(y\), we get \(y \geq x\), and we shade the region above the line \(y = x\). The third inequality restricts \(x\) to all positive values, so we shade the region to the right of the \(y\) axis.

The triangular region shown in Figure 4.10 is the intersection of the three regions and is the solution region of the system of linear inequalities. From the graph, it appears that the solution region has corner points at or near \((0, 0)\), \((0, 6)\), and \((2, 2)\). We will verify these results algebraically.

The coordinates of the first corner point may be found by solving the system of equations formed by the boundary lines that make up the corner.

\[
\begin{align*}
-x + y &= 0 \\
x &= 0
\end{align*}
\]

Adding the first equation to the second equation yields \(y = 0\). Since the second equation tells us that \(x = 0\), the coordinates of the first corner point are \((0, 0)\).

The coordinates of the second corner point may be found by solving the system of equations formed by the boundary lines that make up the corner.

\[
\begin{align*}
2x + y &= 6 \\
x &= 0
\end{align*}
\]

The second equation tells us that \(x = 0\). Substituting this value of \(x\) into the first equation yields

\[
\begin{align*}
2x + y &= 6 \\
2(0) + y &= 6 & \text{Substitute } x = 0 \\
y &= 6
\end{align*}
\]

The coordinates of the second corner point are \((0, 6)\).
The coordinates of the third corner point may be found by solving the system of equations formed by the boundary lines that make up the corner.

\[ 2x + y = 6 \]
\[ -x + y = 0 \]

The second equation may be rewritten as \( y = x \). Substituting this result into the first equation yields

\[ 2x + y = 6 \]
\[ 2x + (x) = 6 \]
\[ 3x = 6 \]
\[ x = 2 \]

Since \( x = y \), the coordinates of the corner point are \((2, 2)\).

It may have seemed superfluous to calculate the coordinates of the corner points algebraically in Example 5 when the coordinates were readily apparent from the graph of the solution region. Despite the apparent redundancy of the procedure, it is a necessary step. Example 6 illustrates the hazards of relying solely upon a graph for the coordinates of the corner points.

**Example 6**

**Finding the Corner Points of a Solution Region**

Graph the solution region for the system of inequalities and determine the coordinates of the corner points.

\[ x + y \leq 5 \]
\[ -5x + 5y \leq 6 \]
\[ y \geq 2 \]

**Solution**

The graph of the solution region is shown in Figure 4.11.

From the graph, it appears that the corner points of the region are at or near \((2, 3)\), \((3, 2)\), and \((0.75, 2)\).

The coordinates of the first corner point may be found by solving the system of equations formed by the boundary lines that make up the corner.

\[ x + y = 5 \]
\[ -5x + 5y = 6 \]

Adding five times the first equation to the second equation yields

\[ 0x + 10y = 31 \]
\[ y = 3.1 \]

Substituting this result back into the first equation yields

\[ x + y = 5 \]
\[ x + (3.1) = 5 \]
\[ x = 1.9 \]

The coordinates of the first corner point are \((1.9, 3.1)\). [From the graph, it looked as if the corner point was \((2, 3)\).]
The coordinates of the second corner point may be found by solving the system of equations formed by the boundary lines that make up the corner.

\[
\begin{align*}
x + y &= 5 \\
y &= 2
\end{align*}
\]

Since the second equation tells us that \( y = 2 \), we may substitute this value into the first equation.

\[
\begin{align*}
x + y &= 5 \\
x + (2) &= 5 \quad \text{Since } y = 2 \\
x &= 3
\end{align*}
\]

The coordinates of the second corner point are \((3, 2)\). Unlike the first corner point, this result agrees with our graphical conclusion.

The coordinates of the third corner point may be found by solving the system of equations formed by the boundary lines that make up the corner.

\[
\begin{align*}
-5x + 5y &= 6 \\
y &= 2
\end{align*}
\]

Since the second equation tells us that \( y = 2 \), we may substitute this value into the first equation.

\[
\begin{align*}
-5x + 5y &= 6 \\
-5x + 5(2) &= 6 \quad \text{Since } y = 2 \\
-5x &= -4 \\
x &= 0.8
\end{align*}
\]

The coordinates of the third corner point are \((0.8, 2)\). [From the graph, it looked as if the corner point was \((0.75, 2)\).]

**Using Technology to Graph Linear Inequalities**

Many graphing calculators can draw the graphs of linear inequalities, as detailed in the following Technology Tip. Often, however, it is quicker to draw the graphs by hand.

**TECHNOLOGY TIP**

**Graphing a System of Linear Inequalities**

1. Enter the linear equations associated with each inequality by using the Editor. (We will use the system \( y \leq -3x + 6 \) and \( y \geq 2x + 4 \) for this example.)
Linear Inequality Applications

Many real-life problems are subject to multiple constraints, such as budget, staffing, resources, and so on. Yet even when subjected to these constraints, the problems often have multiple solutions. As will be shown in Example 7, the linear inequality graphing techniques introduced earlier in this section may often be used to find solutions to real-life problems.

**EXAMPLE 7**

**Using a Linear System of Inequalities to Find the Ideal Work Schedule**

A student earns $8.00 per hour working in a campus computer lab and $10.50 per hour delivering pizza. If he has only 30 hours per week to work and must earn at least $252 during that period, how many hours can he spend at each job in order to earn at least $252?

**SOLUTION**

Let \( c \) be the number of hours the student works in the computer lab and \( p \) be the number of hours he works delivering pizza. He can work at most 30 hours. This is represented by the inequality

\[
c + p \leq 30 \quad \text{The maximum number of work hours is 30}
\]

The amount he earns working in the lab is 8.00\( c \), and the amount of money he earns delivering pizza is 10.50\( p \). His total income must be at least $252. That is,

\[
8c + 10.5p \geq 252 \quad \text{The minimum amount of income is $252}
\]
Solving the inequalities for \( p \) in terms of \( c \), we get the following system of inequalities and its associated graph (Figure 4.12). (We add the restrictions \( p \geq 0 \) and \( c \geq 0 \), since it doesn’t make sense to work a negative number of hours at either job.)

\[
\begin{align*}
    p &\leq -c + 30 \\
    p &\geq -\frac{16}{21}c + 24 \\
    p &\geq 0, \ c \geq 0
\end{align*}
\]

(In decimal form, \( p \geq -0.7619c + 24 \) approximately)

Every point of the solution region represents a combination of hours at the two jobs that will result in earnings of at least $252.

The corner points of the solution region are \((0, 24)\), \((0, 30)\), and \((25.2, 4.8)\). (The last point is found by calculating the intersection of the two boundary lines.) Let’s calculate the student’s weekly earnings at the corner points and some of the other points in the region (see Table 4.2).

<table>
<thead>
<tr>
<th></th>
<th>Lab Hours</th>
<th>Pizza Hours</th>
<th>Weekly Earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corner point</td>
<td>0</td>
<td>24</td>
<td>$252.00</td>
</tr>
<tr>
<td>Corner point</td>
<td>0</td>
<td>30</td>
<td>$315.00</td>
</tr>
<tr>
<td>Corner point</td>
<td>25.2</td>
<td>4.8</td>
<td>$252.00</td>
</tr>
<tr>
<td>Interior point</td>
<td>5</td>
<td>25</td>
<td>$302.50</td>
</tr>
<tr>
<td>Interior point</td>
<td>10</td>
<td>20</td>
<td>$290.00</td>
</tr>
<tr>
<td>Interior point</td>
<td>15</td>
<td>14</td>
<td>$267.00</td>
</tr>
</tbody>
</table>
The weekly earnings vary; however, in every case the number of work hours is less than or equal to 30 hours and the earnings are greater than or equal to $252.

4.1 Summary

In this section, you learned how to graph linear inequalities. You discovered that the solution to a system of linear inequalities is the intersection of the graphs of the solution regions of the individual inequalities.

4.1 Exercises

In Exercises 1–10, graph the solution region of the linear inequality. Then use the graph to determine if point $P$ is in the solution region.

1. $2x + y \leq 6;
   P = (2, 4)$
2. $4x + y \leq 0;
   P = (1, 1)$
3. $x + 5y \leq 10;
   P = (0, 0)$
4. $5x + 6y \leq 30;
   P = (0, 5)$
5. $-2x + 4y \geq -2;
   P = (1, 2)$
6. $x - y \leq 10;
   P = (5, -5)$
7. $5x - 4y \leq 0;
   P = (1, 0)$
8. $-3x - 3y \leq 9;
   P = (2, -1)$
9. $2x - y \geq 8;
   P = (-3, 2)$
10. $7x - 6y \geq 12;
     P = (0, -1)$

In Exercises 11–25, graph the solution region of the system of linear equations. If there is no solution, explain why.

11. $-4x + y \geq 2$
    $-2x + y \geq 1$
    $x \leq 0$
    $y \geq 0$
12. $-5x + y \geq 0$
    $2x + y \leq 4$
    $y \geq 0$
13. $-2x + 6y \leq 8$
    $4x - 12y \leq -6$
14. $10x - y \geq 12$
    $9x - 2y \geq 2$
15. $3x - 2y \leq 4$
    $11x - 20y \geq 2$
16. $9x - 6y \leq 0$
    $4x + 5y \leq 23$
17. $x - y \leq -5$
    $9x + y \leq 25$
18. $2x + 5y \leq 2$
    $3x - 5y \leq 3$
19. $6x + 2y \leq 10$
    $-x - 2y \geq -5$
    $x \geq 0$
    $y \geq 0$
20. $x - y \geq 3$
    $6x + 7y \leq 44$
    $6x - 7y \leq 16$
21. $2x - 4y \geq 16$
    $9x + y \leq -4$
    $-3x + 6y \leq -24$
22. $2x - 2y \geq 0$
    $3x + y \leq 4$
    $5x - y \geq 5$
23. $8x - y \geq 3$
    $x + 2y \leq 11$
    $9x + y \leq 14$
24. $-4x + y \geq 2$
    $-2x + y \geq 1$
    $y \geq 1$
    $x \leq 1$
25. $-5x + y \geq 0$
    $2x + y \leq 4$
    $y \leq 1$
    $x \leq 1$

In Exercises 26–30, set up the system of linear inequalities that can be used to solve the problem. Then graph the system of equations and solve the problem.

26. **Nutritional Content** A 32-gram serving of Skippy® Creamy Peanut Butter contains 150 milligrams of sodium and 17 grams of fat. A 56-gram serving of Bumble Bee® Chunk Light Tuna in Water contains 250 milligrams of sodium and 0.5 gram of fat. (Source: Product labeling.) Some health professionals advise that a person on a 2500-calorie diet should consume no more than 2400 mg of sodium and 80 grams of fat. Graph the region showing all possible serving combinations of peanut butter and tuna that a person could eat and still meet the dietary guidelines.

27. **Nutritional Content** A Nature Valley® Strawberry Yogurt Chewy Granola Bar contains 130 milligrams of sodium and 3.5 grams of fat. A Nature's Choice® Multigrain Strawberry Cereal Bar contains 65 milligrams of sodium and 1.5 grams of fat. (Source: Product labeling.) Some
health professionals advise that a person on a 2500-calorie diet should consume no more than 2400 mg of sodium and 80 grams of fat. Graph the region showing all possible serving combinations of granola bars and cereal bars that a person could eat and still meet the dietary guidelines.

28. **Student Wages** A student earns $15.00 per hour designing web pages and $9.00 per hour supervising a campus tutoring center. She has at most 30 hours per week to work, and she needs to earn at least $300. Graph the region showing all possible work-hour allocations that meet her time and income requirements.

29. **Wages** A salaried employee earns $900 per week managing a copy center. He is required to work a minimum of 35 hours but no more than 45 hours weekly. As a side business, he earns $25 per hour designing brochures for local business clients. In order to maintain his standard of living, he must earn $1100 per week. In order to maintain his quality of life, he limits his workload to 50 hours per week. Given that he has no control over the number of hours he has to work managing the copy center, will he be able to consistently meet his workload and income goals? Explain.

30. **Commodity Prices** Today’s Market Prices (www.todaymarket.com) is a daily fruit and vegetable wholesale market price service. Produce retailers who subscribe to the service can use the wholesale prices to aid them in setting retail prices for the fruits and vegetables they sell.

A 25-pound carton of peaches holds 60 medium peaches or 70 small peaches. In August 2002, the wholesale price for local peaches in Los Angeles was $9.00 per carton for medium peaches and $10.00 per carton for small peaches. (Source: Today’s Market Prices.) A fruit vendor has budgeted up to $100 to spend on peaches. He estimates that weekly demand for peaches is at least 420 peaches but no more than 630 peaches. He wants to buy enough peaches to meet the minimum estimated demand but no more than the maximum estimated demand. Graph the region showing which small- and medium-size peach carton combinations meet his demand and budget restrictions.

Exercises 31–40 are intended to challenge your understanding of the graphs of linear inequalities.

31. Graph the solution region of the system of linear inequalities and identify the coordinates of the corner points.

32. Graph the solution region of the system of linear inequalities and identify the coordinates of the corner points.

33. Graph the solution region of the system of linear inequalities and identify the coordinates of the corner points.

34. Write a system of inequalities whose solution region has the corner points (0, 0), (1, 3), (3, 5), and (2, 1).

35. Write a system of inequalities whose solution region has the corner points (1, 1), (1, 3), (5, 3), and (2, 1).

36. Write a system of inequalities whose unbounded solution region has the corner points (0, 5), (2, 1), and (5, 0).

37. Write a system of inequalities whose unbounded solution region has the corner points (0, 5), (4, 4), and (5, 0).

38. A student concludes that the corner points of a solution region defined by a system of linear inequalities are (0, 0), (1, 1), (0, 2), and (2, 2). After looking at the graph of the region, the instructor immediately concludes that the student is incorrect. How did the instructor know?

39. Is it possible to have a bounded solution region with exactly one corner point? If so, give a system of inequalities whose solution region is bounded and has exactly one corner point.

40. Is it possible to have an unbounded solution region with exactly one corner point? If so, give a system of inequalities whose solution region is unbounded and has exactly one corner point.
Many products, such as printer ink, are sold to business customers at a discount if large quantities are ordered. Profitable businesses want to minimize their supply costs yet have sufficient ink on hand to fulfill their printing requirements. How much ink should they order? The process of acquiring, producing, and distributing supplies can often be made more efficient by setting up and solving systems of linear inequalities.

In this section, we show how a mathematical method called linear programming can help businesses determine the most cost-effective way to manage their resources. We will demonstrate how linear programming is used to optimize an objective function subject to a set of linear constraints. We will also reveal how to find the whole-number solution of an integer programming problem. We will begin our discussion with the following set of definitions.

**LINEAR PROGRAMMING PROBLEM**

A linear equation \( z = ax + by \), called an objective function, may be maximized or minimized subject to a set of linear constraints of the form

\[
\begin{align*}
\text{or } cx + dy & \leq f \\
\text{or } cx + dy & \geq f 
\end{align*}
\]

where \( x \) and \( y \) are variables (called decision variables) and \( a, b, c, d \) and \( f \) are real numbers. A problem consisting of an objective function and a set of linear constraints is called a linear programming problem. The values of \( x \) and \( y \) that optimize (maximize or minimize) the value of the objective function are called the optimal solution. A linear programming problem with the additional constraint that \( x \) and \( y \) are integers is called an integer programming problem.

In Section 4.1, we gave the example of a student earning $10.50 an hour delivering pizza and $8.00 an hour working in a campus computer lab. He had only 30 hours per week to work, and he had to earn at least $252 during that period. We let \( p \) be the number of hours he spent delivering pizza and \( c \) be the number of hours he spent working in the computer lab. We had the constraints

\[
\begin{align*}
c + p & \leq 30 \\
8c + 10.5p & \geq 252 \\
p & \geq 0, c & \geq 0
\end{align*}
\]

which we rewrote as

\[
\begin{align*}
p & \leq -c + 30 \\
p & \geq -\frac{16}{21}c + 24 \\
p & \geq 0, c & \geq 0
\end{align*}
\]
From this scenario, we can set up linear programming problems to address each of the following questions:

1. What is the largest amount of money he can earn?

   **Objective function:** Maximize \( z = 8c + 10.5p \) \quad \text{Total amount earned}

   **Subject to**
   \[
   \begin{align*}
   p &\leq -c + 30 \\
   p &\geq \frac{16}{27}c + 24 \\
   p &\geq 0, \quad c \geq 0
   \end{align*}
   \]

2. What is the least number of hours he can work?

   **Objective function:** Minimize \( z = c + p \) \quad \text{Total hours worked}

   **Subject to**
   \[
   \begin{align*}
   p &\leq -c + 30 \\
   p &\geq \frac{16}{27}c + 24 \\
   p &\geq 0, \quad c \geq 0
   \end{align*}
   \]

3. What is the maximum number of hours he can work in the computer lab?

   **Objective function:** Maximize \( z = c \) \quad \text{Total computer lab hours}

   **Subject to**
   \[
   \begin{align*}
   p &\leq -c + 30 \\
   p &\geq \frac{16}{27}c + 24 \\
   p &\geq 0, \quad c \geq 0
   \end{align*}
   \]

Although each of the three objective functions has the same constraints, the optimal solution to each linear programming problem will differ based upon the objective function. However, all solutions will lie within the solution region of the system of constraints. In the context of linear programming, we call the solution region of the system of constraints the **feasible region** and the points within the region **feasible points**. Which of all the feasible points will optimize each of the objective functions? Testing all of the points in the feasible region would be an impossible task! Fortunately, we don’t have to. The Fundamental Theorem of Linear Programming limits the number of points we have to test.

---

**FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING**

1. If the solution to a linear programming problem exists, it will occur at a corner point.
2. If two adjacent corner points are optimal solutions, then all points on the line segment between them are also optimal solutions.
3. Linear programming problems with bounded feasible regions will always have optimal solutions.
4. Linear programming problems with unbounded feasible regions may or may not have optimal solutions.

---

Recall that the graph of the feasible region for the computer lab and pizza delivery problem had the corner points \((0, 24), (0, 30), (25.2, 4.8)\), as shown in Figure 4.13.
The solution to each one of the linear programming problems will occur at one of these points.

1. What is the largest amount of money he can earn?

Maximize \( z = 8c + 10.5p \).

We evaluate the objective function at each corner point (see Table 4.3).

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computer Lab Hours ((c))</td>
<td>Pizza Delivery Hours ((p))</td>
</tr>
<tr>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>25.2</td>
<td>4.8</td>
</tr>
</tbody>
</table>

The maximum value of \( z \) occurs at corner point \((0, 30)\). The optimal solution is \( c = 0 \) and \( p = 30 \). To maximize his earnings, he should work 0 hours in the lab and 30 hours delivering pizza.

2. What is the least number of hours he can work?

Minimize \( z = c + p \).

We evaluate the objective function at each corner point (see Table 4.4).
The minimum value of $z$ occurs at corner point $(0, 24)$. The optimal solution is $c = 0$ and $p = 24$. To minimize his work hours, he should work 0 hours in the lab and 24 hours delivering pizza.

3. What is the maximum number of hours he can work in the computer lab?

Maximize $z = c$.

We evaluate the objective function at each corner point (see Table 4.5).

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computer Lab Hours ($c$)</td>
<td>Pizza Delivery Hours ($p$)</td>
</tr>
<tr>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>25.2</td>
<td>4.8</td>
</tr>
</tbody>
</table>

The maximum value of $z$ occurs at corner point $(25.2, 4.8)$. The optimal solution is $c = 25.2$ and $p = 4.8$. The maximum amount of time he can work in the computer lab is 25.2 hours. He will still have to work 4.8 hours delivering pizza to reach his earnings goal.

Why do we have to check only the corner points of the feasible region? We’ll address this question by considering a “family” of objective functions. Suppose we are asked to maximize the objective function $z = 3x + y$ subject to the following constraints:

$$
4x + y \leq 12 \\
2x + y \leq 8 \\
x \geq 0 \\
y \geq 0
$$

The graph of the feasible region and the coordinates of the corner points are shown in Figure 4.14.
If we set the variable $z$ in the objective function $z = 3x + y$ to a fixed value $c$, then the graph of the line $3x + y = c$ will pass through all points $(x, y)$ that satisfy the equation $3x + y = c$. If we repeat this for different values of $c$, we will end up with a “family” of objective function lines. The lines will be parallel with different $y$-intercepts. (In Figure 4.15, we set $z$ equal to the following constant values: 2, 4, 6, 8, 10, 12.)

Recall that we want to maximize the objective function subject to the constraints. Observe that although $z = 12$ is the largest value of $z$ shown in the figure, the line $3x + y = 12$ does not intersect the feasible region. Consequently, no feasible solution lies on the line $3x + y = 12$.

The first line above the feasible region that intersects the feasible region is the line $3x + y = 10$. This line intersects the feasible region at a single point: the corner point $(2, 4)$. At this corner point, $z = 10$. All other objective function lines that cross the feasible region take on values of $z$ less than 10. Therefore, the objective function $z = 3x + y$ has an optimal solution at $(2, 4)$. At this point, the
objective function takes on its maximum value: $z = 10$. This value of the objective function is referred to as the **optimal value** for the linear programming problem. Regardless of the objective function, the maximum (or minimum) value of the objective function will occur at a corner point of the feasible region. An argument similar to that given here can be made for any objective function and any feasible region.

The process of solving linear programming problems graphically is summarized in the following box.

### Graphical Method for Solving Linear Programming (LP) Problems

1. Graph the feasible region determined by the constraints.
2. Find the corner points of the feasible region.
3. Find the value of the objective function at each of the corner points.
4. If the feasible region is bounded, the maximum or minimum value of the objective function will occur at one of the corner points.
5. If the feasible region is an unbounded region in the first quadrant and the coefficients of the objective function are positive, then the objective function has a minimum value at a corner point. The objective function will not have a maximum value.

### Example 1: Solving a Linear Programming Problem Graphically

Solve the linear programming problem:

Maximize 
$$z = 6x + 2y$$

Subject to 
$$
\begin{align*}
-3x + y & \geq 2 \\
x + y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
$$

**Solution**

We begin by solving each inequality for $y$:

$$-3x + y \geq 2 \quad \Rightarrow \quad y \geq 3x + 2$$

$$x + y \leq 10 \quad \Rightarrow \quad y \leq -x + 10$$

Graphing the feasible region yields the graph shown in Figure 4.16.

The corner points occur where the boundary lines intersect. The corner points of the feasible region are $(0, 2)$, $(0, 10)$, and $(2, 8)$. Substituting each of these points into the objective function, $z = 6x + 2y$, we get the results in Table 4.6.
The objective function is maximized when and . At that point, .

**EXAMPLE 2**

**Solving an LP Problem with an Unbounded Feasible Region**

Solve the linear programming problem:

Minimize \( z = 2x + 5y \)

Subject to

\[
\begin{align*}
4x + y &\geq 4 \\
x + y &\geq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

**SOLUTION**

The objective function is

\( z = 2x + 5y \)

The constraints are

\[
\begin{align*}
4x + y &\geq 4 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

We draw each boundary line and shade the feasible region, as shown in Figure 4.17.

The region is unbounded and has two corner points. Since the coefficients of the objective function are both positive, the objective function will have a minimum. The first corner point of the feasible region is \((0, 4)\), the \(y\)-intercept of one of the constraints. We determine the coordinates of the second corner point by finding the intersection point of the boundary lines, \(y = x + 1\) and \(y = -4x + 4\).

\[
\begin{align*}
-4x + 4 &= x + 1 \\
5x &= -3 \\
x &= 0.6 \\
y &= 0.6 + 1 \\
y &= 1.6
\end{align*}
\]

The second corner point is \((0.6, 1.6)\). Substituting each of the points into the objective function, \(z = 2x + 5y\), we obtain the values in Table 4.7.
The objective function has a minimum at (0.6, 1.6). At that point, \( z = 9.2 \).

In Example 2, we had an unbounded feasible region. We were able to find a solution because we were asked to minimize the objective function \( z = 2x + 5y \). If we had been asked to maximize the objective function, the problem would have had no solution. For a linear programming problem with an unbounded feasible region and an objective function with positive coefficients, no matter what “optimal solution” we pick, we will always be able to find a point \((x, y)\) in the feasible region that yields a greater “optimal value.”

**EXAMPLE 3**

**Solving a Linear Programming Problem Graphically**

Solve the linear programming problem:

Minimize \( z = -5x + 3y \)

Subject to:

\[
\begin{align*}
6x + y & \geq 6 \\
-2x + y & \geq 1 \\
x & \leq 2 \\
y & \geq 1 \\
x & \geq 0
\end{align*}
\]

**SOLUTION**

After graphing each of the five boundary lines, we will use arrows to show which side of the boundary line will be shaded, as shown in Figure 4.18. This technique is especially helpful when working with a large number of constraints.

We shade the region that has boundary lines with all arrows pointing to the interior of the region. Notice that the line \( y = 1 \) does not form a boundary line of the feasible region. This is okay so long as the feasible region is on the appropriate
side of \( y = 1 \). Since the arrows on the line \( y = 1 \) point toward the side that contains the feasible region, the constraint \( y \geq 1 \) is satisfied.

The first corner point \((0, 6)\) is easily determined, since it is the \( y \)-intercept of the boundary line \( 6x + y = 6 \). The second corner point occurs at the intersection of \( 6x + y = 6 \) and \( -2x + y = 1 \). We must solve the system of equations

\[
\begin{align*}
6x + y &= 6 \\
-2x + y &= 1
\end{align*}
\]

Subtracting the second equation from the first equation yields

\[
8x = 5
\]

\[
x = 0.625
\]

To determine the value of \( y \), we substitute the \( x \) value back into the equation \(-2x + y = 1\).

\[
-2x + y = 1
\]

\[
-2(0.625) + y = 1 \quad \text{Since } x = 0.625
\]

\[
-1.25 + y = 1
\]

\[
y = 2.25
\]

The second corner point is \((0.625, 2.25)\).

The third point occurs at the intersection of \( x = 2 \) and \(-2x + y = 1\). The \( x \) coordinate of the corner point is \( x = 2 \). To determine the \( y \) coordinate, we substitute this result into \(-2x + y = 1\).

\[
-2x + y = 1
\]

\[
-2(2) + y = 1 \quad \text{Since } x = 2
\]

\[
-4 + y = 1
\]

\[
y = 5
\]

The third corner point is \((2, 5)\).

With the corner points identified, we are ready to evaluate the objective function \( z = -5x + 3y \) at each corner point, as shown in Table 4.8.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>( x )</th>
<th>( y )</th>
<th>Objective Function ( z = -5x + 3y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td></td>
<td>18</td>
</tr>
<tr>
<td>0.625</td>
<td>2.25</td>
<td></td>
<td>3.625</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

Since we are looking for the minimum value of the objective function, the optimal solution is \((0.625, 2.25)\) and the optimal value is 3.625.

**Real-Life Applications**

As shown in Examples 4 and 5, many real-life problems can be analyzed using the techniques of linear programming.
Using Linear Programming to Do Investment Analysis

Table 4.9 shows the average annual rate of return on two TIAA-CREF investment accounts over a 10-year period.

<table>
<thead>
<tr>
<th>CREF Variable Annuity Accounts</th>
<th>10-Year Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond Market</td>
<td>7.15%</td>
</tr>
<tr>
<td>Social Choice</td>
<td>10.31%</td>
</tr>
</tbody>
</table>


An investor wants to invest at least $3000 in the Bond Market and Social Choice accounts. He assumes that he will be able to get a return equal to the 10-year average, and he wants the total return on his investment to be at least 9 percent. He assigns each share in the Bond Market account a risk rating of 2 and each share in the Social Choice account a risk rating of 4. The approximate share price at the end of June 2004 was $74 per share for the Bond Market account and $102 per share for the Social Choice account. He will use these prices in his analysis. How many shares of each account should he buy in order to minimize his overall risk? [Note that fractions of shares may be purchased. Also, to make computations easier, round each percentage to the nearest whole-number percent (i.e., 10.31 percent = 10 percent).]

**SOLUTION** Let \( x \) be the number of shares in the Bond Market account and \( y \) be the number of shares in the Social Choice account. The rounded rate of return for the Bond Market account is 7 percent, and that for the Social Choice account is 10 percent. The share price for the Bond Market account is $74, and that for the Social Choice account is $102. Each Bond Market share has a risk rating of 2, and each Social Choice share has a risk rating of 4. We want to minimize the overall risk. That is, we want to minimize \( z = 2x + 4y \).

The amount of money invested in each account is equal to the share price times the number of shares. The amount invested is $74x for the Bond Market account and $102y for the Social Choice account. The total amount invested is given by the equation

\[
74x + 102y \geq 3000 \quad \text{The total amount invested is at least $3000}
\]

The dollar amount of the return on the investment is equal to the product of the rate and the amount invested. The return on the Bond Market account is \((0.07)(74x) = 5.18x\), and the return on the Social Choice account is \((0.10)(102y) = 10.20y\). Since we want to earn at least 9 percent on the total amount of money invested \((74x + 102y)\), the dollar amount of the minimum combined return is \((0.09)(74x + 102y) = 6.66x + 9.18y\). The combined return is given by the equation

\[
5.18x + 10.20y \geq 6.66x + 9.18y \quad \text{The combined return is at least 9 percent of the amount invested}
\]

\[-1.48x + 1.02y \geq 0\]
Combining the objective function and each of the constraints yields the following linear programming problem:

Minimize \[ z = 2x + 4y \]

Subject to

\[
\begin{align*}
74x + 102y &\geq 3000 & \text{The total amount invested is at least$3000}\n-1.48x + 1.02y &\geq 0 & \text{The combined return is at least 9 percent of the amount invested.} \\
x &\geq 0 \\
y &\geq 0 
\end{align*}
\]

The graph of the feasible region is shown in Figure 4.19.

The first corner point is the y-intercept of the line \( 74x + 102y = 3000 \). Since \( \frac{3000}{102} \approx 29.412 \), the corner point is (0, 29.412).

The second corner point is the point of intersection of \( 74x + 102y = 3000 \) and \(-1.48x + 1.02y = 0\). We must solve the system of equations

\[
\begin{align*}
74x + 102y &= 3000 \\
-1.48x + 1.02y &= 0 
\end{align*}
\]

We will find the solution using matrices.

\[
\begin{bmatrix}
74 & 102 & 3000 \\
-1.48 & 1.02 & 0 \\
74 & 102 & 3000 \\
222 & 0 & 3000 \\
0 & 306 & 6000 \\
222 & 0 & 3000 \\
0 & 1 & 19.608 \\
1 & 0 & 13.514 \\
1 & 0 & 13.514 \\
0 & 1 & 19.608 \\
\end{bmatrix}
\]

The second corner point is (13.514, 19.608).
We evaluate the objective function \( z = 2x + 4y \) at each corner point, as shown in Table 4.10.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>0</td>
<td>29.412</td>
</tr>
<tr>
<td>13.514</td>
<td>19.608</td>
</tr>
</tbody>
</table>

The optimal solution is (13.514, 19.608), and the optimal value is 105.460.

The risk is minimized when 13.514 shares of the Bond Market account and 19.608 shares of the Social Choice account are purchased.

Did the rounding of the percentages and share prices affect the solution? Yes. However, since the 10-year average rate of return on each account was not guaranteed, the investor decided that rounding the percentages to whole-number percents and rounding the share prices to whole-dollar amounts was good enough for modeling purposes.

**Integer Programming Problems**

In many real-applications, we have the additional constraint that the objective function input values must be whole numbers. Problems of this type are called **integer programming problems**.

**INTEGER PROGRAMMING PROBLEM**

A linear programming problem with the additional constraint that the decision variables are integers is called an **integer programming problem**.

Integer programming problems containing many input variables and constraints can be very difficult to solve. Even in the two-decision-variable case, solving these types of problems requires several extra steps. Although we won’t require you to solve any of these types of problems in the exercises, we will work an example here to illustrate the errors that may arise when the optimal solution of a linear programming problem is rounded to whole-number values.

**Using Linear Programming to Make Business Decisions**

Marlborough Printer Supplies sells generic replacement ink cartridges for a variety of printers through its eBay store. Shipping is free when 10 or more cartridges are ordered. In 2002, a single black ink cartridge for the Epson Color Stylus 660 printer cost $2.50, and a three-pack of black ink cartridges cost $6. (Source: eBay online store.) The owner of a small business needs to purchase at least 20 ink cartridges and wants to minimize her overall cost. How many single cartridges and how many three-packs should she buy?
Let \( s \) be the number of single cartridges and \( t \) be the number of three-packs. Since 10 or more cartridges will be ordered, shipping will be free. Consequently, the equation of the objective function (the cost function) is

\[
C = 2.5s + 6t
\]

We have the constraints

\[
\begin{align*}
&s + 3t \geq 20 \quad \text{A minimum of 20 cartridges are ordered} \\
&s \geq 0, \ t \geq 0 \quad \text{The number of each type ordered is nonnegative}
\end{align*}
\]

The feasible region is unbounded and has corner points \((0, \frac{2}{3})\) and \((20, 0)\), as shown in Figure 4.20.

Evaluating the objective function \( C = 2.5s + 6t \) at the two corner points yields the data in Table 4.11.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s = 20 )</td>
<td>( t = 0 )</td>
</tr>
<tr>
<td>( s = 0 )</td>
<td>( t = \frac{2}{3} )</td>
</tr>
</tbody>
</table>

Since the smallest value of the objective function is 40, the optimal solution is \((0, \frac{2}{3})\) and the optimal value is 40. However, since we can’t order a fraction of a three-pack, we must find the whole-number solution. Our natural tendency might be to round the optimal solution to \((0, 7)\). However, doing so also alters the optimal value, as shown in Table 4.12.

<table>
<thead>
<tr>
<th>Rounded Optimal Solution</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s = 0 )</td>
<td>( t = 7 )</td>
</tr>
</tbody>
</table>

Using whole-number values for \( s \) and \( t \), is it possible to further reduce the cost? We will investigate this question by adding additional constraints. We create Subproblem 1 by adding the constraint \( t \equiv 6 \) and Subproblem 2 by adding the constraint \( t \geq 7 \), as shown in Figure 4.21. (These are the whole-number values on either side of \( \frac{2}{3} \).

These new constraints split the feasible region into two separate regions. We will solve Subproblem 1 first. The unbounded feasible region of Subproblem 1 has corner points \((20, 0)\) and \((2, 6)\).
The optimal solution for Subproblem 1 is (2, 6), as shown in Table 4.13. Since this is a whole-number solution, it makes sense in the context of the problem. When two single cartridges and six three-packs are purchased, the total ink cost is $41.

The feasible region of Subproblem 2 has the corner point (0, 7) (see Table 4.14).

The optimal solution for Subproblem 2 is (0, 7). Since this is also a whole-number solution, it makes sense in the context of the problem. When no single cartridges and seven three-packs are purchased, the total ink cost is $42.

The optimal whole-number solution for the entire linear programming problem will be the subproblem solution that yields the smallest value of the objective function. The whole-number solution for Subproblem 1, (2, 6), had optimal value $41. The whole-number solution for Subproblem 2, (0, 7), had optimal value $42. Comparing the results of Subproblems 1 and 2, we conclude that the optimal whole-number solution of the entire problem is (2, 6). When two single cartridges and six three-packs are ordered, the overall cost is minimized. (It is important to note that if the business owner spent the extra dollar and ordered no single cartridges and seven three-packs, she would get 21 cartridges instead of 20 cartridges. She may decide that the extra cartridge is worth the extra dollar.)

As shown in Example 5, we must be aware that rounding an optimal solution to whole-number values does not guarantee that we have found the optimal whole-number solution.

### 4.2 Summary

In this section, you learned how a mathematical method called linear programming can help businesses determine the most cost-effective way to manage their resources. You used linear programming to optimize an objective function subject to a set of linear constraints.
4.2 Exercises

In Exercises 1–20, find the optimal solution and optimal value of the linear programming problem. If a solution does not exist, explain why.

1. Minimize \( z = 3x + 7y \)
   Subject to
   \[
   \begin{align*}
   4x + y &\leq 4 \\
   -x + y &\leq 1 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

2. Minimize \( z = 6x + 2y \)
   Subject to
   \[
   \begin{align*}
   6x + y &\geq 16 \\
   -2x + y &\geq 0 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

3. Minimize \( z = 9x + y \)
   Subject to
   \[
   \begin{align*}
   6x + y &\geq 16 \\
   -2x + y &\geq 0 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

4. Maximize \( z = 9x + y \)
   Subject to
   \[
   \begin{align*}
   6x + y &\leq 16 \\
   -2x + y &\leq 0 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

5. Maximize \( z = x + 10y \)
   Subject to
   \[
   \begin{align*}
   6x + y &\leq 16 \\
   -2x + y &\leq 0 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

6. Minimize \( z = 2x - 5y \)
   Subject to
   \[
   \begin{align*}
   4x + y &\leq 12 \\
   -6x + 2y &\leq 24 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

7. Maximize \( z = 2x - 5y \)
   Subject to
   \[
   \begin{align*}
   4x + y &\leq 12 \\
   -6x + 2y &\leq 24 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

8. Minimize \( z = x - y \)
   Subject to
   \[
   \begin{align*}
   -4x + y &\geq 8 \\
   -3x + y &\leq 6 \\
   x &\geq 0 \\
   y &\geq 0
   \end{align*}
   \]

9. Maximize \( z = -2x - y \)
   Subject to
   \[
   \begin{align*}
   -4x + y &\geq 8 \\
   -3x + y &\leq 6 \\
   x &\leq 4 \\
   y &\geq 0
   \end{align*}
   \]

10. Minimize \( z = 5x - y \)
    Subject to
    \[
    \begin{align*}
    -3x + y &\leq 9 \\
    -2x + y &\leq 6 \\
    x &\leq 3 \\
    y &\geq 0
    \end{align*}
    \]

11. Minimize \( z = -2x + 7y \)
    Subject to
    \[
    \begin{align*}
    -x + 4y &\geq 4 \\
    -x + y &\leq 1 \\
    x &\geq 0 \\
    y &\geq 0
    \end{align*}
    \]

12. Minimize \( z = 3x + 5y \)
    Subject to
    \[
    \begin{align*}
    6x + y &\geq 21 \\
    -2x + y &\geq 1 \\
    x &\geq 3 \\
    y &\geq 0
    \end{align*}
    \]

13. Minimize \( z = 11x + 9y \)
    Subject to
    \[
    \begin{align*}
    6x + y &\geq 16 \\
    -2x + y &\geq 0 \\
    x &\geq 0 \\
    y &\geq 2
    \end{align*}
    \]

14. Maximize \( z = 11x + 9y \)
    Subject to
    \[
    \begin{align*}
    6x + y &\geq 16 \\
    -2x + y &\geq 0 \\
    x &\geq 0 \\
    y &\geq 2
    \end{align*}
    \]
15. Maximize \[ z = x + 10y \]
Subject to
\[
\begin{align*}
6x + y & \leq 29 \\
-2x + y & \leq -3 \\
y & \leq 5 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
16. Minimize \[ z = 20x - y \]
Subject to
\[
\begin{align*}
4x + y & \leq 20 \\
-6x + 2y & \leq 40 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
17. Maximize \[ z = 20x - y \]
Subject to
\[
\begin{align*}
4x + y & \leq 20 \\
-6x + 2y & \leq 40 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
18. Minimize \[ z = x - y \]
Subject to
\[
\begin{align*}
2x + y & \geq 0 \\
3x - y & \leq 0 \\
x & \geq 1 \\
y & \geq 1
\end{align*}
\]
19. Maximize \[ z = -2x + y \]
Subject to
\[
\begin{align*}
-4x + 3y & \geq 0 \\
-3x + 4y & \geq -1 \\
x & \leq 4 \\
x & \geq 0 \\
y & \geq 2
\end{align*}
\]
20. Minimize \[ z = x + y \]
Subject to
\[
\begin{align*}
10x + 2y & \geq 8 \\
-20x - 4y & \leq -16 \\
x & \leq 4 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
For Exercises 21–32, identify the objective function and constraints of the linear programming problem. Then solve the problem and interpret the real-world meaning of the results.

21. **Minimum Commodity Cost**

   Today’s Market Prices (www.todaymarket.com) is a daily fruit and vegetable wholesale market price service. Produce retailers who subscribe to the service can use wholesale prices to aid them in setting retail prices for the fruits and vegetables they sell.

   A 25-pound carton of peaches holds 60 medium peaches or 70 small peaches. In August 2002, the wholesale price for local peaches in Los Angeles was $9.00 per carton for medium peaches and $10.00 per carton for small peaches. (Source: Today’s Market Prices.) A fruit vendor sells the medium peaches for $0.50 each and the small peaches for $0.45 each. He estimates that weekly demand for peaches is at least 420 peaches but no more than 630 peaches. He wants to buy enough peaches to meet the minimum estimated demand, but no more than the maximum estimated demand. How many boxes of each size of peaches should he buy if he wants to minimize his wholesale cost?

22. **Painkiller Costs**

   An online drugstore sells Tylenol Extra Strength in a variety of bottle sizes. The 250-caplet bottle costs $15, and the 150-caplet bottle costs $12. (Source: www.drugstore.com.) A family wants to order a supply of at least 750 caplets. How many 150-caplet bottles and how many 250-caplet bottles should the family order if it wants to minimize costs?

23. **Investment Choices**

   The following table shows the average annual rate of return on a variety of TIAA-CREF investment accounts over a 10-year period.

<table>
<thead>
<tr>
<th>CREF Variable Annuity Accounts</th>
<th>10-Year Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond Market</td>
<td>7.15%</td>
</tr>
<tr>
<td>Equity Index</td>
<td>11.34%</td>
</tr>
<tr>
<td>Global Equities</td>
<td>7.24%</td>
</tr>
<tr>
<td>Growth</td>
<td>8.56%</td>
</tr>
<tr>
<td>Money Market</td>
<td>4.22%</td>
</tr>
<tr>
<td>Social Choice</td>
<td>10.31%</td>
</tr>
<tr>
<td>Stock</td>
<td>9.97%</td>
</tr>
</tbody>
</table>


   An investor wants to invest at least $2000 in the Stock and Growth accounts. He assumes that he will be able to get a return equal to the 10-year average, and he wants the total return on his investment to be at least 9 percent. He assigns
each share in the Stock account a risk rating of 6 and each share in the Growth account a risk rating of 7. The approximate share price at the end of June 2004 was $174 per share for the Stock account and $55 per share for the Growth account. He will use these prices in his analysis. How many shares of each account should he buy in order to minimize his overall risk? [Note that fractions of shares may be purchased. Also, to make computations easier, round each percentage to the nearest whole-number percent (i.e., for 9.97 percent, use 10 percent).]

24. **Investment Choices** An investor wants to invest at least $5000 in the Global Equities and Equity Index accounts shown in Exercise 23. She assumes that she will be able to get a return equal to the 10-year average, and she wants the total return on her investment to be at least 10 percent. She assigns each share in the Global Equities account a risk rating of 6 and each share in the Equity Index account a risk rating of 5. The approximate share price at the end of June 2004 was $70 per share for the Global Equities account and $72 per share for the Equity Index account. She will use these prices in her analysis. How many shares of each account should she buy in order to minimize her overall risk? [Note that fractions of shares may be purchased. Also, to make computations easier, round each percentage to the nearest whole-number percent (i.e., for 11.34 percent, use 11 percent).]

25. **Pet Nutrition: Food Cost** PETsMART.com sold the following varieties of dog food in June 2003:
- Nature’s Recipe Venison Meal & Rice Canine, 20 percent protein, $21.99 per 20-pound bag
- PETsMART Premier Oven Baked Lamb Recipe, 25 percent protein, $22.99 per 30-pound bag
(Source: www.petsmart.com.)

A dog breeder wants to make at least 300 pounds of a mix containing at most 22 percent protein. How many bags of each dog food variety should the breeder buy in order to minimize cost? (Hint: Note that each bag is a different weight.)

26. **First Aid Kit Supplies** Safetymax.com sells first aid supplies to businesses. A company that assembles first aid kits for consumers purchases 3500 1” x 3” plastic adhesive bandages and 1800 alcohol wipes from Safetymax.com.

The company assembles standard and deluxe first aid kits for sale to consumers. A **standard** first aid kit contains 40 plastic adhesive bandages and 20 alcohol wipes. A **deluxe** first aid kit contains 50 plastic adhesive bandages and 28 alcohol wipes.

The company makes a profit of $3 from each standard first aid kit sold and $4 from each deluxe first aid kit sold. Assuming that every kit produced will be sold, how many of each type of kit should the company assemble in order to maximize profit?

27. **Furniture Production** In June 2004, an online furniture retailer offered the following items at the indicated prices:
- Teak Double Rocker, $745
- Teak Tennis Bench, $124
(Source: www.outdoordecor.com.)

Suppose that the number of hours required to produce each item is as shown in the following table.

<table>
<thead>
<tr>
<th>Item</th>
<th>Cut</th>
<th>Finish</th>
<th>Package</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rocker</td>
<td>4</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>Bench</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

The company has a maximum of 360 labor hours available in the Cutting Department, a maximum of 730 labor hours available in the Finishing Department, and a maximum of 150 labor hours available in the Packaging Department. Suppose that the company makes a profit of $314 from the sale of each rocker and $57 from the sale of each bench. Assuming that all items produced are sold, how many rockers and how many benches should the company produce in order to maximize profit?

28. **Furniture Production** In June 2004, an online furniture retailer offered the following items at the indicated prices:
- Avalon Teak Armchair, $378
- Teak Tennis Bench, $124
(Source: www.outdoordecor.com.)

Suppose that the number of hours required to produce each item is as shown in the following table.

<table>
<thead>
<tr>
<th>Item</th>
<th>Cut</th>
<th>Finish</th>
<th>Package</th>
</tr>
</thead>
<tbody>
<tr>
<td>Armchair</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Bench</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
29. **Transportation Costs** A high school PTA in southern Florida is planning an overnight trip to Orlando, Florida, for its graduating class. A Plus Transportation, a local charter transportation company, offers the following rates (as of December 2003):

<table>
<thead>
<tr>
<th>Vehicle Capacity</th>
<th>Overnight Charter Rate (for 2 Days of Service)</th>
</tr>
</thead>
<tbody>
<tr>
<td>29</td>
<td>$1000</td>
</tr>
<tr>
<td>49</td>
<td>$1800</td>
</tr>
</tbody>
</table>

*Source: www.buscharter.net.*

The school anticipates that 153 students will go on the trip. Each 29-passenger vehicle requires one chaperone, and each 49-passenger vehicle requires six chaperones. (The chaperones will be traveling with the students.) At most 24 chaperones are available to go on the trip. How many of each type of vehicle should the PTA charter in order to minimize transportation costs?

30. **Transportation Costs** A high school PTA in southern Florida is planning an overnight trip to Orlando, Florida, for its graduating class. A Plus Transportation, a local charter transportation company, offers the following rates (as of December 2003):

<table>
<thead>
<tr>
<th>Vehicle Capacity</th>
<th>Overnight Charter Rate (for 2 Days of Service)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$625</td>
</tr>
<tr>
<td>57</td>
<td>$1995</td>
</tr>
</tbody>
</table>

*Source: www.buscharter.net.*

The school anticipates that 153 students will go on the trip. Each 10-passenger vehicle requires two chaperones, and each 57-passenger vehicle requires four chaperones. (The chaperones will be traveling with the students.) At most 16 chaperones are available to go on the trip. How many of each type of vehicle should the PTA charter in order to minimize transportation costs?

31. **Television Advertising** For the week of July 5-11, 2004, a national media research company estimated that 14,834,000 viewers watched *CSI* and 10,557,000 viewers watched *Law and Order*. *(Source: www.nielsenmedia.com.)*

The amount of money a network can charge for advertising is based in part on the size of the viewing audience. Suppose that a 30-second commercial running on *CSI* costs $3100 per spot and a 30-second commercial running on *Law and Order* costs $2500 per spot. A beverage company is willing to spend up to $68,500 for commercials run during episodes of the two programs. The company requires at least 10 spots to be run on each program. How many spots on each program should be purchased in order to maximize the number of viewers?

32. **Television Advertising** For the week of July 5-11, 2004, a national media research company estimated that 14,834,000 viewers watched *CSI* and 10,557,000 viewers watched *Law and Order*. *(Source: www.nielsenmedia.com.)*

The amount of money a network can charge for advertising is based in part on the size of the viewing audience. Suppose that a 30-second commercial running on *CSI* costs $3100 per spot and a 30-second commercial running on *Law and Order* costs $2500 per spot. An athletic gear company is willing to spend up to $68,500 for commercials run during episodes of the two programs. The company requires at least 10 spots to be run on *CSI* and at least 15 spots on *Law and Order*. How many spots on each program should be purchased in order to maximize the number of viewers?
For Exercises 33–35, use the following data for publicly traded recreational vehicle companies. The information was accurate as of July 16, 2004.

<table>
<thead>
<tr>
<th>Company</th>
<th>Share Price (dollars)</th>
<th>Earnings/Share (dollars)</th>
<th>Dividend/Share (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harley-Davidson, Inc. (HDI)</td>
<td>62.70</td>
<td>2.56</td>
<td>0.40</td>
</tr>
<tr>
<td>Polaris Industries Inc. (PII)</td>
<td>50.50</td>
<td>2.50</td>
<td>0.92</td>
</tr>
<tr>
<td>Winnebago Industries Inc. (WGO)</td>
<td>33.42</td>
<td>1.80</td>
<td>0.20</td>
</tr>
</tbody>
</table>


33. **Investment Choices** An investor has up to $4000 to invest in Harley-Davidson, Inc., and Polaris Industries, Inc. The investor wants to earn at least $60 in dividends while maximizing total earnings. How many shares of each company’s stock should the investor buy? (Assume that portions of shares may be purchased.)

34. **Investment Choices** An investor has up to $10,000 to invest in Harley-Davidson, Inc., and Winnebago, Inc. The investor wants to earn at least $50 in dividends while maximizing total earnings. How many shares of each company’s stock should the investor buy? (Assume that portions of shares may be purchased.)

35. **Investment Choices** An investor has up to $17,000 to invest in Polaris Industries, Inc., and Winnebago, Inc. The investor wants to have total earnings of at least $900 while maximizing total dividends. How many shares of each company’s stock should the investor buy? (Assume that portions of shares may be purchased.)

Exercises 36–40 are intended to challenge your understanding of linear programming.

36. Consider the following linear programming problem.

Maximize \( P = 3x + 4y \)

Subject to \[
\begin{align*}
3x + 5y & \leq 9 \\
x & \geq 0, y & \geq 0
\end{align*}
\]

Can a whole-number solution to a linear programming problem be obtained by simply rounding the noninteger solution to whole-number values? Explain.

37. Is it possible for a corner point \((a, b)\) to simultaneously minimize and maximize an objective function? If yes, give an example.

38. Give an example of a linear programming problem that does not have a solution.

39. Is it possible for the feasible region of a linear programming problem not to have any corner points? If yes, give an example.

40. Given the following linear programming problem, what is the maximum possible number of corner points the feasible region could have?

Maximize \( P = x + y \)

Subject to \[
\begin{align*}
-ax + by & \leq c \\
-dx + fy & \leq g \\
x & \geq 0, y & \geq 0
\end{align*}
\]

Assume \(a, b, c, d, f, \) and \(g\) are positive.
4.3 Solving Standard Maximization Problems with the Simplex Method

Suppose you are a wholesale fruit buyer for a grocery store. You have been given the assignment to purchase five varieties of apples, two varieties of peaches, and three varieties of pears. The grocery store has known budget and space constraints. You are asked to maximize the number of pieces of fruit purchased. This linear programming problem has 10 decision variables and cannot be solved graphically. Although the graphical method works well for linear programming problems with two decision variables, it does not work for problems containing more than two decision variables. In this section, we will introduce an alternative method that may be used to solve linear programming problems with any number of decision variables.

In 1947, George B. Dantzig developed the simplex method to solve linear programming problems. His method has been used to solve linear programming problems with hundreds of decision variables and hundreds of constraints. We will use the method to solve standard maximization problems.

**STANDARD MAXIMIZATION PROBLEM**

A standard maximization problem is a linear programming problem with an objective function that is to be maximized. The objective function is of the form

\[ P = ax + by + cz + \cdots \]

where \( a, b, c, \ldots \) are real numbers and \( x, y, z, \ldots \) are decision variables.

The decision variables are constrained to nonnegative values. Additional constraints are of the form

\[ Ax + By + Cz + \cdots \leq M \]

where \( A, B, C, \ldots \) are real numbers and \( M \) is nonnegative.

Observe that substituting \( x = 0, y = 0, z = 0, \ldots \) into each constraint inequality \( Ax + By + Cz + \cdots \leq M \) yields \( A(0) + B(0) + C(0) + \cdots \leq M \), which simplifies to \( 0 \leq M \). Since the inequality \( 0 \leq M \) is valid for all nonnegative values of \( M \), the origin \((0, 0, \ldots, 0)\) is contained in the solution region of each constraint and, consequently, in the feasible region of the standard maximization problem. Furthermore, if at least one of the constraints \( Ax + By + Cz + \cdots \leq M \) has all nonnegative coefficients, we are guaranteed that the feasible region is bounded. For the vast majority of our real-life applications, the constraints will have nonnegative coefficients.

Graphically speaking, the simplex method starts at the origin and moves from corner point to corner point of the feasible region, each time increasing the
value of the objective function until it attains its maximum value. Remarkably, the method doesn’t require that all corner points be tested, a fact that is appreciated by those solving linear programming problems with hundreds of corner points. Although the simplex method may be used for linear programming problems with any number of decision variables, we will restrict our use to problems with four variables or less. Larger problems are typically solved using computers.

We will introduce the simplex method with a two-variable example before formally listing the steps of the method.

Suppose we are asked to use the simplex method to maximize

$$P = 6x + 2y$$

subject to the constraints

$$\begin{align*}
2x + y &\leq 10 \\
2x + y &\leq 10 \\
x + y &\leq 8 \\
x &\geq 0 \\
y &\geq 0
\end{align*}$$

Since the decision variables are nonnegative and additional constraints are of the form $Ax + By + Cz + \cdots \leq M$ with $M$ nonnegative, this linear programming problem is a standard maximization problem. Consequently, we will be able to maximize the objective function using the simplex method. Although this is not part of the simplex method, we will draw the graph of the feasible region to give you a frame of reference as we work through the problem. The feasible region has corner points $(0, 0), (0, 8), (2, 6),$ and $(5, 0)$ and is shown in Figure 4.22.

The value of the objective function, $P = 6x + 2y$, varies depending upon the corner point. Substituting each corner point $(x, y)$ into $P = 6x + 2y$ and solving for $P$ yields Table 4.15.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective Function Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that the objective function $P = 6x + 2y$ attains a maximum value of 30 at the corner point $(5, 0)$. Using the simplex method to solve the standard maximization problem will yield the same result.

Recall that we are to solve the following standard maximization problem:

Maximize

$$P = 6x + 2y$$

Subject to

$$\begin{align*}
2x + y &\leq 10 \\
x + y &\leq 8 \\
x &\geq 0 \\
y &\geq 0
\end{align*}$$

The first step of the simplex method requires us to convert each of the constraint inequalities into an equation. Since $2x + y \leq 10$, there is some value $s \geq 0$ such that $2x + y + s = 10$. The variable $s$ is called a slack variable.
because it “takes up the slack.” That is, \( s \) adds in whatever value is necessary to make the left-hand side of the equation equal 10. The value of \( s \) will vary depending upon the value of \( x \) and \( y \) (see Table 4.16).

**TABLE 4.16**

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Slack Variable Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

For the constraint inequality \( x + y \leq 8 \), we observe that there is some value \( t \) such that \( x + y + t = 8 \). Again, the value of \( t \) will vary depending upon the value of \( x \) and \( y \) (see Table 4.17).

**TABLE 4.17**

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Slack Variable Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

We rewrite the objective function equation \( P = 6x + 2y \) by moving all terms to the left-hand side of the equation, which yields the equation \( -6x - 2y + P = 0 \). The system of equations representing the linear programming problem created by combining the constraint equations and the objective function equation is

\[
\begin{align*}
2x + y + s &= 10 & \text{Constraint 1} \\
x + y + t &= 8 & \text{Constraint 2} \\
-6x - 2y + P &= 0 & \text{Objective function}
\end{align*}
\]

This is a system of three equations and five unknowns \((x, y, s, t, P)\). Since the number of unknowns exceeds the number of equations, there will be more than one solution to the system, if a solution exists.

We can represent the system of equations using matrix notation. The augmented matrix is called the **initial simplex tableau**.

\[
\begin{bmatrix}
x & y & s & t & P \\
2 & 1 & 1 & 0 & 0 & 10 \\
1 & 1 & 0 & 1 & 0 & 8 \\
-6 & -2 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Constraint 1

Constraint 2

Objective function

The vertical and horizontal lines in the tableau are used to separate the bottom row and rightmost column from the rest of the matrix. This separation will become important when we calculate test ratios (as will be explained later).
The variables of the columns of the simplex tableau that contain exactly one nonzero entry are called **active (basic)** variables. The variables of the columns that contain more than one nonzero entry are called **inactive (nonbasic)** variables. For the initial tableau, the variables \( s, t, \) and \( P \) are the active variables, since their corresponding columns in the tableau contain exactly one nonzero value. The variables \( x \) and \( y \) are the inactive variables in the initial simplex tableau, since their corresponding columns contain more than one nonzero value.

\[
\begin{array}{ccccc|c}
 x & y & s & t & P \\
 2 & 1 & 1 & 0 & 0 & 10 \\
 1 & 1 & 0 & 1 & 0 & 8 \\
 -6 & -2 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Constraint 1

Constraint 2

Objective function

The basic feasible solution of the tableau always corresponds with the origin. The solution is obtained by substituting \( x = 0 \) and \( y = 0 \) into the equations that generated the tableau.

\[
\begin{align*}
2(0) + (0) + s &= 10 \\
(0) + (0) + t &= 8 \\
-6(0) - 2(0) + P &= 0
\end{align*}
\]

Simplifying yields

\[
\begin{align*}
s &= 10 & \text{Constraint 1} \\
t &= 8 & \text{Constraint 2} \\
P &= 0 & \text{Objective function}
\end{align*}
\]

These results are summarized in Table 4.18.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective Function Value</th>
<th>Slack Variable Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x ) ( y ) ( P = 6x + 2y )</td>
<td>( s ) ( t )</td>
<td></td>
</tr>
<tr>
<td>( 0 ) ( 0 ) ( 0 ) ( 10 ) ( 8 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The objective function \( P = 6x + 2y \) has a value of 0 at the corner point \( (0, 0) \).

An alternative and quicker approach to finding the solution may be obtained from the initial tableau itself by crossing out each inactive variable column and reading the result from the tableau. Crossing out an inactive variable column is equivalent to setting the column variable to zero. For the initial simplex tableau, we have

\[
\begin{array}{ccccc|c}
 x & y & s & t & P \\
 2 & 1 & 1 & 0 & 0 & 10 \\
 1 & 1 & 0 & 1 & 0 & 8 \\
 -6 & -2 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Constraint 1

Constraint 2

Objective function

The first row of the resultant matrix is equivalent to

\[
1s + 0t + 0P = 10
\]

\[
s = 10
\]
The second row is equivalent to

\[ 0s + 1t + 0P = 8 \]
\[ t = 8 \]

The bottom row is equivalent to

\[ 0s + 0t + 1P = 0 \]
\[ P = 0 \]

These results are the same as those shown in Table 4.18. Clearly, the corner point \((0, 0)\) does not maximize the value of the objective function.

How can we make \(P\) as large as possible? The coefficients on the \(x\) and \(y\) terms of the objective function \(P = 6x + 2y\) give us a clue. Since the coefficient on the \(x\) term is 6, each 1-unit increase in \(x\) will result in a 6-unit increase in \(P\). Similarly, since the coefficient on the \(y\) term is 2, each 1-unit increase in \(y\) will result in a 2-unit increase in \(P\). Consequently, our initial strategy is to make the \(x\) term as big as possible, since it contributes the most to the value of \(P\). The pivoting process detailed next will achieve this objective.

We will now use row operations to convert the initial simplex tableau into a new tableau. We will first select a pivot column. Then, using row operations, we will make the pivot column look like one of the columns of the \(3 \times 3\) identity matrix. To choose the pivot column, we find the negative number in the bottom row that is furthest away from zero. The corresponding column is the pivot column. In this tableau, the pivot column will be the \(x\) column, since \(-6\) is further away from 0 than \(-2\).

\[
\begin{array}{cccc|c}
 x & y & s & t & P \\
 2 & 1 & 1 & 0 & 0 \quad 10 \\
 1 & 1 & 0 & 1 & 0 \quad 8 \\
 -6 & -2 & 0 & 0 & 1 \quad 0 \\
\end{array}
\]

Why do we pick the column with the negative number that has the largest magnitude? Recall that we are trying to maximize the objective function \(P = 6x + 2y\). As was pointed out earlier, the value of \(x\) contributes more to \(P\) than the value of \(y\) because of the differences in their coefficients. Picking the column with the negative number of the largest magnitude guarantees that we are making the variable that contributes the most to the value of \(P\) as large as possible.

The pivot is the nonzero entry in the pivot column that will remain after we zero out the other column entries. It must be a positive number. If the pivot column contains more than one positive number, we will select the pivot by using a test ratio. To calculate the test ratio, we divide the rightmost entry of each row (excluding the bottom row) by the positive entry in the pivot column of that row.

\[
\begin{array}{cccc|c}
 x & y & s & t & P \\
 2 & 1 & 1 & 0 & 0 \quad 10 \\
 1 & 1 & 0 & 1 & 0 \quad 8 \\
 -6 & -2 & 0 & 0 & 1 \quad 0 \\
\end{array}
\]

10/2 = 5 \(\leftarrow\) Test ratio for first row
8/1 = 8 \(\leftarrow\) Test ratio for second row
Since the $x$ column is the pivot column, these test ratios correspond geometrically with the $x$-intercepts of the constraints (see Figure 4.23).

We select as the pivot the pivot-column entry that has the smallest test ratio. That is, we pick the $x$-intercept on the graph that is closest to the origin to be the pivot. This ensures that we remain in the feasible region. In this case, the first row of the tableau had the smallest test ratio. Consequently, the pivot will be the 2 in the first row and first column.

\[
\begin{array}{cccccc}
\text{Pivot} \\
 x & y & s & t & P \\
\hline
2 & 1 & 1 & 0 & 0 & 10 \\
1 & 1 & 0 & 1 & 0 & 8 \\
-6 & -2 & 0 & 0 & 1 & 0 \\
\end{array}
\]

10/2 = 5 \leftarrow \text{Test ratio for first row}
8/1 = 8 \leftarrow \text{Test ratio for second row}

Why do we calculate test ratios only for positive column entries? When we calculate test ratios for the $x$ column, we are actually determining the values of the $x$-intercepts of the constraints. If the column entry is a negative number, the $x$-intercept of the constraint will be to the left of the origin. However, we are constrained to nonnegative values for both $x$ and $y$. Therefore, determining the value of the $x$-intercepts is necessary only for positive values. (If the column entry is zero, the corresponding line is horizontal and doesn’t have an $x$-intercept.)

Recall that the pivot for this tableau is the 2 in the $x$ column and the first row.

\[
\begin{array}{cccccc}
 x & y & s & t & P \\
\hline
2 & 1 & 1 & 0 & 0 & 10 \\
1 & 1 & 0 & 1 & 0 & 8 \\
-6 & -2 & 0 & 0 & 1 & 0 \\
\end{array}
\]

We will use the pivot to zero out the remaining entries in the column. However, to ensure that all active variables remain nonnegative, row operations must be of the form

\[aR_i \pm bR_p \rightarrow R_i\] with $a > 0$ and $b > 0$
where \( R_r \) is the row to be changed and \( R_p \) is the row containing the pivot. Using the row operations
\[
2R_2 - R_1 \rightarrow R_2
\]
and
\[
R_3 + 3R_1 \rightarrow R_3
\]
we get the new tableau
\[
\begin{bmatrix}
2 & 1 & 1 & 0 & 0 & 10 \\
0 & 1 & -1 & 2 & 0 & 6 \\
0 & 1 & 3 & 0 & 1 & 30
\end{bmatrix}
\]
Columns \( x \) and \( t \) almost look like columns of the identity matrix; however, their nonzero entry is a 2 instead of a 1. Using the row operations
\[
\frac{1}{2}R_1 \rightarrow R_1
\]
and
\[
\frac{1}{2}R_2 \rightarrow R_2
\]
we get
\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 5 \\
0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 3 \\
0 & 1 & 3 & 0 & 1 & 30
\end{bmatrix}
\]
The three columns that look like columns from the \( 3 \times 3 \) identity matrix are \( x, t, \) and \( P \). Each of these columns has exactly one nonzero entry, so their corresponding variables are the active variables. The remaining columns, \( y \) and \( s \), have more than one nonzero entry, so their corresponding variables are the inactive variables. The system of equations represented by the tableau is
\[
\begin{align*}
x + \frac{1}{2}y + \frac{1}{2}s &= 5 \\
\frac{1}{2}y - \frac{1}{2}s + t &= 3 \\
y + 3s + P &= 30
\end{align*}
\]
Setting the inactive variables \( y \) and \( s \) to zero yields the system of equations
\[
\begin{align*}
x + \frac{1}{2}(0) + \frac{1}{2}(0) &= 5 \\
\frac{1}{2}(0) - \frac{1}{2}(0) + t &= 3 \\
1(0) + 3(0) + P &= 30
\end{align*}
\]
which simplifies to \( x = 5, t = 3, \) and \( P = 30 \). The inactive variables \( y \) and \( s \) are both zero. As noted earlier, a quick way to see the values of the active variables is to cross out the columns of the inactive variables and read the resultant values
of the active variables directly from the tableau.

\[
\begin{bmatrix}
x & y & s & t & P \\
1 & 2 & 1 & 0 & 5 \\
0 & 2 & -1 & 1 & 3 \\
0 & 1 & 2 & 0 & 10 \\
\end{bmatrix}
\]

Reading from the tableau, we have

\[
\begin{align*}
x &= 5 \\
t &= 3 \\
P &= 30
\end{align*}
\]

The inactive variables \(y\) and \(s\) are both equal to zero. Since \(x = 5\) and \(y = 0\), this solution corresponds with the corner point \((5, 0)\) of the feasible region. At \((5, 0)\), \(P = 30\).

If there were negative entries in the bottom row of the tableau, we would select a new pivot column and repeat the process. However, since the bottom row contains only nonnegative entries, we are done. The objective function is maximized at \((5, 0)\). This corroborates our earlier conclusion, which was based on Table 4.15.

The simplex method is not intuitive and takes some effort to learn. We will summarize the steps of the process and do several more examples to help you master the method.

### The Simplex Method

1. Verify that the linear programming problem is a standard maximization problem.
2. Add slack variables to convert the constraints into linear equations.
3. Rewrite the objective function in the form \(ax + by + \cdots + P = 0\), where \(a, b, \ldots\) are real numbers and \(x, y, \ldots\) are decision variables. (Make sure the coefficient on \(P\) is positive one.)
4. Set up the initial tableau from the system of equations generated in Steps 2 and 3. Be sure to put the objective function equation from Step 3 in the bottom row of the tableau.
5. Select the pivot column by identifying the negative entry in the bottom row with the largest magnitude.
6. Select the pivot by calculating test ratios for the positive entries in the pivot column and choosing the entry with the smallest test ratio. The test ratio of a row is calculated by dividing the last entry of the row by the entry in the pivot column of the row.
7. Use the pivot to zero out the remaining entries in the pivot column. All row operations must be of the form \(aR_i \pm bR_p \rightarrow R_i\), with \(a > 0\) and \(b > 0\), where \(R_i\) is the row to be changed and \(R_p\) is the pivot row. As needed, multiply a row by a positive number to convert the pivot to a 1.
8. If the bottom row contains all nonnegative entries, cross out the columns of the inactive variables and read the solution from the tableau. If the bottom row contains negative entries, return to Step 5 and repeat the process for the new tableau.
EXAMPLE 1

Using the Simplex Method to Solve a Standard Maximization Problem

Maximize $P = 4x + 3y$ subject to

\[
\begin{align*}
3x + 2y &\leq 12 \\
x + y &\leq 5 \\
x &\geq 0, y &\geq 0
\end{align*}
\]

\SOLUTION We confirm that the problem is a standard maximization problem. Converting the linear programming problem into a system of linear equations, we get

\[
\begin{align*}
3x + 2y + s &= 12 \\
x + y + t &= 5 \\
-4x - 3y + P &= 0
\end{align*}
\]

and the initial simplex tableau

\[
\begin{array}{cccccc}
& x & y & s & t & P \\
\hline
\text{row 1} & 3 & 2 & 1 & 0 & 0 & 12 \\
\text{row 2} & 1 & 1 & 0 & 1 & 0 & 5 \\
\text{row 3} & -4 & -3 & 0 & 0 & 1 & 0
\end{array}
\]

Since $-4$ is the negative number in the bottom row that has the largest magnitude, we select the $x$ column as the pivot column. Because the entries in the first and second rows of the pivot column are positive, we must calculate the test ratios.

\[
\begin{array}{cccccc}
& x & y & s & t & P \\
\hline
\text{row 1} & 3 & 2 & 1 & 0 & 0 & 12 \\
& 1 & 1 & 0 & 1 & 0 & 5 \\
& -4 & -3 & 0 & 0 & 1 & 0
\end{array}
\]

\[
\begin{array}{cccccc}
& x & y & s & t & P \\
\hline
\text{row 1} & 3 & 2 & 1 & 0 & 0 & 12 \\
& 1 & 1 & 0 & 1 & 0 & 5 \\
& -4 & -3 & 0 & 0 & 1 & 0
\end{array}
\]

Since the $x$ column is the pivot column, the test ratios give us the $x$-intercepts of the constraint equations, as shown in Figure 4.24.

\FIGURE 4.24
By picking the column entry with the smallest test ratio, we guarantee that our computations will yield a result that is in the feasible region. The pivot entry is the 3 in the first row and the $x$ column. We zero out the remaining entries in the pivot column with the indicated operations. The resultant tableau is

$$
\begin{array}{cccccc}
 x & y & s & t & P \\
\hline
 3 & 2 & 1 & 0 & 0 & 12 \\
 0 & 1 & -1 & 3 & 0 & 3 \\
 0 & -1 & 4 & 0 & 3 & 48 \\
\end{array}
$$

Further simplifying the tableau yields

$$
\begin{array}{cccccc}
 x & y & s & t & P \\
\hline
 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 4 \\
 0 & \frac{1}{3} & -\frac{1}{3} & 1 & 0 & 1 \\
 0 & -\frac{1}{3} & \frac{4}{3} & 0 & 1 & 16 \\
\end{array}
$$

We cross out the columns of the inactive variables and read the result:

$$
\begin{array}{cccccc}
 x & y & s & t & P \\
\hline
 1 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 4 \\
 0 & \frac{1}{3} & -\frac{1}{3} & 1 & 0 & 1 \\
 0 & -\frac{1}{3} & \frac{4}{3} & 0 & 1 & 16 \\
\end{array}
$$

$x = 4$, $t = 1$, and $P = 16$. The inactive variables $y$ and $s$ are all zero. The corner point $(4, 0)$ yields an objective function value of $P = 16$.

Observe that the fractional entries in the inactive variable columns of the tableau were eliminated when we set the inactive variables to zero. To avoid spending the time calculating fractional entries that will be eliminated anyway, an alternative approach is to cross out the inactive variable columns before converting any active variable column entries to a 1. We then read the resultant equations from the tableau and solve.

$$
\begin{array}{cccccc}
 x & y & s & t & P \\
\hline
 3 & 2 & 1 & 0 & 0 & 12 \\
 0 & -1 & 4 & 3 & 0 & 3 \\
 0 & -1 & 4 & 0 & 3 & 48 \\
\end{array}
$$

$3x = 12$, $3t = 3$, $3P = 48$

$x = 4$, $t = 1$, $P = 16$

Observe that this approach yields the same result and requires fewer computations.

The bottom row of the tableau is equivalent to the equation

$$-y + 4s + 3P = 48$$

$$3P = y - 4s + 48$$

$$P = \frac{1}{3}y - \frac{4}{3}s + 16$$
Observe that since the coefficient on the \( y \) term is positive, increasing the value of \( y \) while leaving the value of \( s \) unchanged will yield a value of \( P \) greater than 16. Consequently, the tableau

\[
\begin{array}{cccc|c}
 x & y & s & t & P \\
 3 & 2 & 1 & 0 & 12 \\
 0 & 1 & -1 & 3 & 3 \\
 0 & -1 & 4 & 0 & 3 \\
\end{array}
\]

is not the final simplex tableau. The negative value in the bottom row indicates that the value of \( P \) may be increased.

We will repeat the simplex method for this new tableau.

\[
\begin{array}{cccc|c}
 x & y & s & t & P \\
 3 & 2 & 1 & 0 & 12 \\
 0 & 1 & -1 & 3 & 3 \\
 0 & -1 & 4 & 0 & 3 \\
\end{array}
\]

The \( y \) column is the pivot column, since it is the column with a negative entry in the bottom row. Because the entries in the first and second rows of the pivot column are positive, we must calculate the test ratios.

\[
\begin{array}{cccc|c}
 x & y & s & t & P \\
 3 & 2 & 1 & 0 & 12 \\
 0 & 1 & -1 & 3 & 3 \\
 0 & -1 & 4 & 0 & 3 \\
\end{array}
\]

The 1 in the second row is the pivot. We simplify the tableau using the indicated row operations.

\[
\begin{array}{cccc|c}
 x & y & s & t & P \\
 3 & 0 & 3 & -6 & 6 \\
 0 & 1 & -1 & 3 & 3 \\
 0 & 0 & 3 & 3 & 48 \\
\end{array}
\]

\[ R_1 - 2R_2 \]

\[ R_3 + R_5 \]

\[ 3x = 6 \]

\[ y = 3 \]

\[ 3P = 51 \]

\[ x = 2 \]

\[ P = 17 \]

Because the bottom row of the tableau is entirely nonnegative, we do not need to repeat the pivoting process. The corner point (2, 3) yields a maximum objective function value of \( P = 17 \).

Observe that each new simplex tableau corresponds with a different corner point. The initial simplex tableau corresponded with (0, 0); the second tableau, with (4, 0); and the final tableau, with (2, 3), as shown in Figure 4.25.
The simplex method moves from a corner point to an adjacent corner point, each time increasing the value of the objective function. We can verify the accuracy of our result by calculating the value of the objective function at each corner point (see Table 4.19).

\[
\begin{array}{ccc}
\text{Corner Points} & \text{Objective Function} \\
\hline
x & y & P = 4x + 3y \\
\hline
0 & 0 & 0 \\
4 & 0 & 16 \\
2 & 3 & 17 \\
0 & 5 & 15 \\
\end{array}
\]

Using the Simplex Method to Make Business Decisions

A store can buy at most 60 packages of three different brands of cheese. The first brand costs $2 per package, the second brand costs $3 per package, and the third brand costs $4 per package. The store can spend up to $120 on cheese. The first brand resells for $5 per package, the second brand resells for $6 per package, and the third brand resells for $8 per package. How many packages of each brand of cheese should the store buy in order to maximize profit? (Note: Profit per item = revenue per item − cost per item.)

**SOLUTION** Let \( x \) be the number of packages of the first brand, \( y \) be the number of packages of the second brand, and \( z \) be the number of packages of the third brand. The total revenue from cheese sales is \( R = 5x + 6y + 8z \). The total cost of the cheese is \( C = 2x + 3y + 4z \). Since profit is the difference between revenue and cost,

\[
P = R - C \\
= (5x + 6y + 8z) - (2x + 3y + 4z) \\
= 3x + 3y + 4z
\]
Since we want to maximize profit, we must maximize $P = 3x + 3y + 4z$ subject to

\[ \begin{align*}
  x + y + z & \leq 60 \quad \text{There are at most 60 packages} \\
  2x + 3y + 4z & \leq 120 \quad \text{At most, $120$ is spent on cheese} \\
  x \geq 0, y \geq 0, z \geq 0 \quad \text{Nonnegative number of each type of cheese is purchased}
\end{align*} \]

The problem is a standard maximization problem. Converting the linear programming problem into a system of linear equations, we get

\[ \begin{align*}
  x + y + z + s &= 60 \quad \text{Package quantity constraint} \\
  2x + 3y + 4z + t &= 120 \quad \text{Budget constraint} \\
  -3x - 3y - 4z + P &= 0 \quad \text{Objective function}
\end{align*} \]

and the initial simplex tableau

\[
\begin{bmatrix}
  x & y & z & s & t & P \\
  1 & 1 & 1 & 1 & 0 & 60 \\
  2 & 3 & 4 & 0 & 1 & 120 \\
  -3 & -3 & -4 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Since $-4$ is the negative number in the bottom row that has the largest magnitude, the $z$ column is the pivot column. Because the entries in the first and second rows of the pivot column are positive, we must calculate the test ratios.

\[
\begin{bmatrix}
  x & y & z & s & t & P \\
  1 & 1 & 1 & 1 & 0 & 60 \\
  2 & 3 & 4 & 0 & 1 & 120 \\
  -3 & -3 & -4 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  60/1 = 60 \\
  120/4 = 30 \\
\end{bmatrix}
\]

The pivot entry is 4, since the second row had the smallest test ratio. We zero out the remaining entries in the pivot column with the indicated row operations. The resultant tableau is

\[
\begin{bmatrix}
  x & y & z & s & t & P \\
  2 & 1 & 0 & 4 & -1 & 120 \\
  2 & 3 & 4 & 0 & 1 & 120 \\
  -1 & 0 & 0 & 0 & 1 & 120 \\
\end{bmatrix}
\]

We could convert the nonzero entries in the active columns to a 1 by multiplying Rows 1 and 2 by $\frac{1}{4}$; however, that would result in messy fractional entries in the tableau. We will hold off on the reduction until we have obtained the final simplex tableau. Since there is still a negative value in the bottom row, we must again select a pivot column and entry. The pivot column $x$ has two positive entries, so we must compute the test ratios.

\[
\begin{bmatrix}
  x & y & z & s & t & P \\
  2 & 1 & 0 & 4 & -1 & 120 \\
  2 & 3 & 4 & 0 & 1 & 120 \\
  -1 & 0 & 0 & 0 & 1 & 120 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  120/2 = 60 \\
  120/2 = 60 \\
\end{bmatrix}
\]
Since both entries have the same test ratio, we may pick either entry to be the pivot. We select the entry in Row 1 as the pivot and zero out the remaining entries in the pivot column with the indicated row operations. The resultant tableau is
\[
\begin{array}{cccccc}
  x & y & z & s & t & P \\
 2 & 1 & 0 & 4 & -1 & 0 & 120 \\
 0 & 2 & 4 & -4 & 2 & 0 & 0 \\
 0 & 1 & 0 & 4 & 1 & 2 & 360 \\
\end{array}
\]
\[R_2 - R_1, \quad 2R_3 + R_1\]

Since the bottom row of the tableau is entirely nonnegative, it is unnecessary to repeat the pivot selection process. Crossing out the inactive variable columns and reading the result yields
\[
\begin{array}{cccccc}
  x & y & z & s & t & P \\
 2 & 1 & 0 & 4 & -1 & 0 & 120 \\
 0 & 2 & 4 & -4 & 2 & 0 & 0 \\
 0 & 1 & 0 & 4 & 1 & 2 & 360 \\
\end{array}
\]
\[2x = 120, \quad 4z = 0, \quad 2P = 360\]
\[x = 60, \quad z = 0, \quad P = 180\]

The corner point \((x, y, z)\) that maximizes the objective function is \((60, 0, 0)\). It is interesting to note that although \(z\) is an active variable, \(z = 0\). Active variables may equal zero, but inactive variables must equal zero.

In the context of the scenario, the store should order 60 packages of the first brand to earn the maximum profit of $180. Even though the profit per package for the third brand ($4 per package) was higher than the profit per package for the first brand ($3 per package), the higher wholesale cost of the third brand ($4 versus $2) limited the number of third-brand items available for sale. Ironically, the lowest-priced brand of cheese resulted in the greatest profit.

In Examples 1 and 2, there was a unique solution to the standard maximization problem; however, on occasion you’ll encounter a standard maximization problem with infinitely many solutions. Example 3 is such a case.

**Example 3**

**Using the Simplex Method to Make Business Decisions**

A store can purchase three brands of binders. The first brand, \(A\), costs $2 per binder. The second brand, \(B\), costs $1 per binder. The third brand, \(C\), costs $2 per binder. The store can spend up to $100 and purchase up to 50 binders. It resells binder \(A\) for $5 per binder, binder \(B\) for $3 per binder, and binder \(C\) for $5 per binder. Recall that profit per binder is revenue per binder minus cost per binder. How many of each type of binder should the store buy if it wants to maximize its profit?

**Solution**

Let \(x\) be the number of binders of the first brand, \(y\) be the number of binders of the second brand, and \(z\) be the number of binders of the third brand. The total revenue from binder sales is \(R = 5x + 3y + 5z\). The total cost of the binders is \(C = 2x + 1y + 2z\). Since profit is the difference between revenue and cost,

\[
P = R - C = (5x + 3y + 5z) - (2x + 1y + 2z) = 3x + 2y + 3z
\]
Since we want to maximize profit, we must maximize \( P = 3x + 2y + 3z \)
subject to

\[
\begin{align*}
  x + y + z &\leq 50 & \text{At most 50 binders are purchased} \\
  2x + y + 2z &\leq 100 & \text{At most $100 is spent on binders} \\
  x &\geq 0, y &\geq 0, z &\geq 0 & \text{A nonnegative number of each brand of binder is purchased}
\end{align*}
\]

The problem is a standard maximization problem. Converting the linear programming problem into a system of linear equations, we get

\[
\begin{align*}
  x + y + z + s &= 50 & \text{Binder quantity constraint} \\
  2x + y + 2z + t &= 100 & \text{Budget constraint} \\
  -3x - 2y - 3z + P &= 0 & \text{Objective function}
\end{align*}
\]

and the initial simplex tableau

\[
\begin{array}{ccccccc}
  x & y & z & s & t & P \\
  1 & 1 & 1 & 1 & 0 & 0 & 50 \\
  2 & 1 & 2 & 0 & 1 & 0 & 100 \\
  -3 & -2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Since \(-3\) is the negative number in the bottom row that has the largest magnitude, we may select either column \(x\) or column \(z\) as the pivot column. We select column \(z\). Because the entries in the first and second rows of the pivot column are positive, we must calculate the test ratios.

\[
\begin{array}{ccccccc}
  x & y & z & s & t & P \\
  1 & 1 & 1 & 1 & 0 & 0 & 50 \\
  2 & 1 & 2 & 0 & 1 & 0 & 100 \\
  -3 & -2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Since the entries have the same test ratio, we may select either one as the pivot. We select the entry in the first row. We zero out the remaining entries in the pivot column with the indicated row operations and obtain the tableau

\[
\begin{array}{ccccccc}
  x & y & z & s & t & P \\
  1 & 1 & 1 & 1 & 0 & 0 & 50 \\
  0 & -1 & 0 & -2 & 1 & 0 & 0 \\
  0 & 1 & 0 & 3 & 0 & 1 & 150 \\
\end{array}
\]

Crossing out the inactive variable columns and reading the resultant equations, we get

\[
\begin{align*}
  x + z &= 50 & t &= 0 & P &= 150 \\
  z &= -x + 50
\end{align*}
\]
There are infinitely many solutions \((x, y, z)\) that maximize the objective function. Each solution is of the form \((n, 0, -n + 50)\) with \(0 \leq n \leq 50\).

In the context of the real-world scenario, the store may purchase any combination of 50 binders from brands \(A\) and \(C\) and still maximize profits. This is because both brands have the same wholesale cost and the same retail price. As a result, the cost, revenue, and profit for each item are the same.

When a standard maximization problem has more than two constraints, additional slack variables will be needed. This will result in additional columns in the initial simplex tableau. We typically use the letters \(u, v,\) and so on for slack variables in addition to the letters \(s\) and \(t\). Even with additional rows or columns, the simplex method is effective in finding the optimal solution to any standard maximization problem.

**Example 4**

Clothing retailers typically mark up the price of apparel by 100 percent. That is, the retail price is typically 200 percent of the wholesale cost. WholesaleFashion.com claims to be the number one Internet business-to-business source for fashion retailers of women’s apparel and shoes. It sells directly to businesses instead of to consumers. In January 2003, WholesaleFashion.com offered a 3/4 Sleeves Stretch Top for $9.50, a Long Sleeve Turtle Neck Top for $12.50, a Heart Neck Tank Top for $9.25, and an American Flag Tank Top for $9.75. (Source: www.wholesalefashion.com.)

A women’s apparel store has up to $1000 to spend on the four items. It has enough rack space for up to 60 tops. It anticipates the demand for tank tops to be greater than the demand for the other tops, so it wants to order at least twice as many tank tops as sleeved tops. The store intends to mark up all of the items by 100 percent. Assuming that all of the items that the store orders will sell, how many of each item should it order if it wants to maximize revenue?

**Solution**

Let \(x\) be the number of 3/4 Sleeves Stretch Tops, \(y\) be the number of Long Sleeve Turtle Neck Tops, \(z\) be the number of Heart Neck Tank Tops, and \(w\) be the number of American Flag Tank Tops. The number of sleeved tops is \(x + y\), and the number of tank tops is \(z + w\). Since the number of tank tops is to be at least twice the number of sleeved tops, we have

\[
\begin{align*}
z + w & \geq 2(x + y) \\
z + w & \geq 2x + 2y \\
2x - 2y + z + w & \geq 0 \\
2x + 2y - z - w & \leq 0 \quad \text{Dividing by \(-1\) reverses the inequality sign}
\end{align*}
\]

We are asked to maximize the revenue from sales

\[
R = 19x + 25y + 18.50z + 19.50w
\]

subject to the constraints

\[
\begin{align*}
x + y + z + w & \leq 60 \quad \text{At most 60 tops are purchased} \\
2x + 2y - z - w & \leq 0 \quad \text{Twice as many tank tops are purchased} \\
9.5x + 12.5y + 9.25z + 9.75w & \leq 1000 \quad \text{At most $1000 is spent on tops} \\
x & \geq 0, y \geq 0, z \geq 0, w \geq 0 \quad \text{A nonnegative number of each type is purchased}
\end{align*}
\]
Rewriting the objective function and adding the slack variables to the constraint inequalities yields the system of equations
\[
\begin{align*}
\text{minimize:} & \quad x + y + z + w + s \\
\text{subject to:} & \quad 2x + 2y - z - w + t = 0 \\
& \quad 9.5x + 12.5y + 9.25z + 9.75w + u = 1000 \\
& \quad -19x - 25y - 18.5z - 19.5w + R = 0
\end{align*}
\]
and the associated initial simplex tableau
\[
\begin{array}{cccccccc}
 & x & y & z & w & s & t & u & R \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 60 \\
2 & 2 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
9.5 & 12.5 & 9.25 & 9.75 & 0 & 0 & 1 & 0 & 1000 \\
-19 & -25 & -18.5 & -19.5 & 0 & 0 & 0 & 1 & 0
\end{array}
\]

The \( y \) column is the pivot column, since \(-25\) is the negative entry in the bottom row with the greatest magnitude. Since the first three entries in the \( y \) column are positive, we must calculate three test ratios.

\[
\begin{array}{cccccccc}
 & x & y & z & w & s & t & u & R \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 60 \\
2 & 2 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
9.5 & 12.5 & 9.25 & 9.75 & 0 & 0 & 1 & 0 & 1000 \\
-19 & -25 & -18.5 & -19.5 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

The pivot entry is the 2 in the second column and second row, since the second row has the smallest test ratio. We zero out the remaining entries in the second column with the indicated operations.

Because of the numerical complexity of the entries in the tableau, we will use the Technology Tips following this example to do the row operations. The resultant tableau is

\[
\begin{array}{cccccccc}
 & x & y & z & w & s & t & u & R \\
0 & 0 & 3 & 3 & 2 & -1 & 0 & 0 & 120 \\
2 & 2 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
-6 & 0 & 31 & 32 & 0 & -12.5 & 2 & 0 & 2000 \\
12 & 0 & -62 & -64 & 0 & 25 & 0 & 2 & 0 \\
\end{array}
\]

The \( w \) column is the new pivot column, since \(-64\) is the negative entry in the bottom row that has the largest magnitude. Since the only positive entries in the \( w \) column are in the first and third rows, we need to calculate only two test ratios.

\[
\begin{array}{cccccccc}
 & x & y & z & w & s & t & u & R \\
0 & 0 & 3 & 2 & -1 & 0 & 0 & 120 \\
2 & 2 & -1 & -1 & 0 & 1 & 0 & 0 \\
-6 & 0 & 31 & 32 & 0 & -12.5 & 2 & 0 & 2000 \\
12 & 0 & -62 & -64 & 0 & 25 & 0 & 2 & 0 \\
\end{array}
\]
Since the entry in the second row of the pivot column was negative, we did not calculate a test ratio for the second row. The pivot entry is the 3 in the \( w \) column and first row. We will zero out the remaining entries in the \( w \) column with the indicated operations. The resultant tableau is

\[
\begin{bmatrix}
0 & 0 & 3 & 3 & 2 & -1 & 0 & 0 & 120 \\
6 & 6 & 0 & 0 & 2 & 2 & 0 & 0 & 120 \\
-18 & 0 & -3 & 0 & -64 & -5.5 & 6 & 0 & 2160 \\
36 & 0 & 6 & 0 & 128 & 11 & 0 & 6 & 7680 \\
\end{bmatrix}
\]

Since all entries in the bottom row are nonnegative, this is the final simplex tableau. Crossing out the inactive variable columns and reading the resultant equations, we get

\[
\begin{align*}
x & = 0 \\
y & = 20 \\
s & = 0 \\
t & = 0 \\
z & = 0 \\
w & = 40 \\
u & = 360 \\
R & = 1280
\end{align*}
\]

The maximum revenue that can be attained is $1280. This will be achieved when 20 Long Sleeve Turtle Neck Tops and 40 American Flag Tank Tops are purchased. Recall that \( u \) was the slack variable associated with the equation for the total cost of the tops. Since \( u = 360 \), the amount spent by the store is $360 less than the maximum amount allowed. That is, $640 is spent on the tops. The additional money remains unspent. It was the space constraint (a maximum of 60 tops), not the budget constraint (a maximum of $1000), that limited how many tops could be ordered.

For large tableaus or tableaus with noninteger entries, it is often helpful to do row operations on the calculator. The next four Technology Tips detail how to do this on the TI-83 Plus.

### Interchanging Two Rows

1. Enter a matrix or simplex tableau using the Matrix Editor. (Press 2nd \( \chi^- \) to access the Matrix Editor.)
2. Close the Matrix Editor to store the matrix, then reopen the Matrix Editor and move the cursor to the MATH menu.

3. Scroll to C: rowSwap( and press ENTER. This operation is used to interchange one row with another row.

4. Type in the matrix name from the Matrix Menu, the first row number, and the second row number. For example, \texttt{rowSwap([A],1,2)} is equivalent to \( R_1 \leftrightarrow R_2 \).

5. Press ENTER to display the new matrix or tableau.

---

**TECHNOLOGY TIP**

### Adding One Row to Another Row

1. Enter a matrix or simplex tableau using the Matrix Editor. (Press 2nd \( \times^{-1} \) to access the Matrix Editor.)
2. Close the Matrix Editor to store the matrix, then reopen the Matrix Editor and move the cursor to the MATH menu.

3. Scroll to D:row+ and press ENTER. This operation is used to add one row to another row.

4. Type in the matrix name from the Matrix Menu, the first row number, and the second row number. The result will be placed in the last row listed. For example, row+([A],2,3) is equivalent to \( R_2 + R_3 \rightarrow R_3 \).

5. Press ENTER to display the new matrix or tableau.

---

**TECHNOLOGY TIP**

**Multiplying a Row by a Nonzero Constant**

1. Enter a matrix or simplex tableau using the Matrix Editor. (Press 2nd \( x^{-1} \) to access the Matrix Editor.)
2. Close the Matrix Editor to store the matrix, then reopen the Matrix Editor and move the cursor to the MATH menu.

3. Scroll to \texttt{E:*row()} and press \texttt{(ENTER)}. This operation is used to multiply a row by a nonzero constant.

4. Type in the multiplier value, the matrix name from the Matrix Menu, and the row to be multiplied. The result will be placed in the last row listed. For example, \texttt{E:*row(-2, [A], 1)} means \(-2R_1 \rightarrow R_1\).

5. Press \texttt{(ENTER)} to display the new matrix or tableau.

---

**TECHNOLOGY TIP**

**Adding a Multiple of One Row to Another Row**

1. Enter a matrix or simplex tableau using the Matrix Editor. (Press \texttt{2nd} \texttt{x^{-1}} to access the Matrix Editor.)

(Continued)
2. Close the Matrix Editor to store the matrix, then reopen the Matrix Editor and move the cursor to the MATH menu.

3. Scroll to \texttt{F:*row+} and press ENTER. This operation is used to add a multiple of one row to another row.

4. Type in the multiplier value, the matrix name, the row to be multiplied, and the row to be added. The result will be placed in the last row listed. For example, \texttt{*row+(4,[A],2,3)} means \(4R_2 + R_3 \rightarrow R_3\).

5. Press ENTER to display the new matrix or tableau.

Recall that simplex method row operations must be of the form \(aR_2 \pm bR_3 \rightarrow R_3\), with both \(a\) and \(b\) positive. Alternatively, we may write this expression as \(\pm bR_2 + aR_3 \rightarrow R_3\). The latter form is more consistent with the calculator functionality and may make using the calculator easier.

**Example 5**

Using the Simplex Method to Make Business Decisions


A women’s apparel store has up to $1000 to spend on the four items. It will mark up the dresses 100 percent but mark up the shoes only 50 percent. The store anticipates that it will sell 80 percent of each type of dress at full price and sell the remaining 20 percent of each type of dress at its cost. It expects to sell 90 percent of each type of shoes at full price and the remaining 10 percent of each shoe type at its cost. It expects to sell a pair of shoes with each dress it sells. (The store will market a dress and shoes as a package.) Additionally, it may sell some shoes separately. How many of each item should the store order to maximize its profit?
**SOLUTION** Let \( x \) be the number of Long Halter Dresses, \( y \) be the number of Long Open Back Dresses, \( z \) be the number of pairs of Thai Silk Pump shoes, and \( w \) be the number of pairs of Glitter Evening High Heels.

The revenue generated by selling 80 percent of each type of dress at full retail price is given by

\[
R = 51.50(0.8x) + 43.50(0.8y)
= 41.20x + 34.80y
\]

The revenue generated by selling the remaining 20 percent of each type of dress at the wholesale cost is given by

\[
R = 25.75(0.2x) + 21.75(0.2y)
= 5.15x + 4.35y
\]

The combined revenue from dress sales is

- Dress revenue \( = (41.20x + 34.80y) + (5.15x + 4.35y) \) 
- \( = 46.35x + 39.15y \)

The retail price of each shoe variety is 150 percent of the wholesale price.

- Thai Silk Pumps \( = 1.5(17.75) \) 
- \( = 26.625 \)
- \( = 26.63 \quad \text{Rounded to the nearest cent} \)

- Glitter Evening High Heels \( = 1.5(17.50) \) 
- \( = 26.25 \)

The revenue generated by selling 90 percent of each shoe variety at full retail price is given by

\[
R = 26.63(0.9z) + 26.25(0.9w)
= 23.97z + 23.63w \quad \text{Rounded to the nearest cent}
\]

The revenue generated by selling 10 percent of each shoe variety at wholesale cost is

\[
R = 17.75(0.1z) + 17.50(0.1w)
= 1.78z + 1.75w \quad \text{Rounded to the nearest cent}
\]

The combined revenue from shoe sales is

- Shoe revenue \( = (23.97z + 23.63w) + (1.78z + 1.75w) \) 
- \( \approx 25.75z + 25.38w \)

The combined revenue from dresses and shoes is the sum of the dress revenue and the shoe revenue.

- Combined revenue \( = \) dress revenue + shoe revenue 
- \( \approx (46.35x + 39.15y) + (25.75z + 25.38w) \) 
- \( \approx 46.35x + 39.15y + 25.75z + 25.38w \)

The total cost of the dresses and shoes is

\[
C = 25.75x + 21.75y + 17.75z + 17.5w
\]
Since profit is the difference in revenue and cost,

\[ P = R - C \]

\[ = (46.35x + 39.15y + 25.75z + 25.38w) \]
\[ - (25.75x + 21.75y + 17.75z + 17.50w) \]
\[ = 20.60x + 17.40y + 8.00z + 7.88w \]

We need to order at least as many pairs of shoes as dresses, so

\[
\text{Number of dresses} \leq \text{number of pairs of shoes} \\
(x + y) \leq (z + w) \\
x + y - z - w \leq 0
\]

We are asked to maximize the profit \( P = 20.6x + 17.4y + 8.00z + 7.88w \) subject to the constraints

\[
x + y - z - w \leq 0 \quad \text{Quantity constraint} \\
25.75x + 21.75y + 17.75z + 17.5w \leq 1000 \quad \text{Budget constraint} \\
x \geq 0, y \geq 0, z \geq 0, w \geq 0
\]

Rewriting the objective function and adding the slack variables to the constraint inequalities yields the following system of equations:

\[
x + y - z - w + s = 0 \\
25.75x + 21.75y + 17.75z + 17.5w + t = 1000 \\
-20.60x - 17.40y - 8.00z - 7.88w + P = 0
\]

and the associated initial simplex tableau

\[
\begin{array}{cccccccc}
1 & 1 & -1 & -1 & 1 & 0 & 0 & P \\
25.75 & 21.75 & 17.75 & 17.50 & 0 & 1 & 0 & 1000 \\
-20.60 & -17.40 & -8.00 & -7.88 & 0 & 0 & 1 & 0
\end{array}
\]

The \( x \) column is the pivot column, since \(-20.60\) is the negative entry in the bottom row that has the largest magnitude. We must calculate test ratios for the first two rows.

\[
\begin{array}{cccccccc}
1 & 1 & -1 & -1 & 1 & 0 & 0 & P \\
25.75 & 21.75 & 17.75 & 17.50 & 0 & 1 & 0 & 1000 \\
-20.60 & -17.40 & -8.00 & -7.88 & 0 & 0 & 1 & 0
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & -1 & -1 & -1 & 1 & 0 & 0 & P \\
25.75 & 21.75 & 17.75 & 17.50 & 0 & 1 & 0 & 1000 \\
-20.60 & -17.40 & -8.00 & -7.88 & 0 & 0 & 1 & 0
\end{array}
\]

The pivot entry is the 1 in the \( x \) column and the first row. We will zero out the remaining entries in the first column with the indicated operations. Because of the numerical complexity of the entries in the tableau, we will use the Technology Tips described in this section to do the row operations.
The resulting tableau is

\[
\begin{bmatrix}
1 & 1 & -1 & -1 & 1 & 0 & 0 & | & 0 \\
-4 & 43.5 & 43.25 & -25.75 & 1 & 0 & 1000 \\
3.2 & -28.60 & -28.48 & 20.60 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\( -25.75R_1 + R_2 \)

\( 20.60R_1 + R_3 \)

(Note: We wrote the row operations in the alternative form \( \pm bR_y + aR_x \rightarrow R_x \).)

The \( z \) column is the new pivot column, since \(-28.60\) is the negative entry in the bottom row that has the largest magnitude. Since there is only one positive entry in the \( z \) column above the bottom row, that entry is the pivot. The pivot entry is the \( 43.5 \) in the \( z \) column and the second row. We will zero out the remaining entries in the \( z \) column with the following row operations written in the alternative form:

\[
R_2 + 43.5R_1 \rightarrow R_1 \\
28.6R_2 + 43.5R_3 \rightarrow R_3
\]

To do this on the calculator requires a sequence of operations. To perform the operation \( R_2 + 43.5R_1 \rightarrow R_1 \), we do the following to the tableau that is on the screen of our calculator:

- **Step 1:** *row(43.5, ans, 1)\)* Multiplies \( R_1 \) by \( 43.5 \)
- **Step 2:** ENTER Returns the modified matrix
- **Step 3:** row + (ans, 2, 1)\)* Adds the first two rows of the modified matrix
- **Step 4:** ENTER Completes the operation

To perform the operation \( 28.6R_2 + 43.5R_3 \rightarrow R_3 \), we do the following to the tableau that is on the screen of our calculator:

- **Step 1:** *row(43.5, ans, 3)\)* Multiplies \( R_3 \) by \( 43.5 \)
- **Step 2:** ENTER Returns the modified matrix
- **Step 3:** *row + (28.6, ans, 2, 3)\)* Multiplies \( R_3 \) by \( 28.6 \) and adds it to the modified \( R_3 \)
- **Step 4:** ENTER Completes the operation

The resultant tableau is

\[
\begin{bmatrix}
43.5 & 39.5 & 0 & -0.25 & 17.75 & 1 & 0 & 1000 \\
0 & -4 & 43.5 & 43.25 & -25.75 & 1 & 0 & 1000 \\
0 & 24.8 & 0 & -1.93 & 159.65 & 28.60 & 43.5 & 28600 \\
\end{bmatrix}
\]

The bottom row contains a negative entry in the \( w \) column, so we must repeat the pivoting process. Since the only positive entry in the pivot column is \( 43.25 \), it is the new pivot. The indicated row operations (in alternative form) will zero out the additional terms in the pivot column.

\[
0.25R_2 + 43.25R_1 \rightarrow R_1 \\
1.93R_2 + 43.25R_3 \rightarrow R_3
\]
To perform the operation $0.25R_2 + 43.25R_1 \rightarrow R_1$, we do the following to the tableau that is on the screen of our calculator:

Step 1: *row(43.25, ans, 1) Multiplies $R_1$ by 43.25
Step 2: ENTER Returns the modified matrix
Step 3: *row + (0.24, ans, 2, 1) Multiplies $R_2$ by 0.25 and adds it to the modified $R_1$
Step 4: ENTER Completes the operation

To perform the operation $1.93R_2 + 43.25R_3 \rightarrow R_3$, we do the following to the tableau that is on the screen of our calculator:

Step 1: *row(43.25, ans, 3) Multiplies $R_1$ by 43.25
Step 2: ENTER Returns the modified matrix
Step 3: *row + (1.93, ans, 2, 3) Multiplies $R_2$ by 1.93 and adds it to the modified $R_3$
Step 4: ENTER Completes the operation

The resultant simplex tableau is

\[
\begin{bmatrix}
1881.38 & 1707.38 & 10.88 & 0 & 761.25 & 43.5 & 0 & 43500 \\
0 & -4 & 43.5 & 43.25 & -25.75 & 1 & 0 & 1000 \\
0 & 1064.88 & 83.96 & 0 & 6855.17 & 1238.88 & 1881.38 & 1238880 \\
\end{bmatrix}
\]

Crossing out the inactive variable columns and reading the resultant equations, we get

\[
\begin{bmatrix}
1881.38 & 1707.38 & 10.88 & 0 & 761.25 & 43.5 & 0 & 43500 \\
0 & -4 & 43.5 & 43.25 & -25.75 & 1 & 0 & 1000 \\
0 & 1064.88 & 83.96 & 0 & 6855.17 & 1238.88 & 1881.38 & 1238880 \\
\end{bmatrix}
\]

Reading from the tableau, we see that

\[
\begin{align*}
1881.38x &= 43500 & y &= 0 & z &= 0 & 43.25w &= 1000 \\
x &= 23.12 \\
w &= 23.12 \\
s &= 0 & t &= 0 & 1881.38P &= 1238880 \\
P &= 658.50
\end{align*}
\]

Since it doesn’t make sense to talk about a fraction of a dress or pair of shoes, we will round the values to the nearest whole number. Rounding to whole numbers doesn’t guarantee the maximum whole number solution. However, if the solution isn’t the maximum whole number solution, it will be close to the optimal solution. We estimate that the maximum profit will be achieved when 23 Long Halter Dresses and 23 pairs of Glitter Evening High Heels are purchased. Since $t = 0$, the amount spent by the store appears to be exactly $1000. (Actually, the cost is $994.75, since we rounded the number of dresses and shoes.) The maximum profit is expected to be $658.50. (Since we rounded down the number of shoes and dresses, the exact profit is $655.04.)

Even though Example 5 required some rounding to make sense of the results, the numerical analysis provided some valuable input to help us make a sound business decision.
4.3 Summary

In this section, you learned the simplex method for solving standard maximization problems. You learned that this is the method of choice in solving problems with more than two decision variables.

4.3 Exercises

In Exercises 1–10, determine whether the problem is a standard maximization problem. If it isn’t, explain why.

1. Maximize $P = 9x + 8y$
   Subject to
   \[
   \begin{align*}
   2x + y &\leq 10 \\
   -x + y &\leq 1
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$

2. Maximize $P = -2x - 5y$
   Subject to
   \[
   \begin{align*}
   4x + 9y &\leq 10 \\
   -11x + y &\leq -21
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$

3. Maximize $P = 6xy$
   Subject to
   \[
   \begin{align*}
   6x + 7y &\leq 13 \\
   -8x - 4y &\leq 12
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$

4. Maximize $P = 4x - 2y + z$
   Subject to
   \[
   \begin{align*}
   2x + y + 4z &\leq 24 \\
   -2x + 3y + 3z &\leq 150
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$
   $z \geq 0$

5. Maximize $P = -1.2x - 2.8y + 4.3z$
   Subject to
   \[
   \begin{align*}
   3.2x + 1.5y + 7.4z &\leq 249.8 \\
   -2.7x + 3.4y + 3.9z &\leq 190.1
   \end{align*}
   \]

6. Maximize $P = 4x + 4y + 9z$
   Subject to
   \[
   \begin{align*}
   -6x - y + 4z &\leq -24 \\
   -9x - 1.5y + 6z &\leq 36
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$
   $z \geq 0$

7. Maximize $P = x + 9y + 8z$
   Subject to
   \[
   \begin{align*}
   -6x - y + 4z &\leq -24 \\
   -9x - 1.5y + 6z &\leq 36
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$
   $z \geq 0$

8. Maximize $P = 4x + 4y + 9z$
   Subject to
   \[
   \begin{align*}
   -3x - 2y + 4z &\leq -22 \\
   5x - 2.5y + 6z &\leq 36
   \end{align*}
   \]
   $x \geq 0$
   $y \geq 0$
   $z \geq 0$

9. Maximize $P = -8x + 2y - z$
   Subject to
   \[
   \begin{align*}
   z &\leq 4 \\
   2x - y &\leq 3 \\
   x + y + z &\leq 0
   \end{align*}
   \]

10. Maximize $P = 5x + 7z$
    Subject to
    \[
    \begin{align*}
    4x + 2y &\leq 16 \\
    4x - 2y &\leq 3 \\
    x \geq 0 \\
    y \geq 0
    \end{align*}
    \]

In Exercises 11–20, solve the standard maximization problems by using the simplex method. Check your answer by graphing the feasible region and calculating the value of the objective function at each of the corner points.

11. Maximize $P = 9x + 8y$
    Subject to
    \[
    \begin{align*}
    2x + y &\leq 10 \\
    -x + y &\leq 1
    \end{align*}
    \]
    $x \geq 0$
    $y \geq 0$
In Exercises 21–30, solve the standard maximization problems by using the simplex method.

21. Maximize \( P = x + y \)
Subject to
\[
\begin{align*}
2x + 3y + 2z &\leq 120 \\
-x + y + z &\leq 60
\end{align*}
\]
Subject to
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

22. Maximize \( P = 4x + 6y + 5z \)
Subject to
\[
\begin{align*}
2x + 3y + 3z &\leq 210 \\
x + y + z &\leq 100
\end{align*}
\]
Subject to
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

23. Maximize \( P = 10x + 6y + 12z \)
Subject to
\[
\begin{align*}
8x + 5y + 9z &\leq 360 \\
x + y + z &\leq 50
\end{align*}
\]
Subject to
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

24. Maximize \( P = x - 2y + 4z \)
Subject to
\[
\begin{align*}
6x - 5y + 10z &\leq 300 \\
2x + 5y + 2z &\leq 500
\end{align*}
\]
Subject to
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

25. Maximize \( P = 5x - 2y + 4z \)
Subject to
\[
\begin{align*}
8x + 5y + 9z &\leq 360 \\
x + y + z &\leq 50
\end{align*}
\]
Subject to
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

26. Maximize \( P = 4x + 4y - 10z \)
Subject to
\[
\begin{align*}
3x + 3y + 6z &\leq 42 \\
2x + y + z &\leq 8
\end{align*}
\]
Subject to
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]
27. Maximize \( P = -x - y + 10z \)
Subject to
\[
\begin{align*}
2x + 2y + 2z &\leq 14 \\
4x + 2y + 2z &\leq 16 \\
3x + 2y + 5z &\leq 23 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

28. Maximize \( P = x + 2y + 3z \)
Subject to
\[
\begin{align*}
x + z &\leq 20 \\
y + 2z &\leq 30 \\
x + 2y + 3z &\leq 60 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

29. Maximize \( P = x + y + z \)
Subject to
\[
\begin{align*}
x + z &\leq 0 \\
y + 2z &\leq 5 \\
x + 2y + 3z &\leq 8 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

30. Maximize \( P = x + 2y + 3z \)
Subject to
\[
\begin{align*}
x - z &\leq 20 \\
y + z &\leq 30 \\
x - 2y + z &\leq 40 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

In Exercises 31–36, set up and solve the standard maximization problem using the simplex method.

31. **Resource Allocation: Beverages** In the mood for a new beverage recipe? iDrink.com provides you with recipes for alcoholic and nonalcoholic drinks based on the ingredients you have on hand.

The Cranberry Cooler calls for 2 ounces of lemon-lime soda and 4 ounces of cranberry juice in addition to other ingredients. Nancy’s Party Punch calls for 32 ounces of cranberry juice and no lemon-lime soda. Jimmy Wallbanger calls for 6 ounces of lemon-lime soda and no cranberry juice. (Source: www.idrink.com.) The Cranberry Cooler recipe yields 1 serving, the Nancy’s Party Punch recipe yields 9 servings, and the Jimmy Wallbanger recipe yields 1 serving.

If you have 2 quarts (64 ounces) of lemon-lime soda and 1 gallon (128 ounces) of cranberry juice, at most how many drink servings can you make?

32. **Resource Allocation: Sandwiches** A plain hamburger requires one ground beef patty and a bun. A cheeseburger requires one ground beef patty, one slice of cheese, and a bun. A double cheeseburger requires two ground beef patties, two slices of cheese, and a bun.

Frozen hamburger patties are typically sold in packs of 12; hamburger buns, in packs of 8; and cheese slices, in packs of 24.

A family is in charge of providing burgers for a neighborhood block party. The family members have purchased 13 packs of buns, 11 packs of hamburger patties, and 3 packs of cheese slices. How many of each type of sandwich should the family prepare if its members want to maximize the number of burgers with cheese?

33. **Resource Allocation: Sandwiches** Repeat Exercise 32, except maximize the number of hamburger patties used.

34. **Resource Allocation: Sandwiches** Repeat Exercise 32, except maximize the number of double cheeseburgers.

35. **Battery Sales** AAA Alkaline Discount Batteries sells low-cost batteries to consumers. In August 2002, the company offered AA batteries at the following prices: 50-pack for $10.00, 100-pack for $18.00, and 600-pack for $96.00. (Source: www.aaaalkalinediscountbatteries.com.)

An electronics store owner wants to buy at most 1000 batteries and spend at most $175. She expects that she’ll be able to resell all of the batteries she orders for $4.00 per 4-pack. How many packs (50-packs, 100-packs, or 600-packs) should she order if she wants to maximize her revenue? What is her maximum revenue?

36. **Battery Sales** Repeat Exercise 35, except maximize profit. Assume that her only cost is the cost of the batteries. What is her maximum profit?

Exercises 37–38 are intended to expand your understanding of the simplex method.

37. Given the standard maximization problem

Maximize \( z = 4x + 4y \)
Subject to
\[
\begin{align*}
-x + y &\leq 4 \\
x + 2y &\leq 14 \\
5x + 2y &\leq 30
\end{align*}
\]

\( x \geq 0, y \geq 0 \)
4.4 Solving Standard Minimization Problems with the Dual

(a) Graph the feasible region and find the coordinates of the corner points.
(b) Solve the standard maximization problem using the simplex method.
(c) Find the feasible solution associated with each simplex tableau and label the point on the graph that is associated with each feasible solution.
(d) Explain what the simplex method does in terms of the graph of the feasible region.

38. Repeat Exercise 37; however, this time pick a different column to be the first pivot column. Compare and contrast your results with those of Exercise 37.

A food distributor has two production facilities: one in Portland, Oregon, and the other in Spokane, Washington. Subject to production limitations and market demands, what shipment plan will minimize the distributor’s shipment costs? Questions such as this may often be answered by solving a standard minimization problem.

In this section, we will introduce standard minimization problems and show how finding the maximum solution for a dual problem leads us to the minimum solution of the minimization problem.

STANDARD MINIMIZATION PROBLEM

A standard minimization problem is a linear programming problem with an objective function that is to be minimized. The objective function is of the form

\[ P = ax + by + cz + \cdots \]

where \( a, b, c, \ldots \) are real numbers and \( x, y, z, \ldots \) are decision variables.

The decision variables are constrained to nonnegative values. Additional constraints are of the form

\[ Ax + By + Cz + \cdots \geq M \]

where \( A, B, C, \ldots \) are real numbers and \( M \) is nonnegative.

We are told that \( M \) is nonnegative. That is, \( M \geq 0 \). Is the origin in the feasible region? Substituting the origin \( x = 0, y = 0, z = 0, \ldots \) into each constraint inequality \( Ax + By + Cz + \cdots \geq M \) yields the inequality \( A(0) + B(0) + C(0) + \cdots \geq M \) which simplifies to \( 0 \geq M \). Since \( M \) is nonnegative, the inequality \( 0 \geq M \) is satisfied if and only if \( M = 0 \). That is, the origin is in the feasible region if and only if \( M = 0 \) for every constraint. If \( M \neq 0 \) for one or more constraints, the origin is not in the feasible region.
Determining If a Linear Programming Problem Is a Standard Minimization Problem

Determine if the linear programming problem is a standard minimization problem.

\[
\text{Minimize } P = 2x + 3y \\
\text{Subject to } \begin{cases} 
3x + y \geq 6 \\
2x - y \geq 4 \\
x \geq 0, y \geq 0
\end{cases}
\]

**SOLUTION** The objective function is to be minimized, the constraints are of the form \(Ax + By + Cz + \cdots \geq M\) with \(M\) nonnegative, and the decision variables are nonnegative. Therefore, the problem is a standard minimization problem.

Determining If a Linear Programming Problem Is a Standard Minimization Problem

Determine if the linear programming problem is a standard minimization problem.

\[
\text{Minimize } P = 4x + 2y \\
\text{Subject to } \begin{cases} 
-3x + 7y \leq -9 \\
7x + 5y \geq 4 \\
x \geq 0, y \geq 0
\end{cases}
\]

**SOLUTION** At first glance, the problem doesn’t look like a standard minimization problem. The first constraint has a less than or equal sign, and the constant term is negative. However, if we multiply the first constraint by \(-1\), it becomes \(3x - 7y \geq 9\). The problem may then be rewritten as

\[
\text{Minimize } P = 4x + 2y \\
\text{Subject to } \begin{cases} 
3x - 7y \geq 9 \\
7x + 5y \geq 4 \\
x \geq 0, y \geq 0
\end{cases}
\]

Therefore, the linear programming problem is a standard minimization problem.

When setting up a linear programming problem, if we make sure that the constant term of each constraint is nonnegative, then we will be able to readily see if the problem is a standard maximization or minimization problem.

The Dual

For a standard minimization problem whose objective function has nonnegative coefficients, we may construct a standard maximization problem called the **dual problem**. By solving the dual problem, we can find the solution to the standard minimization problem. Consider the standard minimization problem,

\[
\text{Minimize } P = 6x + 5y \\
\text{Subject to } \begin{cases} 
x + y \geq 2 \\
2x + y \geq 3 \\
x \geq 0, y \geq 0
\end{cases}
\]
We first construct a matrix for the problem as shown.

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3 \\
6 & 5 & 1
\end{bmatrix}
\]

\[
x + y \geq 2 \\
2x + y \geq 3 \\
6x + 5y = P
\]

The constraints are placed in the first two rows of the matrix, and the coefficients of the objective function are placed in the last row of the matrix. The matrix for the dual problem is found by transposing the matrix. The transpose of a matrix \( A \), which is written \( A^T \), is created by switching the rows and columns of \( A \). In this case, the first row of \( A \) is \([1 \ 1 \ 2] \), so the first column of \( A^T \) is \([1 \ 2] \). Repeating the process for the additional columns of \( A^T \) yields \( A^T = \begin{bmatrix}
1 & 2 & 6 \\
1 & 1 & 5 \\
2 & 3 & 1
\end{bmatrix} \). We construct the dual problem from \( A^T \). We first extract the constraints and the objective function from the matrix.

\[
A = \begin{bmatrix}
1 & 2 & 6 \\
1 & 1 & 5 \\
2 & 3 & 1
\end{bmatrix}
\]

\[
x + 2y \leq 6 \\
x + y \leq 5 \\
2x + 3y = P
\]

The dual problem is

\[
\text{Maximize} \quad P = 2x + 3y
\]

\[
\text{Subject to} \quad \begin{cases}
x + 2y \leq 6 \\
x + y \leq 5 \\
x \geq 0, y \geq 0
\end{cases}
\]

How are the optimal values of the standard minimization problem and its dual problem related? Let’s look at the graphs of the feasible regions (Figures 4.26 and 4.27).
For each problem, we calculate the value of the objective function at each corner point (see Tables 4.20 and 4.21).

<table>
<thead>
<tr>
<th>TABLE 4.20 Standard Minimization Problem</th>
<th>TABLE 4.21 Standard Maximization Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

Observe that the minimum value of the minimization problem objective function is 11. Similarly, the maximum value of the maximization problem objective function is 11. This leads us to the Fundamental Principle of Duality.

**FUNDAMENTAL PRINCIPLE OF DUALITY**

A standard minimization problem has a solution if and only if its dual problem has a solution. If a solution exists, the optimal value of the standard minimization problem and the optimal value of the dual problem are equal.

Although we already know the optimal value of the dual problem, we will demonstrate how using the simplex method to find the maximum value of the dual problem helps us determine the minimum value of the standard minimization problem. Recall that the dual problem is

Maximize $P = 2x + 3y$

Subject to

\[
\begin{align*}
x + 2y &\leq 6 \\
x + y &\leq 5 \\
x &\geq 0, y &\geq 0
\end{align*}
\]

Adding in the slack variables and rewriting the objective function yield the system of equations

\[
\begin{align*}
x + 2y + s & = 6 \\
x + y + t & = 5 \\
-2x - 3y & + P = 0
\end{align*}
\]

and the corresponding initial simplex tableau

\[
\begin{bmatrix}
1 & 2 & 1 & 0 & 0 & | & 6 \\
1 & 1 & 0 & 1 & 0 & | & 5 \\
-2 & -3 & 0 & 0 & 1 & | & 0
\end{bmatrix}
\]
The \(y\) column is the pivot column, since it contains the negative entry in the bottom row with the largest magnitude. We calculate the test ratios and identify the pivot.

\[
\begin{array}{cccccc}
x & y & s & t & P \\
1 & 2 & 1 & 0 & 0 & 6 \\
1 & 0 & 1 & 1 & 0 & 5 \\
-2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}
\]

The 2 in the first row of the \(y\) column is the pivot, since the first row has the smallest test ratio. We zero out the remaining terms in the \(y\) column using the indicated operations.

\[
\begin{array}{cccccc}
x & y & s & t & P \\
1 & 2 & 1 & 0 & 0 & 6 \\
1 & 0 & -1 & 2 & 0 & 4 \\
-1 & 0 & 3 & 0 & 2 & 18 \\
\end{array}
\]

\(2R_2 - R_1\)

\(2R_3 + 3R_1\)

The \(x\) column is the new pivot column, since it is the only column with a negative entry in the bottom row. We calculate the test ratios and identify the pivot.

\[
\begin{array}{cccccc}
x & y & s & t & P \\
1 & 2 & 1 & 0 & 0 & 6 \\
0 & 0 & -1 & 2 & 0 & 4 \\
-1 & 0 & 3 & 0 & 2 & 18 \\
\end{array}
\]

The 1 in the second row of the \(x\) column is the pivot, since the second row has the smallest test ratio. We zero out the remaining entries in the \(x\) column with the indicated operations.

\[
\begin{array}{cccccc}
x & y & s & t & P \\
0 & 2 & 2 & -2 & 0 & 2 \\
1 & 0 & -1 & 2 & 0 & 4 \\
0 & 0 & 2 & 2 & 2 & 12 \\
\end{array}
\]

\(R_1 - R_2\)

\(R_3 + R_2\)

We must perform one additional step to make the nonzero term in each active column equal 1.

\[
\begin{array}{cccccc}
x & y & s & t & P \\
0 & 1 & 1 & -1 & 0 & 1 \\
1 & 0 & -1 & 2 & 0 & 4 \\
0 & 0 & 1 & 1 & 1 & 11 \\
\end{array}
\]

\(1/2 R_1\)

\(1/2 R_3\)

Reading from the tableau, we have

\[
x = 4 \quad y = 1 \quad P = 11
\]

The corner point \((4, 1)\) yields the maximum value of the objective function of the dual problem \(P = 2x + 3y\).

Ironically, the optimal solution of the standard minimization problem is also nested in the final simplex tableau. With all active variable columns containing only 1s and 0s, the values in the bottom row of the \(s\) and \(t\) columns correspond with the \(x\) and \(y\) values, respectively, of the optimal solution of the standard minimization problem. That is, \(x = 1\) and \(y = 1\). The optimal value is the same as that of the dual problem: \(P = 11\). We can verify that the corner point \((1, 1)\) of the standard minimization problem yields an objective function value of 11 by
substituting the point into the objective function of the standard minimization problem \( P = 6x + 5y \).

\[
P = 6x + 5y \\
11 = 6(1) + 5(1) \\
11 = 11
\]

The relationship between the standard minimization problem and its dual problem is somewhat remarkable, and the problem-solving method is straightforward.

### Using Duals to Solve Standard Minimization Problems

1. Find the dual standard maximization problem.
2. Use the simplex method to solve the maximization problem.
3. The maximum value of the objective function of the dual problem is the minimum value of the objective function of the minimization problem.
4. The optimum solution of the minimization problem is given in the entries of the bottom row of the final tableau corresponding with the columns of the slack variables (as long as the entry in the \( P \) column equals 1).

### Example 3: Solving a Standard Minimization Problem

Solve the standard minimization problem.

Minimize \( P = 4x + 5y + 6z \)

Subject to

\[
\begin{align*}
x + y + z & \geq 3 \\
2x - z & \geq 0 \\
x & \geq 0, y & \geq 0, z & \geq 0
\end{align*}
\]

**Solution** We begin by constructing a matrix from the constraints and objective function.

\[
A = \begin{bmatrix}
1 & 1 & 1 & 3 \\
2 & 0 & -1 & 0 \\
4 & 5 & 6 & 1
\end{bmatrix}
\]

Next we find \( A^T \) and extract the objective function and constraints. Note that the constraints are of the form \( ax + by \leq M \) with \( M \) nonnegative.

\[
A^T = \begin{bmatrix}
1 & 2 & 4 \\
1 & 0 & 5 \\
1 & -1 & 6 \\
3 & 0 & 1
\end{bmatrix}
\]

\[
\begin{align*}
x + 2y & \leq 4 \\
x + 0y & \leq 5 \\
x - y & \leq 6 \\
3x + 0y & = P
\end{align*}
\]
The dual problem is

Maximize \[ P = 3x \]
Subject to
\[
\begin{align*}
  x + 2y & \leq 4 \\
  x & \leq 5 \\
  x - y & \leq 6 \\
  x & \geq 0, y \geq 0, z \geq 0 
\end{align*}
\]

The dual is a standard maximization problem. We solve the problem using the simplex method. We begin by adding slack variables to the constraints.

\[
\begin{align*}
  x + 2y + s &= 4 \\
  x + t &= 5 \\
  x - y + u &= 6 \\
  x & \geq 0, y \geq 0, z \geq 0 
\end{align*}
\]

Rewriting the objective function yields \(-3x + P = 0\). The initial simplex tableau is

\[
\begin{array}{cccccc|c}
 & x & y & s & t & u & P \\
 \hline
 1 & 2 & 1 & 0 & 0 & 0 & 4 \\
 1 & 0 & 0 & 1 & 0 & 0 & 5 \\
 1 & -1 & 0 & 0 & 1 & 0 & 6 \\
 -3 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

The \(x\) column is the pivot column, since it is the only column with a negative entry in the bottom row. Mentally calculating the test ratios, we see that the entry in the first row and the \(x\) column is the pivot.

The final tableau is obtained by using the indicated row operations.

\[
\begin{array}{cccccc|c}
 & x & y & s & t & u & P \\
 \hline
 1 & 2 & 1 & 0 & 0 & 0 & 4 \\
 0 & -2 & -1 & 1 & 0 & 0 & 1 \quad R_2 - R_1 \\
 0 & -3 & -1 & 0 & 1 & 0 & 2 \quad R_3 - R_1 \\
 0 & 6 & 3 & 0 & 0 & 1 & 12 \quad R_4 + 3R_1 \\
\end{array}
\]

Reading from the tableau, we see that the objective function of the maximization problem has a maximum value of 12. This occurs when \(x = 4\) and \(y = 0\). According to the fundamental principle of duality, the minimum value of the objective function of the minimization problem is also 12. The solution to the minimization problem may be found by looking at the slack variable columns in the bottom row of the final tableau. The \(x\), \(t\), and \(u\) columns of the final simplex tableau are the \(x\), \(y\), and \(z\) values of the solution to the minimization problem. That is, \((3, 0, 0)\) yields an objective function value of 12. This can be verified by substituting \(x = 3, y = 0,\) and \(z = 0\) into \(P = 4x + 5y + 6z\).

\[
\begin{align*}
P &= 4(3) + 5(0) + 6(0) \\
&= 12
\end{align*}
\]

It is important to note that a standard minimization problem and its dual problem may have different numbers of decision variables. In Example 3, the standard minimization problem had three decision variables, \(x, y,\) and \(z\), whereas the dual problem had only two decision variables, \(x\) and \(y\).
EXAMPLE 4

Solving a Standard Minimization Problem

Solve the standard minimization problem.

Minimize

\[
P = 2x + y + 2z
\]

Subject to

\[
\begin{align*}
x + 2y + z & \geq 6 \\
3y + 2z & \geq 12 \\
x & \geq 0, \; y & \geq 0, \; z & \geq 0
\end{align*}
\]

**SOLUTION**

We have

\[
A = \begin{bmatrix}
1 & 2 & 1 & 6 \\
0 & 3 & 2 & 12 \\
2 & 1 & 2 & 1
\end{bmatrix}
\]

\[
x + 2y + z \geq 6 \\
3y + 2z \geq 12 \\
2x + y + 2z = P
\]

Therefore,

\[
A^T = \begin{bmatrix}
1 & 0 & 2 \\
2 & 3 & 1 \\
1 & 2 & 2 \\
6 & 12 & 1
\end{bmatrix}
\]

\[
x \leq 2 \\
2x + 3y \leq 1 \\
x + 2y \leq 2 \\
6x + 12y = P
\]

The dual problem is

Maximize

\[
P = 6x + 12y
\]

Subject to

\[
\begin{align*}
x & \leq 2 \\
2x + 3y & \leq 1 \\
x + 2y & \leq 2 \\
x & \geq 0, \; y & \geq 0
\end{align*}
\]

Adding the slack variables and rewriting the objective function yields

\[
x + s = 2 \\
2x + 3y + t = 1 \\
x + 2y + u = 2 \\
-6x - 12y + P = 0
\]

The initial simplex tableau is

\[
\begin{bmatrix}
x & y & s & t & u & P \\
1 & 0 & 1 & 0 & 0 & 0 | 2 \\
2 & 3 & 0 & 1 & 0 & 0 | 1 \\
1 & 2 & 0 & 0 & 1 & 0 | 2 \\
-6 & -12 & 0 & 0 & 0 & 1 | 0
\end{bmatrix}
\]

The \( y \) column is the pivot, since it contains the negative entry in the bottom row with the largest magnitude. The pivot is the 3 in the second row and the \( y \) column because the second row’s test ratio of \( \frac{1}{3} \) is smaller than the third row’s test ratio of \( \frac{2}{3} \). Using the indicated row operations yields
To obtain the final simplex tableau, we must ensure that the nonzero entry of each active variable column is equal to 1.

\[
\begin{bmatrix}
  x & y & s & t & u & P \\
  1 & 0 & 1 & 0 & 0 & 2 \\
  2 & 3 & 0 & 1 & 0 & 1 \\
 -1 & 0 & 0 & -2 & 3 & 0 & 4 \\
  2 & 0 & 0 & 4 & 0 & 1 & 4 \\
\end{bmatrix}
\]

\[3R_1 - 2R_2\]
\[R_4 + 4R_3\]

The solution to the dual problem is \( x = 0 \) and \( y = \frac{1}{3} \). The optimal value is 4.

From the bottom row of the final tableau, we read the solution of the minimization problem from the slack variable columns: \( x = 0 \), \( y = 4 \), and \( z = 0 \). The minimal value of the objective function is 4. This can be verified by substituting the solution into the objective function equation of the standard minimization problem.

\[
P = 2x + y + 2z
\]
\[
= 2(0) + (4) + 2(0)
\]
\[
= 4
\]

Substituting the point \((0, 4, 0)\) back into the constraints of the standard minimization problem yields

\[
x + 2y + z \geq 6 \quad 3y + 2z \geq 12
\]
\[
(0) + 2(4) + (0) \geq 6 \quad 3(4) + 2(0) \geq 12
\]
\[
8 \geq 6 \quad 12 \geq 12
\]

We are further convinced that we have found the optimal solution, since the solution satisfies both constraints (as it should). Also, \( 3y + 2z \) was made as small as the constraint would allow.

**Example 5**

**Using a Standard Minimization Problem to Minimize Shipping Costs**

A northwest region canned food distributor has two production facilities: one in Spokane, Washington, and one in Portland, Oregon. For two supermarkets, the Portland facility produces a minimum of 700 cases of canned food per week and the Spokane facility produces a minimum of 300 cases per week. A Seattle-area supermarket requires at least 600 cases per week, and a supermarket in Ellensburg, Washington, requires at least 400 cases per week. Shipping costs (based on a rate of $0.36 per mile) vary based on the point of origin and destination as shown in Table 4.22.
What shipping schedule will minimize the food distributor’s shipping costs?

**SOLUTION**

Let \( x \) be the number of cases shipped from Portland to Seattle.
Let \( y \) be the number of cases shipped from Spokane to Seattle.
Let \( z \) be the number of cases shipped from Portland to Ellensburg.
Let \( w \) be the number of cases shipped from Spokane to Ellensburg.

We need to minimize the shipment cost function,

\[
C = 0.62x + 1.01y + 0.80z + 0.63w.
\]

We have the following constraints:

- \( x + z \geq 700 \) The Portland facility produces at least 700 cases
- \( y + w \geq 300 \) The Spokane facility produces at least 300 cases
- \( x + y \geq 600 \) The Seattle-area supermarket needs at least 600 cases
- \( z + w \geq 400 \) The Ellensburg supermarket needs at least 400 cases
- \( x \geq 0, y \geq 0, z \geq 0, w \geq 0 \) The number of cases shipped is nonnegative

This is a standard minimization problem. We will solve it by using the dual and the simplex method. We have

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 700 \\
0 & 1 & 0 & 1 & 300 \\
1 & 1 & 0 & 0 & 600 \\
0 & 0 & 1 & 1 & 400 \\
0.62 & 1.01 & 0.80 & 0.63 & 1
\end{bmatrix}
\]

Thus

\[
A^T = \begin{bmatrix}
1 & 0 & 1 & 0 & 0.62 \\
0 & 1 & 1 & 0 & 1.01 \\
1 & 0 & 0 & 1 & 0.80 \\
0 & 1 & 0 & 1 & 0.63 \\
700 & 300 & 600 & 400 & 1
\end{bmatrix}
\]

The dual problem is

\[
\text{Maximize } P = 700x + 300y + 600z + 400w
\]

\[
\begin{align*}
x + z & \leq 0.62 \\
y + z & \leq 1.01 \\
x + w & \leq 0.80 \\
y + w & \leq 0.63
\end{align*}
\]

Subject to

\[
x \geq 0, y \geq 0, z \geq 0, w \geq 0
\]
The initial tableau is given by

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0.62 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1.01 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0.80 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0.63 \\
-700 & -300 & -600 & -400 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

The pivot is the 1 in the first row of the \( x \) column of the tableau. The second tableau is generated with the indicated row operations. The 1 in the third row of the \( w \) column is the new pivot.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0.62 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1.01 \\
0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0.18 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0.63 \\
0 & -300 & 100 & -400 & 700 & 0 & 0 & 0 & 1 & 434
\end{bmatrix}
\]

R\(_3\) - R\(_1\)

R\(_5\) + 700R\(_1\)

The third tableau is generated with the indicated row operations. We may choose either the \( y \) column or the \( z \) column to be the new pivot column. We pick the 1 in the fourth row of the \( y \) column to be the new pivot.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0.62 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1.01 \\
0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0.18 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0.45 \\
0 & -300 & -300 & 0 & 300 & 0 & 400 & 0 & 1 & 506
\end{bmatrix}
\]

R\(_4\) - R\(_3\)

R\(_5\) + 400R\(_3\)

The final simplex tableau is generated with the indicated row operations.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0.62 \\
0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0.56 \\
0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0.18 \\
0 & 1 & 1 & 0 & 1 & 0 & -1 & 1 & 0.45 \\
0 & 0 & 0 & 0 & 600 & 0 & 100 & 300 & 1 & 641
\end{bmatrix}
\]

R\(_5\) + 300R\(_4\)

The optimal solution of the dual problem is \( x = 0.62 \), \( y = 0.45 \), \( z = 0 \), and \( w = 0.18 \) with optimal value \( P = 641 \). We read the optimal solution for the minimization problem from the bottom row of the final tableau. When \( x = 600 \), \( y = 0 \), \( z = 100 \), and \( w = 300 \), the optimal value of 641 is obtained. In the context of the problem, the total shipping cost is minimized when 600 cases are shipped from Portland to Seattle, no cases are shipped from Spokane to Seattle, 100 cases are shipped from Portland to Ellensburg, and 300 cases are shipped from Spokane to Ellensburg. The total shipping cost is $641. (Note: This result assumes that the minimum number of cases were produced at each facility.)
EXAMPLE 6

Using a Standard Minimization Problem to Minimize Training Costs

Many companies with large government contracts use a competitive bidding process to hire subcontractors to do much of the work. A company has a contract to create at least 1000 hours of training and plans to hire three subcontractors (Trainum, Teachum, and Schoolum) to help with the work. Each subcontractor requires a contract for at least 200 hours of training development. Trainum charges $250 per hour of training development, Teachum charges $300 per hour of training development, and Schoolum charges $250 per hour of training development. How many hours should the company allocate to each subcontractor in order to minimize costs?

SOLUTION

Let \( x \) be the number of hours allocated to Trainum.
Let \( y \) be the number hours allocated to Teachum.
Let \( z \) be the number of hours allocated to Schoolum.

We need to minimize the training development cost function,

\[
C = 250x + 300y + 250z,
\]

subject to the following constraints:

\[
\begin{align*}
x + y + z & \geq 1000 & \text{At least 1000 training hours are produced} \\
x & \geq 200 & \text{Trainum is allocated at least 200 hours} \\
y & \geq 200 & \text{Teachum is allocated at least 200 hours} \\
z & \geq 200 & \text{Schoolum is allocated at least 200 hours}
\end{align*}
\]

This is a standard minimization problem. We will solve it by using the dual and the simplex method.

The dual problem is

Maximize \( P = 1000x + 200y + 200z + 200w \)

Subject to

\[
\begin{align*}
x + y & \leq 250 \\
x + z & \leq 300 \\
x + w & \leq 250 \\
x & \geq 0, y & \geq 0, z & \geq 0, w & \geq 0
\end{align*}
\]

The initial simplex tableau is

\[
\begin{bmatrix}
x & y & z & w & s & t & u & P \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 250 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 300 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 250 \\
-1000 & -200 & -200 & -200 & 0 & 0 & 0 & 250
\end{bmatrix}
\]

The pivot column is the \( x \) column, since \(-1000\) is the negative entry in the bottom row with the largest magnitude. The test ratios of the first and third rows are both equal to 250. Since this is smaller than the test ratio for the second row...
(300), either the first or the third entry in the $x$ column may be selected as the pivot. We pick the first entry. Using the indicated row operations, we obtain the second tableau.

$$
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 250 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 50 \\
0 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & 800 & -200 & -200 & 1000 & 0 & 0 & 1 & 250000 \\
\end{array}
\quad R_2 - R_1
\quad R_3 - R_1
\quad R_4 + 1000R_1
$$

Since the magnitude of the negative entries in the bottom row is the same, we may pick either the $z$ column or the $w$ column to be the pivot column. We pick the $w$ column. The pivot is the 1 in the third row. Using the indicated row operation, we get the third tableau. The new pivot is the entry in the second row of the $z$ column.

$$
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 250 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 50 \\
0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 600 & -200 & 0 & 800 & 0 & 200 & 1 & 250000 \\
\end{array}
\quad R_4 + 200R_3
$$

Using the indicated row operation, we obtain the final tableau.

$$
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 250 \\
0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 50 \\
0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 400 & 0 & 0 & 600 & 200 & 200 & 1 & 260000 \\
\end{array}
\quad R_4 + 200R_3
$$

The optimal solution for the dual problem is $x = 250$, $y = 0$, $z = 50$, and $w = 0$. The optimal value is 260,000.

An optimal solution for the minimization problem is $x = 600$, $y = 200$, and $z = 200$ with optimal value 260,000. In the context of the problem, the training development costs are minimized if Trainum develops 600 hours of training, Teachum develops 200 hours of training, and Schoolum develops 200 hours of training. The minimum training costs are $260,000. Since the hourly training development costs for Trainum and Schoolum are the same, another optimal solution is $(200, 200, 600)$. In fact, any point of the form $(t, 200, 800 - t)$ for $0 \leq t \leq 800$ will yield an optimal solution.

As shown in Example 6, a standard minimization problem may have more than one optimal solution. Each solution, however, will generate the same optimal value. The method of using the dual guarantees that we will find an optimal solution, if there is one.

**Using a Standard Minimization Problem to Minimize Food Costs**

PETsMART.com, an online retailer, allows pet owners to quickly compare nutrition and pricing for different brands of pet food sold on its web site. A June 2003 query revealed the data given in Table 4.23.
A dog breeder wants to create at least 300 pounds of a dog food mix that is at least 26 percent protein and 4 percent fiber while minimizing dog food cost. How many 20-pound bags of each type of dog food should the breeder buy?

**SOLUTION**

Let \( x \) be the number of bags of the Eukanuba brand.
Let \( y \) be the number of bags of the Nutro brand.
Let \( z \) be the number of bags of the Science Diet brand.

To make computations easier, we will convert the percentages per bag to pounds per bag by multiplying the percentage per bag by 20 pounds and dividing by 100, as shown in Table 4.24.

### TABLE 4.24

<table>
<thead>
<tr>
<th>Brand</th>
<th>Protein (pounds)</th>
<th>Fiber (pounds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eukanuba Adult Maintenance Formula</td>
<td>5.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Nutro Natural Choice Plus</td>
<td>5.4</td>
<td>0.6</td>
</tr>
<tr>
<td>Science Diet Active Formula Canine Maintenance</td>
<td>5.3</td>
<td>0.7</td>
</tr>
</tbody>
</table>

The total weight of the mix is \( 20x + 20y + 20z \) pounds.
The total amount of protein in the mix is \( 5.0x + 5.4y + 5.3z \) pounds. We want the mix to be at least 26 percent protein. Thus we require that

\[
0.26(20x + 20y + 20z) \leq 5.0x + 5.4y + 5.3z
\]

\[
5.2x + 5.2y + 5.2z \leq 5.0x + 5.4y + 5.3z
\]

\[
0.2x - 0.2y - 0.1z \leq 0
\]

\[
-0.2x + 0.2y + 0.1z \geq 0
\]
The total amount of fiber in the mix is $1.0x + 0.6y + 0.7z$ pounds. We want the mix to be at least 4 percent fiber. Thus we require that
\[
0.04(20x + 20y + 20z) \leq 1.0x + 0.6y + 0.7z
\]
\[
0.8x + 0.8y + 0.8z \leq 1.0x + 0.6y + 0.7z
\]
\[-0.2x + 0.2y + 0.1z \leq 0
\]
\[0.2x - 0.2y - 0.1z \geq 0
\]
For computational ease, we round the prices to the nearest dollar. We must minimize $C = 19x + 18y + 18z$ subject to the following constraints:
\[
20x + 20y + 20z \geq 300 \quad \text{At least 300 pounds must be purchased}
\]
\[-0.2x + 0.2y + 0.1z \geq 0 \quad \text{The mix is at least 26 percent protein}
\]
\[0.2x - 0.2y - 0.1z \geq 0 \quad \text{The mix is at least 4 percent fiber}
\]
\[x \geq 0, \quad y \geq 0, \quad z \geq 0 \quad \text{A nonnegative number of bags are purchased}
\]
We have
\[
A = \begin{bmatrix}
20 & 20 & 20 & 300 \\
-0.2 & 0.2 & 0.1 & 0 \\
0.2 & -0.2 & -0.1 & 0 \\
19 & 18 & 18 & 1
\end{bmatrix}
\text{ and } A^T = \begin{bmatrix}
20 & -0.2 & 0.2 & 19 \\
20 & 0.2 & -0.2 & 18 \\
20 & 0.1 & -0.1 & 18 \\
300 & 0 & 0 & 1
\end{bmatrix}
\]

The dual problem is
\[
\begin{align*}
\text{Maximize} & \quad P = 300x + 0y + 0z \\
\text{Subject to} & \quad 20x - 0.2y + 0.2z \leq 19 \\
& \quad 20x + 0.2y - 0.2z \leq 18 \\
& \quad 20x + 0.1y - 0.1z \leq 18 \\
& \quad x \geq 0, \quad y \geq 0, \quad z \geq 0
\end{align*}
\]

with the corresponding initial tableau
\[
\begin{bmatrix}
\begin{array}{cccccc}
20 & -0.2 & 0.2 & 1 & 0 & 0 & 19 \\
20 & 0.2 & -0.2 & 0 & 1 & 0 & 18 \\
20 & 0.1 & -0.1 & 0 & 0 & 1 & 18 \\
-300 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\end{bmatrix}
\]

The $x$ column is the pivot column. Either the second or the third entry in the $x$ column may be used as a pivot. Using the entry in the second row of the $x$ column as a pivot and the indicated row operations, we obtain the second tableau. The entry in the third row of the $z$ column is the new pivot, since it has the smallest test ratio.
\[
\begin{bmatrix}
\begin{array}{cccccc}
0 & -0.4 & 0.4 & 1 & -1 & 0 & 0 \ R_1 - R_2 \\
20 & 0.2 & -0.2 & 0 & 1 & 0 & 18 \ R_1 - R_2 \\
0 & 0.1 & 0 & -1 & 1 & 0 & 0 \ R_3 - R_2 \\
0 & 3 & -3 & 0 & 15 & 0 & 1 \ 270 \ \ R_4 + 15R_2
\end{array}
\end{bmatrix}
\]
The indicated row operations give the next tableau. The new pivot is the 3 in the first row of the $t$ column.

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 3 & -4 & 0 & 1 \\
20 & 0 & 0 & 0 & -1 & 2 & 0 & 18 \\
0 & -0.1 & 0.1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -15 & 30 & 1 & 270
\end{bmatrix}
$$

The indicated row operations give the next tableau.

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 3 & -4 & 0 & 1 \\
60 & 0 & 0 & 1 & 0 & 2 & 0 & 55 \\
0 & -0.3 & 0.3 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 5 & 0 & 10 & 1 & 275
\end{bmatrix}
$$

The final simplex tableau is obtained by converting the nonzero entry of each column with a single nonzero entry to a 1 by using the indicated row operations.

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 3 & -4 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & \frac{1}{60} & 0 & \frac{1}{30} \\
0 & -1 & 1 & 0 & \frac{10}{3} & 0 & -\frac{10}{3} & 0 \\
0 & 0 & 0 & 5 & 0 & 10 & 1 & 275
\end{bmatrix}
$$

Setting the inactive variable columns $s$ and $u$ to 0 yields

$$
\begin{align*}
x &= \frac{11}{12} \\
y &= \frac{10}{3} \\
z &= \frac{10}{3}
\end{align*}
$$

Any solution of the form \(\left(\frac{11}{12}, r, r + \frac{10}{3}\right)\) for nonnegative $r$ maximizes the objective function of the dual problem. The optimal value is 275.

The minimization problem has the solution $x = 5$, $y = 0$, and $z = 10$. When 5 bags of the Eukanuba brand and 10 bags of the Science Diet brand are purchased, the total cost is $275. This is the minimum cost for a mix that meets the breeder’s nutrition requirements.

### 4.4 Summary

In this section, you learned how to solve standard minimization problems using the notion of the dual problem. You discovered that finding the maximum solution of a dual problem leads us to the minimum solution of the minimization problem.
4.4 Exercises

In Exercises 1–10, determine if the problem is a standard minimization problem. If it isn’t, explain why.

1. Minimize \( P = 9x + 8y \)
   Subject to \[
   \begin{align*}
   2x + y & \geq 10 \\
   -x + y & \geq 1 \\
   x & \geq 0 \\
   y & \geq 0 
   \end{align*}
   \]

2. Minimize \( P = -2x - 5y \)
   Subject to \[
   \begin{align*}
   4x + 9y & \geq 10 \\
   -11x + y & \geq -21 \\
   x & \geq 0 \\
   y & \geq 0 
   \end{align*}
   \]

3. Minimize \( P = 6xy \)
   Subject to \[
   \begin{align*}
   6x + 7y & \geq 13 \\
   -8x - 4y & \geq 12 \\
   x & \geq 0 \\
   y & \geq 0 
   \end{align*}
   \]

4. Minimize \( P = 4x - 2y + z \)
   Subject to \[
   \begin{align*}
   2x + y + 4z & \geq 24 \\
   -2x + 3y + 3z & \geq 150 \\
   x & \geq 0 \\
   y & \geq 0 \\
   z & \geq 0 
   \end{align*}
   \]

5. Minimize \( P = -1.2x - 2.8y + 4.3z \)
   Subject to \[
   \begin{align*}
   3.2x + 1.5y + 7.4z & \geq 249.8 \\
   -2.7x + 3.4y + 3.9z & \geq 190.1 
   \end{align*}
   \]

6. Minimize \( P = 4x + 4y + 9z \)
   Subject to \[
   \begin{align*}
   -6x - y + 4z & \leq -24 \\
   -9x - 1.5y + 6z & \geq 36 \\
   x & \geq 0 \\
   y & \geq 0 \\
   z & \geq 0 
   \end{align*}
   \]

7. Minimize \( P = x + 9y + 8z \)
   Subject to \[
   \begin{align*}
   -6x - y + 4z & \leq -24 \\
   -9x - 1.5y + 6z & \geq 36 \\
   x & \geq 0 \\
   y & \geq 0 \\
   z & \geq 0 
   \end{align*}
   \]

8. Minimize \( P = 4x + 4y + 9z \)
   Subject to \[
   \begin{align*}
   -3x - 2y + 4z & \leq -22 \\
   5x - 2.5y + 6z & \geq 36 \\
   x & \geq 0 \\
   y & \geq 0 \\
   z & \geq 0 
   \end{align*}
   \]

9. Minimize \( P = -8x + 2y - z \)
   Subject to \[
   \begin{align*}
   z & \geq 4 \\
   x - y & \geq 3 \\
   2x - z & \geq 5 \\
   x + y + z & \leq 0 
   \end{align*}
   \]

10. Minimize \( P = 5x + 7z \)
    Subject to \[
    \begin{align*}
    4x + 2y & \geq 16 \\
    4x - 2y & \geq 3 \\
    x & \geq 0 \\
    y & \geq 0 
    \end{align*}
    \]

In Exercises 11–20, find the transpose of the given matrix.

11. \( A = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 5 & 9 & 7 & 3 \\ 8 & 1 & 0 & 1 \end{bmatrix} \)
12. \( A = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 4 & 5 & 1 & 1 \end{bmatrix} \)
13. \( A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \end{bmatrix} \)
14. \( A = \begin{bmatrix} 8 & 2 & 1 \\ 2 & 5 & 7 \\ 1 & 7 & 1 \end{bmatrix} \)
15. \( A = \begin{bmatrix} 9 & 3 & -1 & 4 \\ 7 & 2 & -2 & 1 \end{bmatrix} \)
16. \( A = \begin{bmatrix} -5 & 4 & -7 & 3 & 9 \\ 3 & 7 & 9 & 3 & 1 \end{bmatrix} \)
17. \( A = \begin{bmatrix} 9 & 2 & 18 \\ -3 & -2 & 6 \\ 4 & 0 & 0 \\ 5 & 6 & 1 \end{bmatrix} \)
18. \( A = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 \\ 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 5 & 1 \end{bmatrix} \)
19. \[ A = \begin{bmatrix} 0 & 2 & 1 \\ -5 & 4 & 7 \\ 8 & 1 & 0 \\ -1 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \]

20. \[ A = [2 \quad 4 \quad 6 \quad 5 \quad 1] \]

In Exercises 21–30, do the following:

(i) Write the dual problem for the given standard minimization problem.

(ii) Solve the dual problem using the simplex method.

(iii) Use the final simplex tableau of the dual problem to solve the standard minimization problem.

(iv) Check your answer by graphing the feasible region of the standard minimization problem and calculating the value of the objective function at each of the corner points.

21. Minimize \[ P = x + 2y \]

   Subject to \[ \begin{cases} 3x + y \geq 6 \\ 2x + 3y \geq 11 \\ x \geq 0, y \geq 0 \end{cases} \]

22. Minimize \[ P = 4x + 2y \]

   Subject to \[ \begin{cases} 5x + 7y \geq 19 \\ -2x + 3y \geq 5 \\ x \geq 0, y \geq 0 \end{cases} \]

23. Minimize \[ P = 9x + 6y \]

   Subject to \[ \begin{cases} 5x + 2y \geq 30 \\ 10x - 5y \geq 15 \\ x \geq 0, y \geq 0 \end{cases} \]

24. Minimize \[ P = 4x + 4y \]

   Subject to \[ \begin{cases} 3x + 4y \geq 18 \\ 5x + 3y \geq 19 \\ x \geq 0, y \geq 0 \end{cases} \]

25. Minimize \[ P = 5x + 20y \]

   Subject to \[ \begin{cases} -5x + 2y \geq 10 \\ 5x + y \geq 25 \\ x \geq 0, y \geq 0 \end{cases} \]

26. Minimize \[ P = 2x + 5y \]

   Subject to \[ \begin{cases} x + y \geq 9 \\ 2x - 2y \geq 6 \\ x \geq 0, y \geq 0 \end{cases} \]

27. Minimize \[ P = 3x + 2y \]

   Subject to \[ \begin{cases} x + y \geq 5 \\ 3x + 2y \geq 11 \\ 4x - y \geq 0 \\ x \geq 0, y \geq 0 \end{cases} \]

28. Minimize \[ P = 9x + 7y \]

   Subject to \[ \begin{cases} 5x + 3y \geq 8 \\ 3x + 5y \geq 8 \\ 8x - 8y \geq 0 \\ x \geq 0, y \geq 0 \end{cases} \]

29. Minimize \[ P = 6x + 5y \]

   Subject to \[ \begin{cases} x + y \geq 4 \\ x - y \geq 2 \\ 2x - y \geq 6 \\ x \geq 0, y \geq 0 \end{cases} \]

30. Minimize \[ P = 5x + 7y \]

   Subject to \[ \begin{cases} x + y \geq 4 \\ x - y \geq 2 \\ 2x - y \geq 6 \\ x \geq 0, y \geq 0 \end{cases} \]

In Exercises 31–40, set up and solve the standard minimization problem.

31. **Pet Nutrition: Food Cost**

    PETsMART.com sold the following varieties of dog food in June 2003. The price shown is for an 8-pound bag.

    **Pro Plan Adult Chicken & Rice Formula,**
    25 percent protein, 3 percent fiber, $7.99

    **Pro Plan Adult Lamb & Rice Formula,**
    28 percent protein, 3 percent fiber, $7.99

    **Pro Plan Adult Turkey & Barley Formula,**
    26 percent protein, 3 percent fiber, $8.49

    (Source: www.petsmart.com.)

    A dog breeder wants to make at least 120 pounds of a mix containing at least 27 percent protein and at least 3 percent fiber. How many 8-pound bags of each dog food variety should the breeder buy in order to minimize cost? (Round prices up to the nearest dollar.)
32. **Pet Nutrition: Food Cost**

PETsMART.com sold the following varieties of dog food in June 2003.

**Authority Chicken Adult Formula,** 32 percent protein, 3 percent fiber, $19.99 per 33-pound bag

**Bil-Jac Select Dog Food,** 27 percent protein, 4 percent fiber, $18.99 per 18-pound bag

**Iams Minichunks,** 26 percent protein, 4 percent fiber, $8.99 per 8-pound bag

(Source: www.petsmart.com.)

A dog kennel wants to make at least 2178 pounds of a mix containing at least 29 percent protein and at least 3.5 percent fiber. How many bags of each dog food variety should the kennel buy in order to minimize cost? (Round prices up to the nearest dollar.)

33. **Pet Nutrition: Food Cost**

PETsMART.com sold the following varieties of dog food in June 2003.

**Nature’s Recipe Venison Meal & Rice Canine,** 20 percent protein, 10 percent fat, $21.99 per 20-pound bag

**Nutro Max Natural Dog Food,** 27 percent protein, 16 percent fat, $12.99 per 17.5-pound bag

PETsMART Premier Oven Baked Lamb Recipe, 25 percent protein, 14 percent fat, $22.99 per 30-pound bag

(Source: www.petsmart.com.)

A dog breeder wants to make at least 175 pounds of a mix containing at least 25 percent protein and at least 14 percent fat. How many bags of each dog food variety should the breeder buy in order to minimize cost? (Round prices up to the nearest dollar.)

34. **Pet Nutrition: Fat Content**

PETsMART.com sold the following varieties of dog food in June 2003.

**Nature’s Recipe Venison Meal & Rice Canine,** 20 percent protein, 10 percent fat, $21.99 per 20-pound bag

**Nutro Max Natural Dog Food,** 27 percent protein, 16 percent fat, $12.99 per 17.5-pound bag

PETsMART Premier Oven Baked Lamb Recipe, 25 percent protein, 14 percent fat, $22.99 per 30-pound bag

(Source: www.petsmart.com.)

A dog breeder wants to make at least 210 pounds of a mix containing at least 25 percent protein. How many bags of each dog food variety should the breeder buy in order to minimize fat content?

35. **Food Distribution Cost**

Wal-Mart Stores, Inc., has food distribution centers in Monroe, Georgia, and Shelbyville, Tennessee, and Wal-Mart Supercenters in Birmingham, Alabama, and Scottsboro, Alabama. (Source: www.walmart.com.) Suppose that the Monroe distribution center must ship at least 600 cases of peanut butter weekly and the Shelbyville distribution center must ship at least 400 cases of peanut butter weekly.* If the Birmingham store requires at least 700 cases of peanut butter weekly and the Scottsboro store requires at least 300 cases of peanut butter weekly, what shipment plan will minimize the distribution cost?

**Estimated Distribution Cost per Case**

<table>
<thead>
<tr>
<th></th>
<th>Monroe</th>
<th>Shelbyville</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birmingham</td>
<td>$0.68</td>
<td>$0.57</td>
</tr>
<tr>
<td>Scottsboro</td>
<td>$0.65</td>
<td>$0.32</td>
</tr>
</tbody>
</table>

*Distribution costs and amounts are hypothetical.

36. **Food Distribution Cost**

Wal-Mart Stores, Inc., has food distribution centers in Monroe, Georgia, and Shelbyville, Tennessee, and Wal-Mart Supercenters in Birmingham, Alabama, and Calhoun, Georgia. (Source: www.walmart.com.) Suppose that the Monroe distribution center must ship at least 600 cases of pickles weekly and the Shelbyville distribution center must ship at least 700 cases of pickles weekly.* If the Birmingham store requires at least 1000 cases of pickles weekly and the Calhoun store requires at least 300 cases of pickles weekly, what shipment plan will minimize the distribution cost?

**Estimated Distribution Cost per Case**

<table>
<thead>
<tr>
<th></th>
<th>Monroe</th>
<th>Shelbyville</th>
</tr>
</thead>
<tbody>
<tr>
<td>Birmingham</td>
<td>$0.68</td>
<td>$0.57</td>
</tr>
<tr>
<td>Calhoun</td>
<td>$0.39</td>
<td>$0.50</td>
</tr>
</tbody>
</table>

*Distribution costs and amounts are hypothetical.
37. **Food Distribution Cost**  
Wal-Mart Stores, Inc., has food distribution centers in Monroe, Georgia, and Shelbyville, Tennessee, and Wal-Mart Supercenters in Scottsboro, Alabama, and Calhoun, Georgia. (Source: www.walmart.com.) Suppose that the Monroe distribution center must ship at least 400 cases of potato chips weekly and the Shelbyville distribution center must ship at least 200 cases of potato chips weekly.* If the Scottsboro store requires at least 300 cases of potato chips weekly and the Calhoun store requires at least 300 cases of potato chips weekly, what shipment plan will minimize the distribution cost?

**Estimated Distribution Cost per Case**

<table>
<thead>
<tr>
<th></th>
<th>Monroe</th>
<th>Shelbyville</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calhoun</td>
<td>$0.39</td>
<td>$0.50</td>
</tr>
<tr>
<td>Scottsboro</td>
<td>$0.65</td>
<td>$0.32</td>
</tr>
</tbody>
</table>

*Distribution costs and amounts are hypothetical.

38. **Food and Entertainment**  
It costs $35 for a family to dine out and $10 for the family to eat at home. Eating out has a fun rating of 10 points, while eating at home has a fun rating of 2 points.

It costs $32 for a family to play a game of miniature golf and $30 to watch a movie. Miniature golf has a fun rating of 8 points, and watching a movie has a fun rating of 6 points.

The family must eat at least 21 meals weekly and must go out for food or miniature golf at least five times weekly. The family wants to earn at least 82 fun points per week. How many times a week should the family participate in each activity in order to minimize food and entertainment costs?

39. **Marital Harmony**  
It costs a couple 3 hours and $200 to go clothes shopping together. She gives shopping a fun rating of 10 points, while he gives it a fun rating of 1 point (a total of 11 fun points for the couple).

It costs a couple 4 hours and $250 to go to a major league baseball playoff game. She gives the game a fun rating of 4 points, while he gives it a fun rating of 10 points (a total of 14 fun points for the couple).

The couple wants to spend at least 10 hours together (shopping and watching baseball) while earning at least 36 fun points. She insists that they go shopping at least twice. How shall they spend their time if they want to minimize their financial costs?

40. **Marital Discord**  
It costs a couple 3 hours to go clothes shopping together. He gives shopping with her a fun rating of 6 points while she gives it a fun rating of 1 point (a total of 7 fun points for the couple).

It costs a couple 4 hours to go to a major league baseball game. She gives going to a game with him a fun rating of 7 points while he gives it a fun rating of 2 points (a total of 9 fun points for the couple).

The couple wants to earn at least 30 fun points. She refuses to go shopping with him unless he goes with her to at least one game. How shall they spend their time if they want to minimize their time together? (Things are not going well in the relationship.)

### 4.5 Solving General Linear Programming Problems with the Simplex Method

- Solve general linear programming problems with the simplex method

An investor plans to invest at most $3000 in the three publicly traded recreational vehicle companies shown in Table 4.25.
He wants to earn at least $50 in dividends while maximizing the number of shares purchased. How many shares of each company should he purchase?

At first glance, this looks like a standard maximization problem. However, a closer analysis reveals that it is not. While the investment constraint is of the form \( Ax + By + Cz + \cdots \leq M \), the dividend constraint is of the form \( Ax + By + Cz + \cdots \geq M \) instead of \( Ax + By + Cz + \cdots \leq M \). This is a general linear programming problem or a linear programming problem with mixed constraints.

In this section, we will show how to solve general linear programming problems. We will demonstrate how minimization problems may be solved by maximizing the negative of the objective function. We will return to the recreational vehicle investment problem in Example 3.

Recall that the feasible region of any standard maximization problem always includes the origin. Graphically speaking, the simplex method starts at the origin and moves from corner point to adjacent corner point, each time increasing the value of the objective function until the maximum value is reached. The simplex method works with standard maximization problems because we are guaranteed that the origin is in the feasible region. But what if the origin is outside of the feasible region? The feasible region of a linear programming problem with mixed constraints often does not contain the origin.

A linear programming problem with mixed constraints has constraints in two or more of the following forms: \( Ax + By + Cz + \cdots \leq M \), \( Ax + By + Cz + \cdots \geq M \), or \( Ax + By + Cz + \cdots = M \). In all cases, \( M \geq 0 \). Consider the following linear programming problem with mixed constraints.

Maximize \( P = 3x + 2y \)

Subject to \[
\begin{align*}
-x + 2y & \leq 4 \\
2x - y & \leq 4 \\
x + y & \geq 5 \\
x & \geq 0, \ y & \geq 0
\end{align*}
\]

Since the constraint \( x + y \geq 5 \) contains a \( \geq \) sign instead of a \( \leq \) sign, this is not a standard maximization problem. It is a problem with mixed constraints. The graph of the feasible region is shown in Figure 4.28.
Notice that the origin is not contained within the feasible region. Our strategy for solving linear programming problems with mixed constraints is a two-stage process.

Stage 1: Get inside the feasible region.
Stage 2: Solve the problem with the simplex method.

Since our problem contains only two decision variables, we have the luxury of observing how the process works graphically. This is not the case in problems involving three or more decision variables.

We set up the problem for the simplex method as usual by adding in slack variables to constraints of the form \( Ax + By + Cz + \cdots \leq M \).

\[
\begin{align*}
-x + 2y + s &= 4 \\
2x - y + t &= 4
\end{align*}
\]

Can we add in slack variables to constraints of the form \( Ax + By + Cz + \cdots \geq M \)? Let’s see. We have the constraint \( x + y \geq 5 \). Recall that slack variables add in what is necessary to make the inequality an equality. By definition, slack variables must be nonnegative. What must we do to the inequality \( x + y \geq 5 \) to make it an equality? Adding a slack variable to the inequality will increase instead of decrease the value of the left-hand side. Since the left-hand side of the inequality is greater than or equal to 5, we must subtract some nonnegative value \( u \) in order to make \( x + y \) equal to 5. That is,

\[
x + y - u = 5
\]

Since this variable takes away the surplus, it is referred to as a \textbf{surplus variable}.

We have

\[
\begin{align*}
-x + 2y + s &= 4 \\
2x - y + t &= 4 \\
x + y - u &= 5 \\
-3x - 2y + P &= 0
\end{align*}
\]
and the corresponding initial tableau

\[
\begin{bmatrix}
x & y & s & t & u & P \\
-1 & 2 & 1 & 0 & 0 & 4 \\
2 & -1 & 0 & 1 & 0 & 4 \\
1 & 1 & 0 & 0 & -1 & 0 \\
-3 & -2 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Setting the \( x \) and \( y \) variables equal to zero yields the solution to this tableau.

\[
x = 0 \quad y = 0 \quad s = 4 \quad t = 4 \quad -u = 5 \quad P = 0
\]

\[u = -5\]

Recall that all decision, slack, and surplus variables are required to be nonnegative. Since \( u < 0 \), this solution is not in the feasible region. This may be easily seen graphically (Figure 4.29). The origin is not in the feasible region.

![Figure 4.29](image)

In the initial tableau, columns \( s, t, u \), and \( P \) were the active variable columns. Since the active variable \( u \) was negative, we want to modify the tableau in such a way as to make \( u \) an inactive variable. We place a \( ^* \) on the left of the tableau to highlight the rows above the bottom row that resulted in negative variable values. In this case, the third row was the only one with a negative variable value.

\[
\begin{bmatrix}
x & y & s & t & u & P \\
-1 & 2 & 1 & 0 & 0 & 4 \\
2 & -1 & 0 & 1 & 0 & 4 \\
* & 1 & 1 & 0 & 0 & -1 \\
-3 & -2 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Since all surplus variables in the initial tableau are negative, each row of the initial tableau containing surplus variables will be starred. We select the column with the largest positive entry in the starred row as the pivot column. Since the positive values in the starred row are equal, we may select either the \( x \) column or the \( y \) column to be the pivot column. We select the \( x \) column. To determine the
pivot, we calculate the test ratios and select the row with the smallest test ratio. This will ensure that all nonnegative variables will remain nonnegative.

\[
\begin{bmatrix}
-1 & 2 & 1 & 0 & 0 & 4 \\
2 & -1 & 0 & 1 & 0 & 4 \\
1 & 1 & 0 & -1 & 0 & 5 \\
-3 & -2 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Pivot column

The 2 in the second row of the \(x\) column is the pivot. We modify the initial tableau with the indicated operations and star any rows corresponding with negative active variables.

\[
\begin{bmatrix}
0 & 3 & 2 & 1 & 0 & 0 & 12 \\
2 & -1 & 0 & 1 & 0 & 4 \\
0 & 3 & 0 & -1 & -2 & 0 & 6 \\
0 & -7 & 0 & 3 & 0 & 2 & 12 \\
\end{bmatrix}
\]

\[
2x = 4 \quad y = 0 \quad 2s = 12 \quad t = 0 \quad -2u = 6 \quad 2P = 12 \\
x = 2 \quad s = 6 \quad u = -3 \quad P = 6
\]

We starred the third row, since the value of \(u\) is negative. Although \(u\) is still negative, it is less negative than it was before. We will continue the process until \(u\) is nonnegative. However, before continuing, let’s examine what is happening graphically (Figure 4.30).

We have moved from the origin to the \(x\)-intercept of one of the constraints. However, we still are outside of the feasible region. This is due to the fact that the surplus variable \(u\) is still negative. We will pivot again and further increase the value of \(u\).
The largest value in the starred row is the 3 in the \( y \) column. The \( y \) column is the new pivot column.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
0 & 3 & 2 & 1 & 0 & 0 & 12 \\
2 & -1 & 0 & 1 & 0 & 0 & 4 \\
* & 3 & 0 & -1 & -2 & 0 & 6 \\
0 & -7 & 0 & 3 & 0 & 2 & 12 \\
\end{array}
\]

\[
\frac{12}{3} = 4 \\
\frac{6}{3} = 2
\]

The pivot is the 3 in the third row of the \( y \) column, since that row has the smallest test ratio. We zero out the remaining terms in the column with the indicated operations and read the solution from the tableau.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
0 & 0 & 2 & 2 & 0 & 6 \\
6 & 0 & 0 & 2 & -2 & 18 \\
0 & 3 & 0 & -1 & -2 & 6 \\
0 & 0 & 0 & 2 & -14 & 678 \\
\end{array}
\]

\[
6x = 18 \\
3y = 6 \\
2s = 6 \\
t = 0 \\
u = 0 \\
6P = 78 \\
x = 3 \\
y = 2 \\
s = 3 \\
P = 13
\]

Since none of the variables are negative, this solution is in the feasible region (Figure 4.31). Since we are now in the feasible region, this completes Stage 1 of the process.

Stage 2 allows us to apply the simplex method to find the optimal solution. The \( u \) column is the pivot column. The 2 in the first row of that column is the pivot, since it is the only nonnegative value in the column.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
0 & 0 & 2 & 2 & \boxed{3} & 0 & 6 \\
6 & 0 & 0 & 2 & -2 & 0 & 18 \\
0 & 3 & 0 & -1 & -2 & 0 & 6 \\
0 & 0 & 0 & 2 & -14 & 6 & 78 \\
\end{array}
\]
We zero out the remaining terms in the $u$ column using the indicated operations.

$$
\begin{array}{cccc|c}
 x & y & s & t & u & P \\
 0 & 0 & 2 & 2 & 0 & 6 \\
 6 & 0 & 2 & 4 & 0 & 24 \\
 0 & 3 & 2 & 1 & 0 & 12 \\
 0 & 0 & 14 & 16 & 6 & 120 \\
\end{array}
$$

We have

$$
\begin{align*}
6x &= 24 \\
3y &= 12 \\
s &= 0 \\
t &= 0 \\
2u &= 6 \\
6P &= 120 \\
x &= 4 \\
y &= 4 \\
u &= 3 \\
P &= 20
\end{align*}
$$

Since the bottom row of the tableau does not contain any negative values, this is the final simplex tableau. The solution that maximizes the objective function $P = 3x + 2y$ is $x = 4$ and $y = 4$ (Figure 4.32). The maximum value of the objective function is $P = 20$.

The following box details the steps used to solve the general linear programming problem.

### Solving General Linear Programming Problems

**Stage 1: Get inside the feasible region**

1. Star all rows that correspond with a negative value of a decision, slack, or surplus variable.
2. Identify the largest positive entry in the starred row. The corresponding column is the pivot column.
3. Calculate the test ratios for the positive entries above the bottom row of the pivot column.
4. Pick the pivot column entry with the smallest test ratio as the pivot.
5. Row reduce the tableau using operations of the form $aR_c \pm bR_p \rightarrow R_c$ with $a$ and $b$ positive. (Recall that $R_c$ is the row we want to change and $R_p$ is the pivot row.)
6. From the tableau, calculate the value of all decision, slack, and surplus variables. If any of the variables are negative, repeat Steps 1 through 5 for the new tableau. Otherwise, go to Stage 2.

**Stage 2: Solve the maximization problem with the simplex method.**

### Example 1: Solving a Linear Programming Problem with No Optimal Solution

Graph the feasible region associated with the given linear programming problem. Then solve the problem using the two-stage method described previously, indicating on the graph the solution that corresponds with each tableau.

Maximize $P = 3x + 10y$

Subject to

$$
\begin{align*}
4x - y &\geq 11 \\
x + 2y &\geq 5 \\
-x + y &\leq 1 \\
x &\geq 0, y &\geq 0
\end{align*}
$$

FIGURE 4.32
The graph of the feasible region is shown in Figure 4.33. We can see from the graph that the feasible region is unbounded. Since the objective function has positive coefficients and the region is unbounded, the objective function will not have a maximum value. Nevertheless, we will proceed with the two-stage method to discover what indicators in the tableau let us know that there is no objective function maximum.

We add the slack and surplus variables and rewrite the objective function. We have

\[4x - y - s = 11\]
\[x + 2y - t = 5\]
\[-x + y + u = 1\]
\[-3x - 10y + P = 0\]

and the corresponding initial tableau

\[
\begin{array}{cccccc}
 x & y & s & t & u & P \\
 4 & -1 & -1 & 0 & 0 & 0 & 11 \\
 1 & 2 & 0 & -1 & 0 & 0 & 5 \\
 -1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 -3 & -10 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Reading from the tableau, we have

\[x = 0\quad y = 0\quad -s = 11\quad -t = 5\quad u = 1\quad P = 0\]
\[s = -11\quad t = -5\]

Since both \(s\) and \(t\) are negative, we will star the corresponding rows.

\[
\begin{array}{cccccc}
 x & y & s & t & u & P \\
 * & 4 & -1 & -1 & 0 & 0 & 0 & 11 \\
 * & 1 & 2 & 0 & -1 & 0 & 0 & 5 \\
 -1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 -3 & -10 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

We may choose either starred row to work with. We pick the row corresponding with the variable \(t\) (second row). The largest value in the labeled columns of the second row is the 2 in the \(y\) column. Consequently, the \(y\) column is the pivot column. We calculate the test ratios and locate the pivot.

\[
\begin{array}{cccccc}
 x & y & s & t & u & P \\
 * & 4 & -1 & -1 & 0 & 0 & 0 & 11 \\
 * & 1 & 2 & 0 & -1 & 0 & 0 & 5 \\
 -1 & 1 & 0 & 0 & 1 & 0 & 1 \\
 -3 & -10 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Pivot column

\[\frac{5}{2} = 2.5\]
\[\frac{1}{1} = 1\]
We zero out the remaining terms in the pivot column using the indicated operations and star the rows that correspond with negative variable values.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
  \ast & 3 & 0 & -1 & 0 & 1 & 0 & 12 & R_1 + R_1 \\
  \ast & 3 & 0 & 0 & -1 & -2 & 0 & 3 & R_2 - 2R_3 \\
  & -1 & 1 & 0 & 0 & 1 & 0 & 1 \\
  & -13 & 0 & 0 & 0 & 10 & 1 & 10 & R_4 + 10R_3 \\
\end{array}
\]

\[
\begin{array}{ccc}
x = 0 & y = 1 & -s = 12 & -t = 3 & u = 0 & P = 10 \\
& s = -12 & t = -3 \\
\end{array}
\]

We observe that although \( t \) remains negative, it has become less negative. We will again select the largest value in a labeled column of the second row. The 3 in the \( x \) column is the largest value. Consequently, the \( x \) column is the pivot column.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
  \ast & 3 & 0 & -1 & 0 & 1 & 0 & 12 & 12/3 = 4 \\
  \ast & \frac{\Delta}{3} & 0 & 0 & -1 & -2 & 0 & 3 & 3/3 = 1 \\
  & -1 & 1 & 0 & 0 & 1 & 0 & 1 \\
  & -13 & 0 & 0 & 0 & 10 & 1 & 10 & \uparrow \\
\end{array}
\]

The 3 in the second row of the \( x \) column is the pivot, since the second row has the smallest test ratio. We zero out the remaining entries in the \( x \) column using the indicated operations. We then star the row corresponding to the negative variable.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
  \ast & 0 & 0 & -1 & 1 & 3 & 0 & 9 & R_1 - R_2 \\
  & 3 & 0 & 0 & -1 & -2 & 0 & 3 \\
  & 0 & 3 & 0 & -1 & 1 & 0 & 6 & 3R_1 + R_2 \\
  & 0 & 0 & 0 & -13 & 4 & 3 & 69 & 3R_4 + 13R_2 \\
\end{array}
\]

\[
\begin{array}{cccc}
3x = 3 & 3y = 6 & -s = 9 & t = 0 \\
& u = 0 & 3P = 69 \\
\end{array}
\]

\[
\begin{array}{cccc}
x = 1 & y = 2 & s = -9 & \end{array}
\]

\[
\begin{array}{cccc}
P = 23 & \end{array}
\]

Observe that the surplus variable \( t \) is now nonnegative. We need to continue the process to make \( s \) nonnegative. The largest value in the labeled columns of the starred row is the 3 in the \( u \) column. Consequently, the \( u \) column is the pivot column.

\[
\begin{array}{cccccc}
  x & y & s & t & u & P \\
  \ast & 0 & 0 & -1 & 1 & \frac{\Delta}{3} & 0 & 9 & 9/3 = 3 \\
  & 3 & 0 & 0 & -1 & -2 & 0 & 3 \\
  & 0 & 3 & 0 & -1 & 1 & 0 & 6 & 6/1 = 6 \\
  & 0 & 0 & 0 & -13 & 4 & 3 & 69 \uparrow \\
\end{array}
\]
Using the 3 in the first row of the \( u \) column as a pivot, we zero out the remaining column entries using the indicated operations.

\[
\begin{bmatrix}
0 & 0 & -1 & 1 & 3 & 0 & 9 \\
9 & 0 & -2 & -1 & 0 & 0 & 27 \\
0 & 9 & 1 & 4 & -4 & 0 & 9 \\
0 & 0 & 4 & -3 & 0 & 9 & 171
\end{bmatrix}
\]

\[3R_2 + 2R_1, \quad 3R_1 - R_1, \quad 3R_4 - 4R_1\]

\[9x = 27, \quad 9y = 9, \quad s = 0, \quad t = 0, \quad 3u = 9, \quad 9P = 171\]

\[x = 3, \quad y = 1, \quad u = 3, \quad P = 19\]

Since all decision, slack, and surplus variables are nonnegative, we are in the feasible region. We can now proceed to Stage 2 of the process: applying the simplex method. Before going on, let’s observe what has happened graphically (see Figure 4.34).

The blue dots on the graph indicate the solutions to the various tableaus. We started at \((0, 0)\), proceeded to \((0, 1)\), then to \((1, 2)\), and finally to \((3, 1)\).

Returning to the tableau, we apply the simplex method. The only negative entry in the bottom row is in the \( t \) column, so the \( t \) column is our pivot column.

The only positive entry in the \( t \) column is the 1 in the first row, so that is our pivot.

\[
\begin{bmatrix}
0 & 0 & -1 & 1 & 3 & 0 & 9 \\
9 & 0 & -2 & -1 & 0 & 0 & 27 \\
0 & 9 & 1 & 4 & -4 & 0 & 9 \\
0 & 0 & 4 & -3 & 0 & 9 & 171
\end{bmatrix}
\]

We zero out the remaining entries in the \( t \) column with the indicated operations.

\[
\begin{bmatrix}
0 & 0 & -1 & 1 & 3 & 0 & 9 \\
9 & 0 & -3 & 0 & 3 & 0 & 36 \\
0 & 9 & -3 & 0 & 12 & 0 & 45 \\
0 & 0 & 4 & -3 & 0 & 9 & 558
\end{bmatrix}
\]

\[R_2 + R_1, \quad R_1 + 4R_1, \quad R_4 + 43R_1\]

\[9x = 36, \quad 9y = 45, \quad s = 0, \quad t = 9, \quad u = 0, \quad 9P = 558\]

\[x = 4, \quad y = 5, \quad P = 62\]

The \( s \) column is the new pivot column. However, we are unable to select a pivot because every entry in the \( s \) column is negative. This signifies that the objective function has no optimal solution. Figure 4.35 shows what has happened graphically.

We increased the value of the objective function by moving from the corner point \((3, 1)\) to the corner point \((4, 5)\). Out of all of the corner points, this is the corner point that yields the largest value of the objective function. However, since the region is unbounded, we can continue to increase the value of the objective function indefinitely.
LINEAR PROGRAMMING PROBLEMS WITH NO OPTIMAL SOLUTIONS

If none of the entries in the pivot column of a simplex tableau are positive, the corresponding feasible region is unbounded and the objective function has no optimal solution.

EXAMPLE 2
Solving a General Linear Programming Problem

Solve the general linear programming problem:

Maximize \( P = 4x + 3y + 2z \)

Subject to

\[
\begin{align*}
-x - y + z &\geq 1 \\
-2x + y + z &\geq 3 \\
-x + y + z &\leq 3 \\
x &\geq 0, y &\geq 0, z &\geq 0
\end{align*}
\]

SOLUTION

We add the slack and surplus variables and rewrite the objective function.

\[
\begin{align*}
-x - y + z - s &= 1 \\
-2x + y + z - t &= 3 \\
-x + y + z + u &= 3 \\
-4x - 3y - 2z + P &= 0
\end{align*}
\]

The corresponding initial tableau is shown. We calculate the basic solution and star the rows with negative variable values.

\[
\begin{array}{ccccccccc}
x & y & z & s & t & u & P \\
-1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 1 & 0 & -1 & 0 & 0 & 3 \\
-1 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\
-4 & -3 & -2 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

We may work with either starred row. We choose the second row. (This row corresponds to \( t = -3 \).) The largest positive entries in the labeled columns of the second row are the 1 in the \( y \) column and the 1 in the \( z \) column. We may pick either column as the pivot column. We pick the \( y \) column. We calculate the test ratios and locate the pivot. The test ratios are equal, so we may pick either the second or the third row as the pivot row. We choose the second row.

\[
\begin{array}{ccccccccc}
x & y & z & s & t & u & P \\
-1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 1 & 0 & -1 & 0 & 0 & 3 \\
-1 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\
-4 & -3 & -2 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
x & y & z & s & t & u & P \\
-1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \\
-2 & 1 & 1 & 0 & -1 & 0 & 0 & 3 \\
-1 & 1 & 1 & 0 & 0 & 1 & 0 & 3 \\
-4 & -3 & -2 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Pivot column
We zero out the $y$ column using the indicated operations and star the row corresponding with a negative variable value.

\[
\begin{array}{rrrrrr}
  x & y & z & s & t & u & P \\
  \ast & -3 & 0 & 2 & -1 & -1 & 0 & 0 & 4 & R_1 + R_2 \\
  & -2 & 1 & 1 & 0 & -1 & 0 & 0 & 3 & R_3 - R_2 \\
  & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \text{ } \\
  & -10 & 0 & 1 & 0 & -3 & 0 & 1 & 9 & R_4 + 3R_2 \\
\end{array}
\]

\[
x = 0 \quad y = 3 \quad z = 0 \quad -s = 4 \quad t = 0 \quad u = 0 \quad P = 9 \\
s = -4
\]

The 2 in the $z$ column is the largest positive value in the starred row. Consequently, the $z$ column is the pivot column. We calculate the test ratios and locate the pivot.

\[
\begin{array}{rrrrrr}
  x & y & z & s & t & u & P \\
  -3 & 0 & 2 & -1 & -1 & 0 & 0 & 4 & 4/2 = 2 \\
  -2 & 1 & 1 & 0 & -1 & 0 & 0 & 3 & 3/1 = 3 \\
  1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \text{ } \\
  -10 & 0 & 1 & 0 & -3 & 0 & 1 & 9 & \text{ } \\
\end{array}
\]

We zero out the $z$ column with the indicated operations.

\[
\begin{array}{rrrrrr}
  x & y & z & s & t & u & P \\
  -3 & 0 & 2 & -1 & -1 & 0 & 0 & 4 & 2R_2 - R_1 \\
  -1 & 2 & 0 & 1 & -1 & 0 & 0 & 2 & \text{ } \\
  1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \text{ } \\
  -17 & 0 & 0 & 1 & -5 & 0 & 2 & 14 & 2R_4 - R_1 \\
\end{array}
\]

\[
x = 0 \quad 2y = 2 \quad 2z = 4 \quad s = 0 \quad t = 0 \quad u = 0 \quad 2P = 14 \quad P = 7 \\
y = 1 \quad z = 2 \quad \text{ } \\
\]

Since all of the decision, slack, and surplus variables are positive, we are in the feasible region. We may now move to Stage 2 and apply the simplex method. Since the $x$ column contains the negative value in the bottom row with the largest magnitude, it is the pivot column. Since there is only one positive entry in the $x$ column, it is the pivot.
We zero out the remaining terms in the $x$ column using the indicated operations.

\[
\begin{bmatrix}
0 & 0 & 2 & -1 & 2 & 3 & 0 & 4 \\
0 & 2 & 0 & 1 & 0 & 1 & 0 & 2 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 12 & 17 & 2 & 14
\end{bmatrix}
\]

$R_1 + 3R_3$

$R_2 + R_3$

$R_4 + 17R_1$

$x = 0$  \hspace{1cm} 2$y = 2$  \hspace{1cm} 2$z = 4$  \hspace{1cm}  s = 0$  \hspace{1cm}  t = 0  \hspace{1cm}  u = 0  \hspace{1cm}  2P = 14$

$y = 1$  \hspace{1cm}  z = 2$  \hspace{1cm}  P = 7$

Since all of the entries in the bottom row of the tableau are nonnegative, this solution is the optimal solution. When $x = 0$, $y = 1$, and $z = 2$, the objective function $P = 4x + 3y + 2z$ attains a maximum value of 7.

Figure 4.36 shows the graph of the constraints and the optimal solution. Each constraint is a plane. The planes intersect at $(0, 1, 2)$.

**EXAMPLE 3**

Using Linear Programming to Make Investment Decisions

An investor plans to invest at most $3000 in the three publicly traded recreational vehicle companies shown in Table 4.26. (Share prices are rounded to the nearest dollar, and dividends per share are rounded to the nearest dime.)

<table>
<thead>
<tr>
<th>Company</th>
<th>Share Price (dollars)</th>
<th>Dividends/Share (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harley-Davidson, Inc. (HDI)</td>
<td>63</td>
<td>0.40</td>
</tr>
<tr>
<td>Polaris Industries Inc. (PII)</td>
<td>51</td>
<td>0.90</td>
</tr>
<tr>
<td>Winnebago Industries Inc. (WGO)</td>
<td>33</td>
<td>0.20</td>
</tr>
</tbody>
</table>

*Source: moneycentral.msn.com. (Accurate as of July 16, 2004.)*
He wants to earn at least $50 in dividends while maximizing the number of shares purchased. How many shares of each company should he purchase?

**SOLUTION** Let \( x \) be the number of shares of Harley-Davidson, \( y \) be the number of shares of Polaris Industries, and \( z \) be the number of shares of Winnebago Industries. We must maximize \( P = x + y + z \) subject to

\[
\begin{align*}
63x + 51y + 33z &\leq 3000 & \text{The total amount invested is at most $3000} \\
0.4x + 0.9y + 0.2z &\geq 50 & \text{Total dividends are at least $50} \\
x &\geq 0, y &\geq 0, z &\geq 0
\end{align*}
\]

We add in the slack and surplus variables and rewrite the objective function.

\[
\begin{align*}
63x + 51y + 33z + s &= 3000 \\
0.4x + 0.9y + 0.2z - t &= 50 \\
-x - y - z + P &= 0
\end{align*}
\]

We then write the initial tableau and star the row corresponding with a negative variable value.

\[
\begin{array}{cccccc}
 x & y & z & s & t & P \\
\hline
63 & 51 & 33 & 1 & 0 & 0 & 3000 \\
0.4 & 0.9 & 0.2 & 0 & -1 & 0 & 50 \\
-1 & -1 & -1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

We zero out the \( y \) column with the indicated operations and star the row corresponding with a negative variable value.

\[
\begin{array}{cccccc}
 x & y & z & s & t & P \\
\hline
63 & 51 & 33 & 1 & 0 & 0 & 3000 \\
0.4 & 0.9 & 0.2 & 0 & -1 & 0 & 50 \\
-1 & -1 & -1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

The largest entry in the starred row is the 0.9 in the \( y \) column; hence, the \( y \) column is the pivot column. We calculate the test ratios and locate the pivot.

\[
\begin{array}{cccccc}
 x & y & z & s & t & P \\
\hline
63 & 51 & 33 & 1 & 0 & 0 & 3000 \\
0.4 & 0.9 & 0.2 & 0 & -1 & 0 & 50 \\
-1 & -1 & -1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

We zero out the \( y \) column with the indicated operations and star the row corresponding with a negative variable value.
Since all of the decision, slack, and surplus variables are nonnegative, we are in the feasible region and may move to Stage 2.

The negative entry in the bottom row of a labeled column with the largest magnitude is in the \( t \) column. Consequently, the \( t \) column is the pivot column. Since 51 is the only positive entry in the \( t \) column, it is the pivot. We zero out the \( t \) column with the indicated operations.

The \( z \) column is the new pivot column, since it contains a negative entry in the bottom row. We calculate the test ratios and identify the pivot.

We zero out the \( z \) column with the indicated operations.

The investor should purchase 53.85 shares of Polaris Industries and 7.69 shares of Winnebago Industries in order to maximize the number of shares while simultaneously earning dividends of at least $50.

Minimization Problems

Examples 1, 2, and 3 showed how to maximize an objective function of a linear programming problem with mixed constraints. What if we want to minimize the objective function? Fortunately, with one minor modification to the objective function, the same procedure works. We begin by observing the relationship between a function \( f \) and the function \( -f = -1 \cdot f \) shown in Figure 4.37.
Observe that the maximum of $f$ and the minimum of $-f$ both occur at $x = a$. Similarly, the minimum of $f$ and the maximum of $-f$ both occur at $x = b$. This relationship holds true for all functions. Consequently, if we want to minimize an objective function $P$, we need only maximize the function $-P$.

**Example 4**

Solving a General Linear Programming Problem

Solve the general linear programming problem:

Minimize $P = 3x + 5y + 2z$

Subject to

$$
\begin{align*}
-x + 2y - z &\leq 0 \\
-2x + 4y + z &\geq 3 \\
-x + y + z &\leq 3 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
$$

**Solution**

We first introduce a new function $C = -P$.

$$
C = -P = -(3x + 5y + 2z) = -3x - 5y - 2z
$$

The solution that maximizes $C$ will minimize $P$. Therefore, the linear programming problem may be rewritten as

Maximize $C = -3x - 5y - 2z$

Subject to

$$
\begin{align*}
x + 2y - z &\leq 0 \\
-2x + 4y + z &\geq 3 \\
-x + y + z &\leq 3 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
$$
We add in the slack and surplus variables and rewrite the objective function. The system of equations is
\[
\begin{align*}
  x + 2y - z + s &= 0 \\
  -2x + 4y + z - t &= 3 \\
  -x + y + z + u &= 3 \\
  3x + 5y + 2z + C &= 0
\end{align*}
\]
and has the corresponding tableau
\[
\begin{bmatrix}
  x & y & z & s & t & u & C \\
  1 & 2 & -1 & 1 & 0 & 0 & 0 \\
  -2 & 4 & 1 & 0 & -1 & 0 & 0 \\
  -1 & 1 & 1 & 0 & 0 & 1 & 0 \\
  3 & 5 & 2 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
\[
x = 0 \quad y = 0 \quad z = 0 \quad s = 0 \quad -t = 3 \quad u = 3 \quad C = 0
\]
\[
t = -3
\]
Since the surplus variable \( t \) is negative, we star the corresponding row. The \( y \) column is the pivot column because it contains the largest positive entry in the starred row. Calculating the test ratios, we select the pivot.
\[
\begin{bmatrix}
  x & y & z & s & t & u & C \\
  1 & 2 & -1 & 1 & 0 & 0 & 0 \\
  -2 & 4 & 1 & 0 & -1 & 0 & 0 \\
  -1 & 1 & 1 & 0 & 0 & 1 & 0 \\
  3 & 5 & 2 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
\[
\begin{align*}
  0/2 &= 0 \\
  3/4 &= 0.75 \\
  3/1 &= 3
\end{align*}
\]
We zero out the remaining terms in the \( y \) column using the indicated operations and star the row corresponding with the negative variable value.
\[
\begin{bmatrix}
  x & y & z & s & t & u & C \\
  1 & 2 & -1 & 1 & 0 & 0 & 0 \\
  * & -4 & 0 & 3 & -2 & -1 & 0 & 0 & 3 \\
  -3 & 0 & 3 & -1 & 0 & 2 & 0 & 6 \\
  1 & 0 & 9 & -5 & 0 & 0 & 2 & 0 \\
\end{bmatrix}
\]
\[
x = 0 \quad 2y = 0 \quad z = 0 \quad s = 0 \quad -t = 3 \quad 2u = 6 \quad 2C = 0
\]
\[
y = 0 \quad t = -3 \quad u = 3 \quad C = 0
\]
The largest positive value in the starred row is the 3 in the \( z \) column. Consequently, the \( z \) column is the pivot column. We calculate the test ratios and select the pivot.
\[
\begin{bmatrix}
  x & y & z & s & t & u & C \\
  1 & 2 & -1 & 1 & 0 & 0 & 0 \\
  -4 & 0 & 3 & -2 & -1 & 0 & 0 & 3 \\
  * & -3 & 0 & 3 & -1 & 0 & 2 & 0 & 6 \\
  1 & 0 & 9 & -5 & 0 & 0 & 2 & 0 \\
\end{bmatrix}
\]
\[
\begin{align*}
  3/3 &= 1 \\
  6/3 &= 2
\end{align*}
\]
We zero out the remaining terms in the \( z \) column with the indicated operations.

\[
\begin{array}{ccccccc}
   x & y & z & s & t & u & C \\
   \begin{bmatrix}
      -1 & 6 & 0 & 1 & -1 & 0 & 0 \\
      -4 & 0 & 3 & -2 & -1 & 0 & 0 \\
      1 & 0 & 0 & 1 & 1 & 2 & 0 \\
   \end{bmatrix}
   & 3R_1 + R_2 \\
   & R_1 - R_2 \\
   & R_4 - 3R_3 \\
\end{array}
\]

\( x = 0 \quad 6y = 3 \quad 3z = 3 \quad s = 0 \quad t = 0 \quad 2u = 3 \quad 2C = -9 \)

\( y = \frac{1}{2} \quad z = 1 \quad u = \frac{3}{2} \quad C = -\frac{9}{2} \)

Since all decision, slack, and surplus variables are nonnegative, we are now in the feasible region. (There is no restriction on the sign of the objective function. In this case, \( C \) is negative.) We may now move to Stage 2 of the process. However, since no entry in the bottom row of a labeled column is negative, this is the final tableau. The objective function \( C = -3x - 5y - 2z \) is maximized at \( \left( 0, \frac{1}{2}, 1 \right) \).

Consequently, the objective function \( P = 3x + 5y + 2z \) is minimized at \( \left( 0, \frac{1}{2}, 1 \right) \).

The maximum value of \( C \) is \( -\frac{9}{2} \). Since \( P = -C \), the minimum value of \( P \) is \( \frac{9}{2} \).

So far, the general linear programming problems we have demonstrated have included only linear inequalities as constraints. However, some linear programming problems have linear equations as constraints. Any linear equation may be written as a system of linear inequalities. For example, consider the linear equation \( 2x + 3y = 5 \). This equation is equivalent to the following system of linear inequalities:

\[
\begin{align*}
2x + 3y & \leq 5 \\
2x + 3y & \geq 5
\end{align*}
\]

When we are given a linear equation as a constraint, we will rewrite it as a system of linear inequalities, as demonstrated in Example 5.

### EXAMPLE 5

**Using Linear Programming to Make Investment Decisions**

An investor has $5000 to invest in three mutual funds: Bond, Index, and Growth. The Bond fund is expected to earn 7 percent; the Index fund, 11 percent; and the Growth fund, 9 percent. The investor wants to earn an annual return of at least 10 percent while minimizing the amount invested in the Index fund. How much should she invest in each account?

**SOLUTION** Let \( x \) be the amount invested in the Bond fund, \( y \) be the amount invested in the Index fund, and \( z \) be the amount invested in the Growth fund. We must solve the following linear programming problem.

Minimize \( P = y \)

Subject to

\[
\begin{align*}
x + y + z &= 5000 \\
0.07x + 0.11y + 0.09z &\geq 0.1(5000) \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

\$5000 is invested

The return is at least 10 percent of $5000
We rewrite the first constraint as two inequalities and simplify the second constraint. We then create a new objective function. Since \( P = y \), \( C = -y \).

The new linear programming problem is

\[
\begin{align*}
\text{Maximize} & \quad C = -y \\
\text{Subject to} & \quad \begin{cases} 
  x + y + z \leq 5000 \\
  x + y + z \geq 5000 \\
  0.07x + 0.11y + 0.09z \geq 5 \\
  x \geq 0, y \geq 0, z \geq 0
\end{cases}
\end{align*}
\]

We add the surplus and slack variables and rewrite the objective function.

\[
\begin{align*}
  x + y + z + s &= 5000 \\
  x + y + z - t &= 5000 \\
  0.07x + 0.11y + 0.09z - u &= 500 \\
  y + C &= 0
\end{align*}
\]

We write the initial tableau and star the appropriate rows.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( s )</th>
<th>( t )</th>
<th>( u )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>*1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*0.07</td>
<td>0.11</td>
<td>0.09</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\( x = 0 \) \( \quad y = 0 \) \( \quad z = 0 \) \( \quad s = 5000 \) \( \quad -t = 5000 \) \( \quad -u = 500 \) \( \quad C = 0 \) \( \quad t = -5000 \) \( \quad u = -500 \)

Although we may use either starred row, we pick the second row of the tableau because the entries are simpler. We may pick the \( x \), \( y \), or \( z \) column as the pivot column. We pick the \( x \) column. We calculate the test ratios and identify the pivot.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( s )</th>
<th>( t )</th>
<th>( u )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.07</td>
<td>0.11</td>
<td>0.09</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\( x = 0 \) \( \quad y = 0 \) \( \quad z = 0 \) \( \quad s = 5000 \) \( \quad -t = 5000 \) \( \quad -u = 500 \) \( \quad C = 0 \) \( \quad t = -5000 \) \( \quad u = -500 \)

We may pick either the first or the second term in the \( x \) column to be the pivot. We pick the first term and zero out the remaining entries with the indicated operations. We then star the appropriate row.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( s )</th>
<th>( t )</th>
<th>( u )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*0</td>
<td>0.04</td>
<td>0.02</td>
<td>-0.07</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\( x = 5000 \) \( \quad y = 0 \) \( \quad z = 0 \) \( \quad s = 0 \) \( \quad -t = 0 \) \( \quad -u = 150 \) \( \quad C = 0 \) \( \quad t = 0 \) \( \quad u = -150 \)
The \( y \) column is the pivot column, since the largest positive entry in the starred row is the 0.04 in the \( y \) column. We calculate the test ratios and locate the pivot.

\[
\begin{bmatrix}
    x & y & z & s & t & u & C \\
    1 & 1 & 1 & 1 & 0 & 0 & 0 & 5000 \\
    0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
    \star & 0.04 & 0.02 & -0.07 & 0 & -1 & 0 & 150 \\
    0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

5000/1 = 5000

150/0.04 = 3750

We zero out the \( y \) column with the indicated operations.

\[
\begin{bmatrix}
    x & y & z & s & t & u & C \\
    0.04 & 0 & 0.02 & 0.11 & 0 & 1 & 0 & 50 \\
    0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
    0 & 0.04 & 0.02 & -0.07 & 0 & -1 & 0 & 150 \\
    0 & 0 & -0.02 & 0.07 & 0 & 1 & 0.04 & -150 \\
\end{bmatrix}
\]

0.04\( x = 50 \)

0.04\( y = 150 \)

\( z = 0 \)

\( s = 0 \)

\( -t = 0 \)

\( u = 0 \)

0.04\( C = -150 \)

\( x = 1250 \)

\( y = 3750 \)

\( t = 0 \)

\( C = -3750 \)

Since all decision, slack, and surplus variables are nonnegative, we are in the feasible region and may proceed to Stage 2.

Since the only negative entry in the bottom row of a labeled column is the \(-0.02\) in the \( z \) column, the \( z \) column is the pivot column. We calculate the test ratios and identify the pivot.

\[
\begin{bmatrix}
    x & y & z & s & t & u & C \\
    0.04 & 0 & 0.02 & 0.11 & 0 & 1 & 0 & 50 \\
    0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
    0 & 0.04 & 0.02 & -0.07 & 0 & -1 & 0 & 150 \\
    -0.04 & 0.04 & 0 & -0.18 & 0 & -2 & 0 & 100 \\
\end{bmatrix}
\]

\( 50/0.02 = 2500 \)

\( 150/0.04 = 3750 \)

We zero out the remaining terms in the pivot column with the indicated operations.

\[
\begin{bmatrix}
    x & y & z & s & t & u & C \\
    0.04 & 0 & 0.02 & 0.11 & 0 & 1 & 0 & 50 \\
    0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
    -0.04 & 0.04 & 0 & -0.18 & 0 & -2 & 0 & 100 \\
    0.04 & 0 & 0 & 0.18 & 0 & 2 & 0.04 & -100 \\
\end{bmatrix}
\]

\( R_1 - R_1 \)

\( R_1 + R_1 \)

\( x = 0 \)

0.04\( y = 100 \)

0.02\( z = 50 \)

\( s = 0 \)

\( -t = 0 \)

\( u = 0 \)

0.04\( C = -100 \)

\( y = 2500 \)

\( z = 2500 \)

\( t = 0 \)

\( C = -2500 \)

Since all entries in the bottom row are positive, this is the final tableau. The objective function \( C \) reaches its maximum value when \( x = 0, y = 2500, \) and \( z = 2500 \). The maximum value is \( C = -2500 \). The objective function \( P = -C \) attains its minimum value of 2500 at the same point. In the context of the problem, a 10 percent return is earned when $0 is invested in the Bond fund, $2500 is invested in the Index fund, and $2500 is invested in the Growth fund. This solution minimizes the amount of money invested in the Index fund.
In Exercises 1–5, rewrite the general linear programming problem as a system of equations with slack and surplus variables and a rewritten objective function.

1. Maximize \( P = 4x + 2y + 3z \)
   \[
   \begin{align*}
   2x + y + z & \leq 10 \\
   x + z & \geq 3 \\
   x - 2z & \geq 1 \\
   x \geq 0, y \geq 0, z \geq 0
   \end{align*}
   \]

2. Maximize \( P = 2x + 10y + z \)
   \[
   \begin{align*}
   -x + 3y & \leq 2 \\
   3x + z & \geq 4 \\
   5x - 4z & \geq 1 \\
   x \geq 0, y \geq 0, z \geq 0
   \end{align*}
   \]

3. Maximize \( P = 3x - y + 9z \)
   \[
   \begin{align*}
   -12x + 8y + 2z & \leq 15 \\
   8x + 5y - 2z & \geq 10 \\
   3x - 2y + 6z & \geq 5 \\
   x \geq 0, y \geq 0, z \geq 0
   \end{align*}
   \]

4. Maximize \( P = x - z \)
   \[
   \begin{align*}
   -x + y + z & \leq 12 \\
   2x + y - 2z & \leq 1 \\
   x \geq 0, y \geq 0, z \geq 0
   \end{align*}
   \]

5. Maximize \( P = 10x + 9y + z \)
   \[
   \begin{align*}
   4x - 3y - 2z & \leq 12 \\
   -2x + 4y + 6z & \leq 12 \\
   3x - 2y + 4z & \geq 12 \\
   x \geq 0, y \geq 0, z \geq 0
   \end{align*}
   \]

In Exercises 6–10, star the rows of the tableau that correspond with negative variable values.

6. \[
   \begin{array}{cccccccc}
   x & y & s & t & u & P \\
   \hline
   -1 & 2 & 1 & 0 & 0 & 4 \\
   2 & -3 & 0 & 1 & 0 & 6 \\
   4 & 1 & 0 & 0 & -1 & 0 \\
   -6 & -2 & 0 & 0 & 0 & 1 \\
   \end{array}
   \]

7. \[
   \begin{array}{cccccccc}
   x & y & s & t & u & P \\
   \hline
   9 & 4 & -1 & 0 & 0 & 10 \\
   2 & 5 & 0 & 1 & 0 & 8 \\
   1 & 1 & 0 & 0 & -1 & 12 \\
   5 & 4 & 0 & 0 & 0 & 1 \\
   \end{array}
   \]

8. \[
   \begin{array}{cccccccc}
   x & y & s & t & u & P \\
   \hline
   -1 & 2 & -1 & 0 & 0 & 4 \\
   2 & 0 & -2 & 1 & 0 & 6 \\
   4 & 0 & 1 & 0 & -5 & 5 \\
   -6 & 0 & 3 & 0 & 0 & 4 \\
   \end{array}
   \]

9. \[
   \begin{array}{cccccccc}
   x & y & s & t & u & P \\
   \hline
   0 & 6 & -2 & 0 & 0 & 16 \\
   5 & 1 & 0 & 1 & 0 & 1 \\
   0 & 3 & 0 & 4 & -4 & 6 \\
   0 & 10 & 0 & 5 & 0 & 3 \\
   \end{array}
   \]

10. \[
    \begin{array}{cccccccc}
    x & y & s & t & u & P \\
    \hline
    0 & 3 & 0 & 0 & 3 & 15 \\
    0 & 0 & 1 & 1 & 2 & 0 \\
    2 & 0 & 0 & -2 & -4 & 4 \\
    0 & 0 & 0 & 6 & 3 & 1 \\
    \end{array}
    \]

In this section, you learned how to solve general linear programming problems. You discovered that minimization problems may be solved by maximizing the negative of the objective function.
In Exercises 11–20, graph the feasible region of the general linear programming problem. Then solve the problem by using the methods demonstrated in the section. For each tableau, indicate on the graph of the feasible region the point that corresponds with the solution of the tableau.

11. Maximize \[ P = 2x - 3y \]
Subject to \[
\begin{align*}
2x + y & \leq 10 \\
x + y & \geq 3 \\
3x - y & \geq 1 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

12. Maximize \[ P = 4x + y \]
Subject to \[
\begin{align*}
2x + y & \leq 10 \\
2x + 3y & \geq 14 \\
-2x + y & \leq 2 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

13. Maximize \[ P = 5x - 8y \]
Subject to \[
\begin{align*}
2x + y & \leq 13 \\
9x - 3y & \leq 6 \\
-x + 2y & \geq 6 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

14. Maximize \[ P = 6x + 3y \]
Subject to \[
\begin{align*}
-x + y & \geq 0 \\
2x + y & \geq 9 \\
x + 2y & \geq 6 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

15. Maximize \[ P = -3x + 4y \]
Subject to \[
\begin{align*}
3x + 12y & \geq 12 \\
2x + y & \leq 8 \\
x + y & \leq 6 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

16. Minimize \[ P = 3x + 5y \]
Subject to \[
\begin{align*}
-5x + 2y & \leq 2 \\
-x + y & \leq 4 \\
-7x + 4y & \geq 4 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

17. Minimize \[ P = 7x + 4y \]
Subject to \[
\begin{align*}
3x + y & \geq 10 \\
x + y & \geq 6 \\
3x + 2y & \leq 17 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

18. Minimize \[ P = -3x + 8y \]
Subject to \[
\begin{align*}
-2x + y & \geq 6 \\
2x + y & \geq 8 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

19. Minimize \[ P = -x + y \]
Subject to \[
\begin{align*}
x + y & \geq 1 \\
2x - 3y & \geq 2 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

20. Maximize \[ P = 4x + 2y \]
Subject to \[
\begin{align*}
5x + 2y & \geq 17 \\
x - 2y & \leq 1 \\
x & \geq 0, y & \geq 0
\end{align*}
]\]

In Exercises 21–35, solve the general linear programming problem. If there is no solution, so state.

21. Maximize \[ P = 3x + 2y + z \]
Subject to \[
\begin{align*}
x + y & \leq 10 \\
x + z & \leq 5 \\
y + z & \geq 4 \\
x & \geq 0, y & \geq 0, z & \geq 0
\end{align*}
]\]

22. Maximize \[ P = 5x + 6y + 2z \]
Subject to \[
\begin{align*}
x + 2y & \leq 10 \\
2x + z & \leq 8 \\
y + 2z & \geq 6 \\
x & \geq 0, y & \geq 0, z & \geq 0
\end{align*}
]\]

23. Maximize \[ P = x - y + z \]
Subject to \[
\begin{align*}
4x + 3y + 2z & \geq 12 \\
2x + 2y + z & \leq 12 \\
x + y + 4z & \geq 4 \\
x & \geq 0, y & \geq 0, z & \geq 0
\end{align*}
]\]

24. Maximize \[ P = 6x + y \]
Subject to \[
\begin{align*}
5x + 2y + 4z & \leq 23 \\
x + y + z & \geq 4 \\
x - y + z & \geq 4 \\
x & \geq 0, y & \geq 0, z & \geq 0
\end{align*}
]\]

25. Maximize \[ P = 2z \]
Subject to \[
\begin{align*}
x + 2y + 3z & \geq 12 \\
3x + y + 2z & \leq 12 \\
2x + 3y + z & \leq 12 \\
x & \geq 0, y & \geq 0, z & \geq 0
\end{align*}
]\]
26. Maximize \[ P = 2x + y + z \]
Subject to
\[
\begin{align*}
-x + y + z &\leq 5 \\
 x - y + z &\leq 7 \\
-x - y + z &\geq 3 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

27. Maximize \[ P = 6x - 2y + 6z \]
Subject to
\[
\begin{align*}
-x - y + z &\leq 1 \\
x + y + z &\leq 7 \\
2x - 3y + z &\leq 12 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

28. Minimize \[ P = 4y + 9z \]
Subject to
\[
\begin{align*}
-2x - 3y + z &\geq 0 \\
4x + y + z &\geq 10 \\
-4x + 2y + z &\geq 3 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

29. Minimize \[ P = -x - y + 4z \]
Subject to
\[
\begin{align*}
-2x + 3y + z &\geq 4 \\
2x + 4y + z &\geq 4 \\
-4x + y + z &\leq 12 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

30. Minimize \[ P = 5x + 4y + 3z \]
Subject to
\[
\begin{align*}
2x - 5y + z &\geq 4 \\
2x + 4y + z &\leq 13 \\
z &\geq 3 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

31. Minimize \[ P = 9x - 3y - 3z \]
Subject to
\[
\begin{align*}
-3x - 4y + z &\leq 4 \\
2x - 5y + z &\leq 14 \\
3x - 2y + z &\geq 8 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

32. Minimize \[ P = x - y + 3z \]
Subject to
\[
\begin{align*}
2x + 2y - z &\leq 4 \\
5x - 4y + z &\geq 16 \\
 x - 4y + z &\geq 5 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

33. Minimize \[ P = 6x - 2y + 4z \]
Subject to
\[
\begin{align*}
8x + y - z &\leq 8 \\
x - y + z &\geq 1 \\
x - y + z &\leq 10 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

34. Minimize \[ P = x + 2y + 3z \]
Subject to
\[
\begin{align*}
x + y + z &\geq 6 \\
x - y + z &\geq 2 \\
x + z &\leq 4 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

35. Maximize \[ P = 4x + 3y + 2z \]
Subject to
\[
\begin{align*}
x + y + z &\geq 8 \\
-2x + z &\geq 1 \\
3y + 5z &\leq 27 \\
x &\geq 0, y \geq 0, z \geq 0
\end{align*}
\]

In Exercises 36–45, use the techniques demonstrated in the section to set up and solve each problem.

36. **Commodity Prices** Today’s Market Prices (www.todaymarket.com) is a daily fruit and vegetable wholesale market price service. Produce retailers who subscribe to the service can use wholesale prices to aid them in setting retail prices for the fruits and vegetables they sell.

A 25-pound carton of peaches holds 60 medium peaches or 70 small peaches. In August 2002, the wholesale price for local peaches in Los Angeles was $9.00 per carton for medium peaches and $10.00 per carton for small peaches. (Source: Today’s Market Prices.) A fruit vendor has budgeted up to $100 to spend on peaches. He estimates that weekly demand for peaches is no more than 660 peaches. He wants to buy at least four boxes of each size of peach. Subject to these constraints, how many boxes of each size of peach should he buy in order to maximize the number of peaches available for sale?

37. **Battery Sales** AAA Alkaline Discount Batteries sells low-cost batteries to consumers. In August 2002, the firm offered AA batteries at the following prices: 50-pack for $10.00, 100-pack for $18.00, and 600-pack for $96.00. (Source: www.aaaalkalinediscountbatteries.com.)

An electronics store owner wants to buy at least 900 batteries and spend at most $150. She expects that she’ll be able to resell all of the batteries she orders for $1.50 each. How many packs (50-packs, 100-packs, or 600-packs) should she order if she wants to maximize her revenue? What is her maximum revenue? (Hint: At most one 600-pack may be purchased without exceeding the $150 limit.)
38. **Resource Allocation: Food** An ice cream parlor wants to make three different types of ice cream: vanilla, strawberry, and peach-cherry. The parlor has 120 cups of cream, 48 eggs, and 32 cups of sugar on hand. The vanilla ice cream recipe calls for 4 cups of cream, 1 egg, and 0.75 cup of sugar. The strawberry ice cream recipe calls for 2 cups of cream, 2 eggs, and 0.75 cup of sugar. The peach-cherry ice cream recipe calls for 4 cups of cream, 1 egg, and 1.25 cups of sugar. Each recipe yields 1.5 quarts of ice cream. The parlor needs at least 18 quarts of vanilla and at least 6 quarts of each of the other varieties. The parlor wants to maximize the amount of cream produced. How many batches of each variety of ice cream does the parlor need to produce?

39. **Resource Allocation: Food** A delicatessen makes three types of pudding: rice, tapioca, and vanilla. The deli has 108 cups of milk, 150 cups of sugar, and 84 eggs on hand. The rice pudding recipe requires 12 cups of milk, 1.5 cups of sugar, and 9 eggs and yields 24 servings. The tapioca pudding recipe requires 12 cups of milk, 1.5 cups of sugar, and 9 eggs and yields 18 servings. The vanilla pudding recipe requires 6 cups of milk, 1.5 cups of sugar, and 6 eggs and yields 12 servings. The deli requires that the sum of the number of batches of tapioca pudding and the number of batches of rice pudding be at least two. How many batches of each recipe must the deli produce in order to maximize the number of servings?

40. **Advertising** A large company advertises through magazines, radio, and television. For every $10,000 spent on magazine advertising, the company estimates that it reaches 150,000 people. For every $10,000 spent on radio advertising, the company estimates that it reaches 250,000 people. For every $10,000 spent on television advertising, the company estimates that it reaches 300,000 people. The company has at most $2.5 million to spend on advertising. It requires that at least twice as much be spent on radio as on television and that the amount spent on magazines be at least $150,000 more than the amount spent on radio. How much should the company spend on each type of advertising in order to maximize the number of people reached?

41. **Nutrition** The nutritional content of canned beans varies based on the type of bean and the manufacturer. A 1/2-cup serving of Fred Meyer Pinto Beans contains 7 grams of fiber, 1 gram of sugar, and 6 grams of protein. A 1/2-cup serving of Fred Meyer Kidney Beans contains 11 grams of fiber, 1 gram of sugar, and 8 grams of protein. A 1/2-cup serving of Trader Joe’s Cuban Style Black Beans contains 2 grams of fiber, 1 gram of sugar, and 6 grams of protein. (Source: Package labeling.) A bean dish is to be made using the three bean varieties. The dish can contain at most 3 cups of beans but must include at least 39 grams of fiber and 42 grams of protein. How much of each type of bean should be included in the dish in order to minimize the amount of kidney beans used? (Fractions of cups may be used.)

42. **Nutrition** The nutritional content of canned vegetables varies based on the type of vegetable and the manufacturer. A 1/2-cup serving of S&W Cut Blue Lake Green Beans contains 2 grams of fiber, 2 grams of sugar, and 20 calories. A 1/2-cup serving of Safeway Golden Sweet Whole Kernel Corn contains 2 grams of fiber, 6 grams of sugar, and 80 calories. A 1/2-cup serving of Pot O’Gold Sliced Carrots contains 2 grams of fiber, 4 grams of sugar, and 30 calories. (Source: Package labeling.) A dish is to be made using the three types of vegetables. The dish can contain at most 5 cups of vegetables but must include at least 18 grams of fiber and at least 32 grams of sugar (for flavor). How much of each type of vegetable should be included in the dish in order to minimize the amount of calories in the dish?

43. **Resource Allocation: Schools** A city has two elementary schools. The first school has a maximum enrollment of 500 students, and the second school has a maximum enrollment of 420 students. The city is divided into two regions: North and South. There are at least 400 students in the North region and at least 430 students in the South region. The annual transportation cost varies by region as shown in the table.

<table>
<thead>
<tr>
<th>Transportation Costs</th>
<th>School 1 (cost per student)</th>
<th>School 2 (cost per student)</th>
</tr>
</thead>
<tbody>
<tr>
<td>North</td>
<td>120</td>
<td>180</td>
</tr>
<tr>
<td>South</td>
<td>100</td>
<td>150</td>
</tr>
</tbody>
</table>
Based on these constraints, what is the minimum possible transportation cost, and under what conditions does it occur?

44. **Resource Allocation: Schools**  A city has two high schools. The first school has a maximum enrollment of 900 students, and the second school has a maximum enrollment of 550 students. The city is divided into two regions: Inner City and Suburbs. There are at least 500 students in the Inner City region and at least 800 students in the Suburbs region. The annual transportation cost varies by region as shown in the table.

<table>
<thead>
<tr>
<th>Transportation Costs</th>
<th>School 1 (cost per student)</th>
<th>School 2 (cost per student)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner City</td>
<td>220</td>
<td>260</td>
</tr>
<tr>
<td>Suburbs</td>
<td>200</td>
<td>280</td>
</tr>
</tbody>
</table>

Based on these constraints, what is the minimum possible transportation cost, and under what conditions does it occur?

45. **Transportation Costs**  A furniture company has warehouses in Phoenix, Arizona, and Las Vegas, Nevada. The company has customers in Kingman, Arizona, and Flagstaff, Arizona. Its Phoenix warehouse has 500 desks in stock, and its Las Vegas warehouse has 200 desks in stock. Its Kingman customer needs at least 250 desks, and its Flagstaff customer needs at least 400 desks. Based on a rate of $0.01 per mile per item, the cost of delivery per item is shown in the table.

<table>
<thead>
<tr>
<th></th>
<th>Phoenix</th>
<th>Las Vegas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kingman</td>
<td>$1.85</td>
<td>$1.05</td>
</tr>
<tr>
<td>Flagstaff</td>
<td>$1.45</td>
<td>$2.51</td>
</tr>
</tbody>
</table>

Subject to these constraints, under what conditions are the company’s delivery costs minimized?

**Exercises 46–50 are intended to challenge your understanding of general linear programming problems.**

46. Maximize  
\[
P = x + y + z + w \]
\[
\begin{align*}
x + y + z & \leq 9 \\
y + z + w & \leq 8 \\
x + y + w & \geq 7 \\
x & \geq 0, y \geq 0, z \geq 0, w \geq 0
\end{align*}
\]

Subject to

47. Minimize  
\[
P = x + y + z + w \]
\[
\begin{align*}
x + y + z & \leq 6 \\
y + z + w & \leq 8 \\
x + z + w & \geq 9 \\
x + y + w & \geq 7 \\
x & \geq 0, y \geq 0, z \geq 0, w \geq 0
\end{align*}
\]

48. Create a two-variable linear programming problem with three or more mixed constraints with the property that the feasible region consists of exactly one point. Provide evidence that the feasible region contains exactly one point.

49. Create a three-variable linear programming problem with three or more mixed constraints with the property that the feasible region is empty. Provide evidence that the feasible region is empty.

50. Explain what happens graphically in Stage 1 of the general linear programming problem-solving process.

---

**Section 4.1**  In Exercises 1–4, graph the solution region of the linear inequality. Then use the graph to determine if point \( P \) is a solution.

1. \( 4x - 2y \leq 6; P = (2, 9) \)
2. \( 3x + 5y \leq 0; P = (1, 7) \)
3. \( 9x - 8y \leq 12; P = (8, 9) \)
4. \( 7x + 6y \leq 42; P = (3, 4) \)
In Exercises 5–8, graph the solution region of the system of linear equations. If there is no solution, explain why.

5. \(-4x + 3y \geq 2\)  \(\begin{align*}
-3x + 2y &\geq 1 \\
2x + y &\leq 4
\end{align*}\)

6. \(-10x + y \geq 0\)

7. \(2x + 4y \leq 8\)

8. \(x + y \leq 8\)

\(6x - 2y \leq -6\)  \(-x + y \leq 0\)  \(4x - 2y \leq 8\)

In Exercise 9, set up and graphically solve the system of linear inequalities.

9. **Wages** A salaried employee earns $800 per week managing a copy center. She is required to work a minimum of 40 hours but no more than 50 hours weekly. As a side business, she earns $30 per hour designing brochures for local business clients. In order to maintain her standard of living, she must earn $1000 per week. In order to maintain her quality of life, she limits her workload to 50 hours per week. Given that she has no control over the number of hours she will have to work managing the copy center, will she be able to consistently meet her workload and income goals? Explain.

**Section 4.2** In Exercises 10–13, find the optimal solution to the linear programming problem, if it exists. If a solution does not exist, explain why.

10. Minimize \(z = 5x - 7y\)

Subject to \(\begin{align*}
4x + y &\geq 4 \\
-x + y &\geq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}\)

11. Minimize \(z = 9x + 4y\)

Subject to \(\begin{align*}
6x + y &\geq 16 \\
-2x + y &\geq 0 \\
x &\geq 0 \\
y &\geq 0
\end{align*}\)

12. Maximize \(z = 6x + 10y\)

Subject to \(\begin{align*}
6x + y &\leq 16 \\
-2x + y &\leq 0 \\
x &\geq 0 \\
y &\geq 0
\end{align*}\)

13. Maximize \(z = 5x - y\)

Subject to \(\begin{align*}
6x + 2y &\leq 16 \\
-3x - y &\leq 10 \\
x &\geq 0 \\
y &\geq 0
\end{align*}\)

For Exercise 14, identify the objective function and constraints of the linear programming problem. Then solve the problem and interpret the real-world meaning of your results.

14. **Family Food Storage** A family wants to purchase at least 75 pounds of beans. A #10 can of pinto beans weighs 5.0 pounds and costs $2.75. A #10 can of white beans weighs 5.3 pounds and costs $2.88. (Source: Kent Washington Cannery.) The family wants to buy at least 25 pounds of pinto beans and at least 53 pounds of white beans. It has budgeted $50 and, because of limited storage space, wants to minimize the number of cans purchased. How many cans of each type of beans should the family purchase?

**Section 4.3** In Exercises 15–16, determine if the problem is a standard maximization problem. If it isn’t, explain why.

15. Maximize \(P = -2x + 8y\)

Subject to \(\begin{align*}
2x + y &\leq 1 \\
-x + y &\leq -20
\end{align*}\)

16. Maximize \(P = -9x + 8y\)

Subject to \(\begin{align*}
42x + 19y &\leq 10 \\
-11x + 19y &\leq 21
\end{align*}\)

In Exercises 17–18, solve the standard maximization problems by using the simplex method. Check your answer by graphing the feasible region and calculating the value of the objective function at each of the corner points.

17. Maximize \(P = -2x + 10y\)

Subject to \(\begin{align*}
2x + y &\leq 10 \\
-x + y &\leq 1 \\
x &\geq 0 \\
y &\geq 0
\end{align*}\)
In Exercises 19–20, solve the standard maximization problems by using the simplex method.

19. Maximize \( P = 6x + 4y + 5z \)
\[
\begin{align*}
2x + 3y + 2z &\leq 120 \\
x + y + z &\leq 60 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

20. Maximize \( P = 4x - y + z \)
\[
\begin{align*}
2x + 3y + 3z &\leq 210 \\
x + y + z &\leq 100 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

In Exercises 19–20, solve the standard maximization problems by using the simplex method.

21. **Battery Sales** AAA Alkaline Discount Batteries sells low-cost batteries to consumers. In August 2002, the company offered AAA batteries at the following prices: 100-pack for $18.00, 600-pack for $96.00, and 1200-pack for $180.00.
(Source: www.aaaalkalinediscountbatteries.com.)

An electronics store owner wants to buy at most 1800 batteries and spend at most $300. She expects that she’ll be able to resell all of the batteries she orders for $1.00 each. How many packs (100-packs, 600-packs, or 1200-packs) should she order if she wants to maximize her revenue? What is her maximum revenue?

22. **Battery Sales** Repeat Exercise 21 except maximize profit. Assume that her only cost is the cost of the batteries. What is her maximum profit?

In Exercises 23–25, determine if the problem is a standard minimization problem. If it isn’t, explain why.

23. Minimize \( P = 11x + 8y + 2z \)
\[
\begin{align*}
2x + y + z &\geq 10 \\
x + y - z &\geq 1 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

24. Maximize \( P = -2x - 5y + 7z \)
\[
\begin{align*}
x + 2y + z &\geq 11 \\
-11x + y &\leq -21 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

25. Maximize \( P = 4x - 2y \)
\[
\begin{align*}
6x + 7y &\geq 13 \\
-8x - 4y &\geq -12 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

In Exercises 26–28, find the transpose of the given matrix.

26. \( A = \begin{bmatrix} 7 & -2 & 0 & 4 \\ 5 & 3 & 7 & 6 \\ -1 & 1 & 2 & 1 \end{bmatrix} \)

27. \( A = \begin{bmatrix} 12 & 10 & 8 \\ 14 & 12 & 10 \\ 16 & 14 & 12 \end{bmatrix} \)

28. \( A = \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 9 \\ 1 & 1 & 1 & 1 & 11 \end{bmatrix} \)

In Exercises 29–32, do the following:

(i) Write the dual problem for the given standard minimization problem.

(ii) Solve the dual problem using the simplex method.

(iii) Use the final simplex tableau of the dual problem to solve the standard minimization problem.
(iv) Check your answer by graphing the feasible region of the standard minimization problem and calculating the value of the objective function at each of the corner points.

29. Minimize \( P = 4x + 2y \)
Subject to
\[
\begin{align*}
4x + 5y & \geq 20 \\
2x + 3y & \geq 11 \\
x & \geq 0, \ y & \geq 0
\end{align*}
\]

30. Minimize \( P = x + 6y \)
Subject to
\[
\begin{align*}
2x + 9y & \geq 19 \\
x & \geq 0, \ y & \geq 0
\end{align*}
\]

31. Minimize \( P = 2x + 3y \)
Subject to
\[
\begin{align*}
5x + 2y & \geq 26 \\
10x - 5y & \geq 25 \\
x & \geq 0, \ y & \geq 0
\end{align*}
\]

32. Minimize \( P = 3x + 5y \)
Subject to
\[
\begin{align*}
9x + 4y & \geq 36 \\
3x + 2y & \geq 18 \\
x & \geq 0, \ y & \geq 0
\end{align*}
\]

In Exercises 33–34, set up and solve the standard minimization problem.

33. **Pet Nutrition: Food Cost**

PETsMART.com sold the following varieties of dog food in June 2003.

**Authority Chicken Adult Formula**, 32 percent protein, 3 percent fiber, $19.99 per 33-pound bag

**Bil-Jac Select Dog Food**, 27 percent protein, 4 percent fiber, $18.99 per 18-pound bag

**Iams Minichunks**, 26 percent protein, 4 percent fiber, $8.99 per 8-pound bag (Source: www.petsmart.com)

A dog breeder wants to make at least 300 pounds of a mix containing at least 24 percent protein. How many bags of each dog food variety should the breeder buy in order to minimize cost? (To make computations easier, round the price of each bag to the nearest dollar.)

34. **Pet Nutrition: Fat Content**

PETsMART.com sold the following varieties of dog food in June 2003.

**Nature’s Recipe Venison Meal & Rice Canine**, 20 percent protein, 10 percent fat, $21.99 per 20-pound bag

**Nutro Max Natural Dog Food**, 27 percent protein, 16 percent fat, $12.99 per 17.5-pound bag

**PETsMART Premier Oven Baked Lamb Recipe**, 25 percent protein, 14 percent fat, $22.99 per 30-pound bag (Source: www.petsmart.com)

A dog kennel wants to make at least 330 pounds of a mix containing at least 24 percent protein. How many bags of each dog food variety should the kennel buy in order to minimize fat content?

**Section 4.5** In Exercises 35–36, rewrite the general linear programming problem as a system of equations with slack and surplus variables and a rewritten objective function.

35. Maximize \( P = 2x + 3y + z \)
Subject to
\[
\begin{align*}
2x + 4y + z & \leq 10 \\
x + 2z & \geq 3 \\
x & \geq 0, \ y & \geq 0, \ z & \geq 0
\end{align*}
\]

36. Maximize \( P = x - 10y + 6z \)
Subject to
\[
\begin{align*}
-2x + 2y + 3z & \leq 2 \\
5x + 2y + 3z & \geq 4 \\
x & \geq 0, \ y & \geq 0, \ z & \geq 0
\end{align*}
\]

In Exercises 37–38, solve the general linear programming problem. If there is no solution, so state.

37. Maximize \( P = 4x + y + 10z \)
Subject to
\[
\begin{align*}
x + 3y + 2z & \leq 5 \\
2x + y + 2x & \geq 4 \\
x & \geq 0, \ y & \geq 0, \ z & \geq 0
\end{align*}
\]

38. Minimize \( P = 4x + y + 10z \)
Subject to
\[
\begin{align*}
x + 3y + 2z & \leq 5 \\
2x + y + 2x & \geq 4 \\
x & \geq 0, \ y & \geq 0, \ z & \geq 0
\end{align*}
\]
What to do

1. Find the wholesale and retail prices of three items of personal interest.
2. Estimate the number of cubic feet each item (or box of items) occupies.
3. Find a place in your home or workplace where you could store the items. Calculate the number of cubic feet in the storage area.
4. Assume that you may spend up to 100 times the retail price of the most expensive item in purchasing a combination of the three items.
5. Assume that you will be able to sell at retail price all of the items you order at wholesale. Determine how many of the items you should order if you want to maximize your profit subject to the spending and storage area constraints.

Where to Look for Prices

Wholesale Prices
Fruits: www.todaymarket.com
Vegetables: www.todaymarket.com
Flowers: www.flowersales.com
Computer parts: www.tcwo.com
Batteries: www.aaaalkalinediscountbatteries.com
Various items: Costco, Sam’s Club, etc.
Search the Internet at www.yahoo.com and type in the key word “wholesale”.

Retail Prices
Local grocery stores, clothing stores, etc.
Advertisements from newspapers
Search the Internet and type in a key word for your product of interest.