

Chapter 9 Ingredients of Multivariable Change: Models, Graphs, Rates



9.1 Multivariable Functions and Contour Graphs

Although Excel can easily draw 3-dimensional surfaces, they are often difficult to mathematically analyze. Therefore, we often use 2-dimensional contour curves to investigate 3-dimensional surfaces.

9.1.1 SKETCHING CONTOUR CURVES When given a multivariable function with two input variables, you can draw contour graphs. We illustrate with the function that appears in Example 2 of Section 9.1 of *Calculus Concepts*:

$$H(v, t) = (10.45 + 10\sqrt{v} - v)(33 - t) \text{ kilogram-calories}$$

per square meter of body surface area per hour for wind speed v in meters per second.

Reproduce the table shown in Example 2 of Section 9.1. Initially, this may seem like a daunting task; however, the table can be creating in just a few steps. Use the **Edit: Fill: Series** command to create the values of A and t shown in Column B and Row 2, respectively.

In cell C3, enter “ $=(10.45+10*\text{SQRT}(C\$2)-C\$2)*(33-\$B3)$ ”. It is important that you place the “\$” signs where shown. This will ensure that all values for A come from column B and all values for t come from row 2.

Copy the formula in cell C3 and to the remaining cells in the table. The cell references will automatically update.

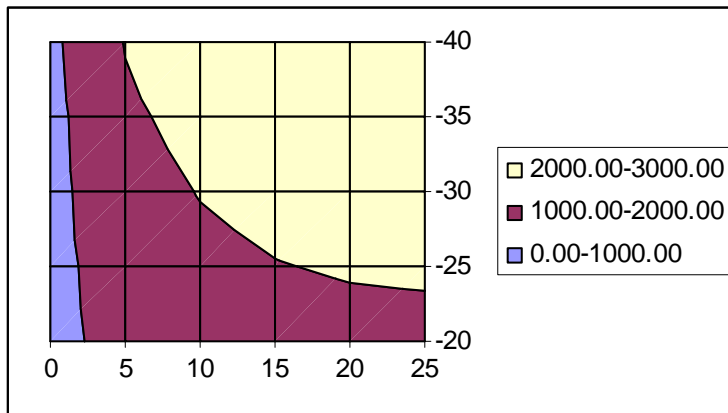
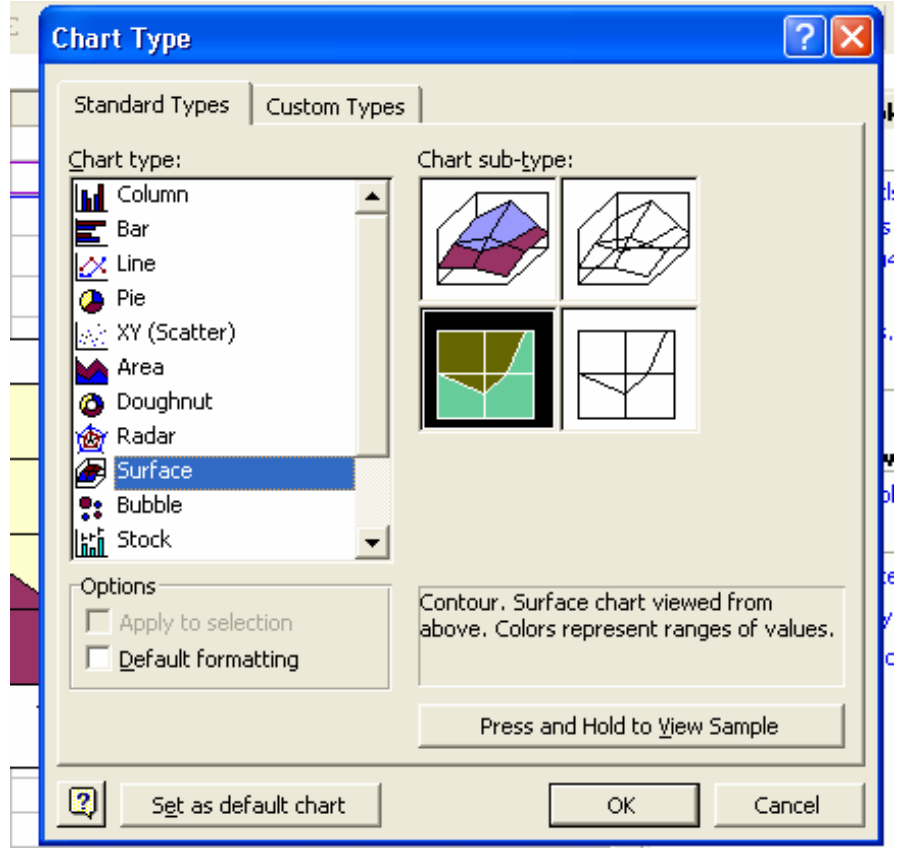
	B	C	D	E	F	G	H
		0	5	10	15	20	25
	-20	553.85	1473.97	1699.86	1811.53	1864.08	1878.85
	-25	606.10	1613.02	1860.22	1982.43	2039.94	2056.10
	-30	658.35	1752.07	2020.58	2153.33	2215.80	2233.35
	-35	710.60	1891.13	2180.95	2324.23	2391.65	2410.60
	-40	762.85	2030.18	2341.31	2495.13	2567.51	2587.85

Unlike most graphing calculators, Excel has the capability to easily draw the complete contour graph represented by the data. Additionally, Excel allows you to color code the various bands based on the multivariable output value.

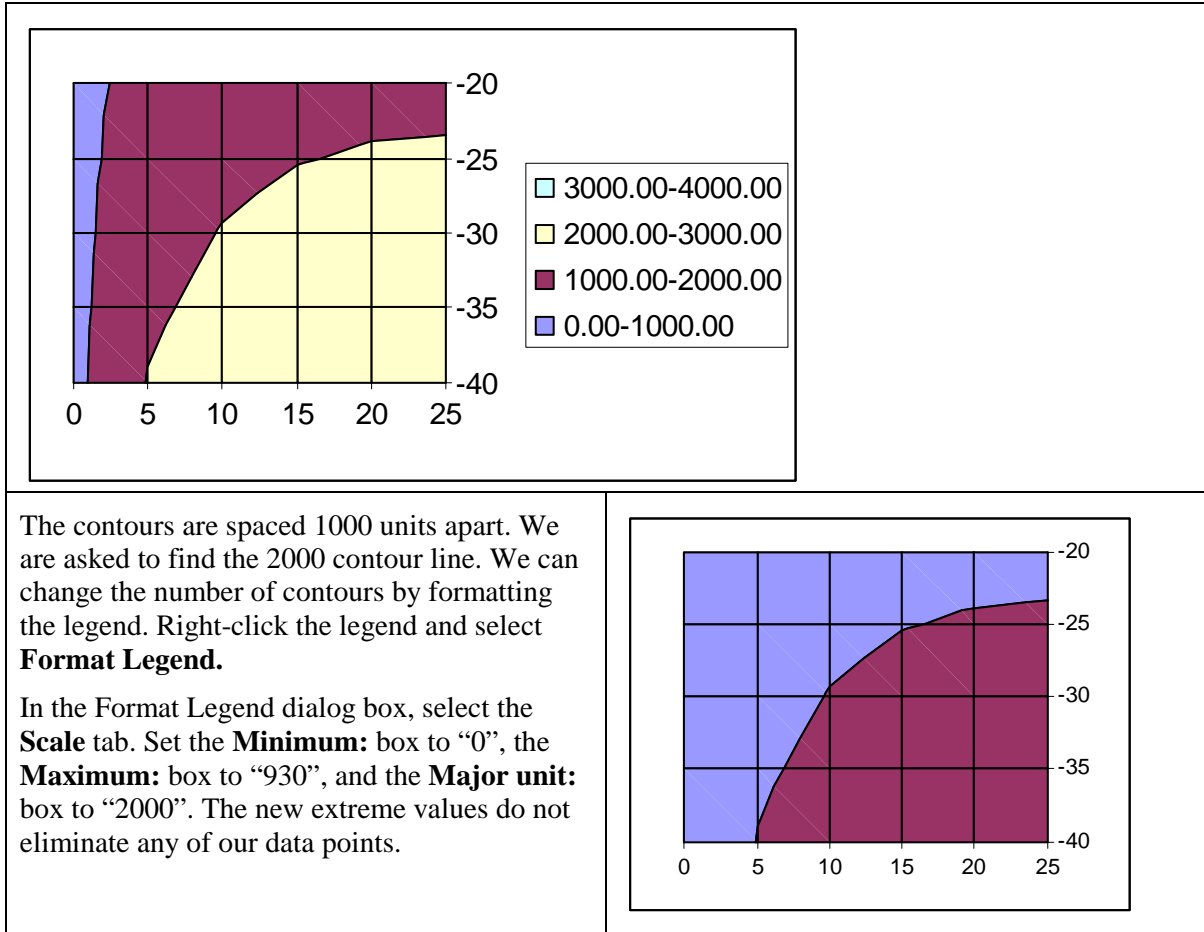
Select **Insert: Chart** from the menu bar.

In the Chart Wizard – Chart Type dialog box, select **Chart type: Standard Types: Surface and Chart sub-type: Contour**. (Chart sub-type Wireframe Contour works nicely when superimposing a contour onto a table.)

Click the **Next** button three times and create the chart.



Notice that the units on the vertical axis are shown in decreasing order as we ascend. To reverse this order, right-click the vertical axis, select **Format Axis: Scale**, and chose the reverse scale option.



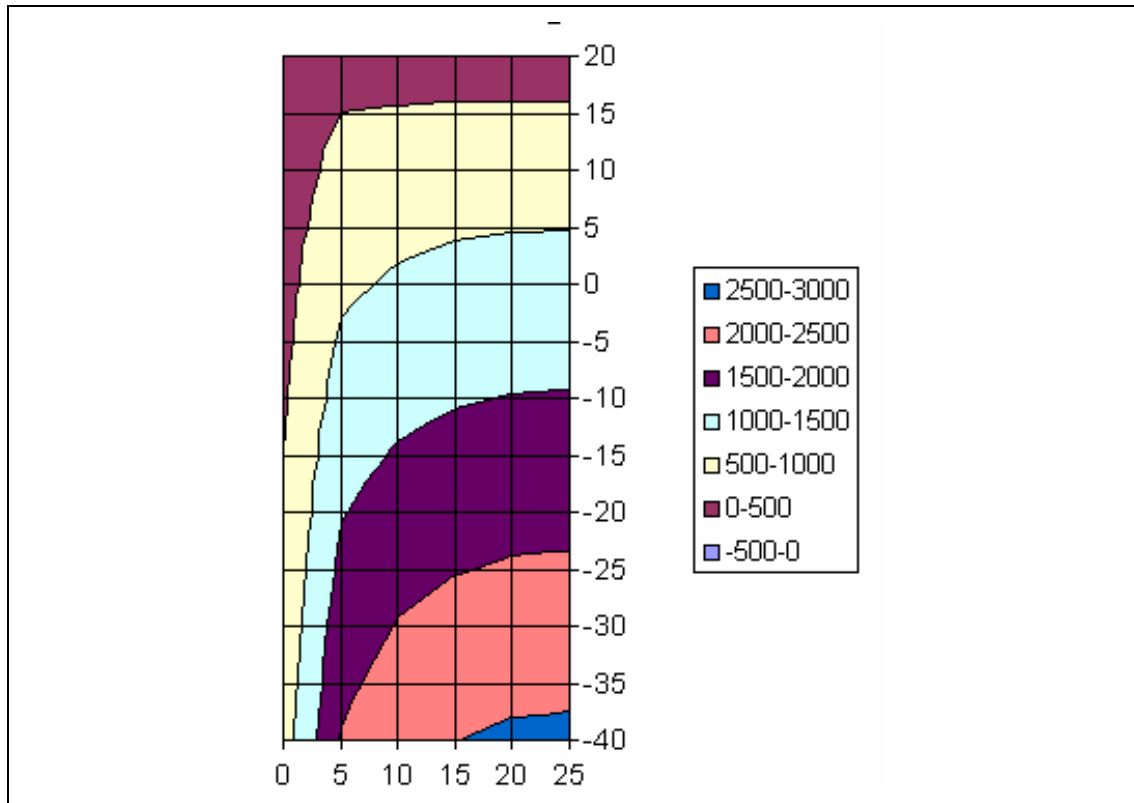
The contours are spaced 1000 units apart. We are asked to find the 2000 contour line. We can change the number of contours by formatting the legend. Right-click the legend and select **Format Legend**.

In the Format Legend dialog box, select the **Scale** tab. Set the **Minimum:** box to “0”, the **Maximum:** box to “930”, and the **Major unit:** box to “2000”. The new extreme values do not eliminate any of our data points.

In Example 3 of Section 9.1 of the text, we are asked to extend the contour graph for air temperatures from -40°C to 20°C and plot contours for heat-loss levels at intervals of 500 kilogram-calories per square meter per hour.

First, extend the table by entering the new inputs in Column B. Excel should automatically extend the formula generated portion of the table. However, you will need to extend the **Source Data** for the chart, by right-clicking on the chart and updating **Source Data: Data Range**.

We can change the number of contour levels by right-clicking on the legend and changing options under **Format Legend: Scale**.



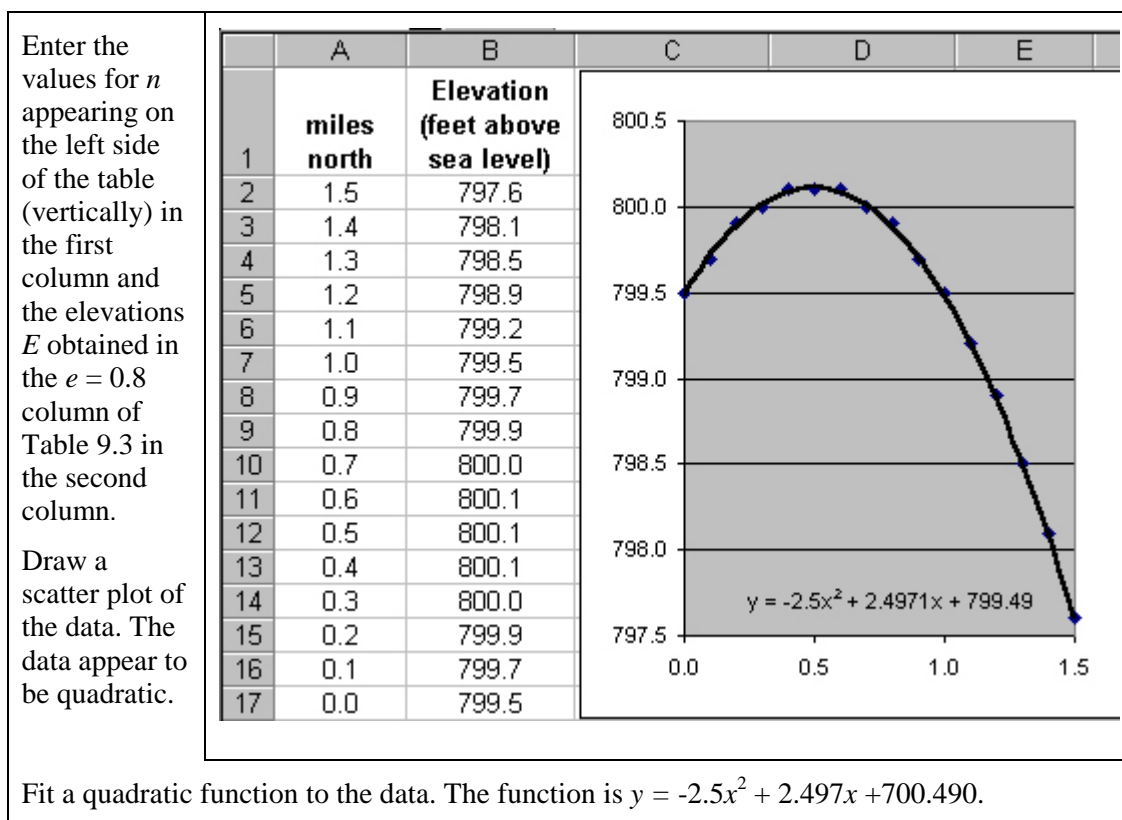
9.2 Cross-Sectional Models and Rates of Change

For a multivariable function with two input variables, obtain a cross-sectional model by entering the data in two columns of a table and then fitting the appropriate function as indicated in previous chapters of this *Guide*. Unless you are told otherwise, we assume that the data are given in a table with the values of the *first* input variable listed *horizontally* across the top of the table and the values of the *second* input variable listed *vertically* down the left side of the table.

9.2.1a FINDING A CROSS-SECTIONAL MODEL FROM DATA (HOLDING THE FIRST INPUT VARIABLE CONSTANT) Using the elevation data in Table 9.4 of Section 9.2 in *Calculus Concepts*, find the cross-sectional model $E(0.8, n)$ as described below. Remember that “rows” go from left to right horizontally and “columns” go from top to bottom vertically.

n (miles)	1.5	1.4	1.3	1.2	1.1	1.0	0.9	0.8
Elevation (feet above sea level)	797.6	798.1	798.5	798.9	799.2	799.5	799.7	799.9
n (miles)	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
Elevation (feet above sea level)	800.0	800.1	800.1	800.1	800.0	799.9	799.7	799.5

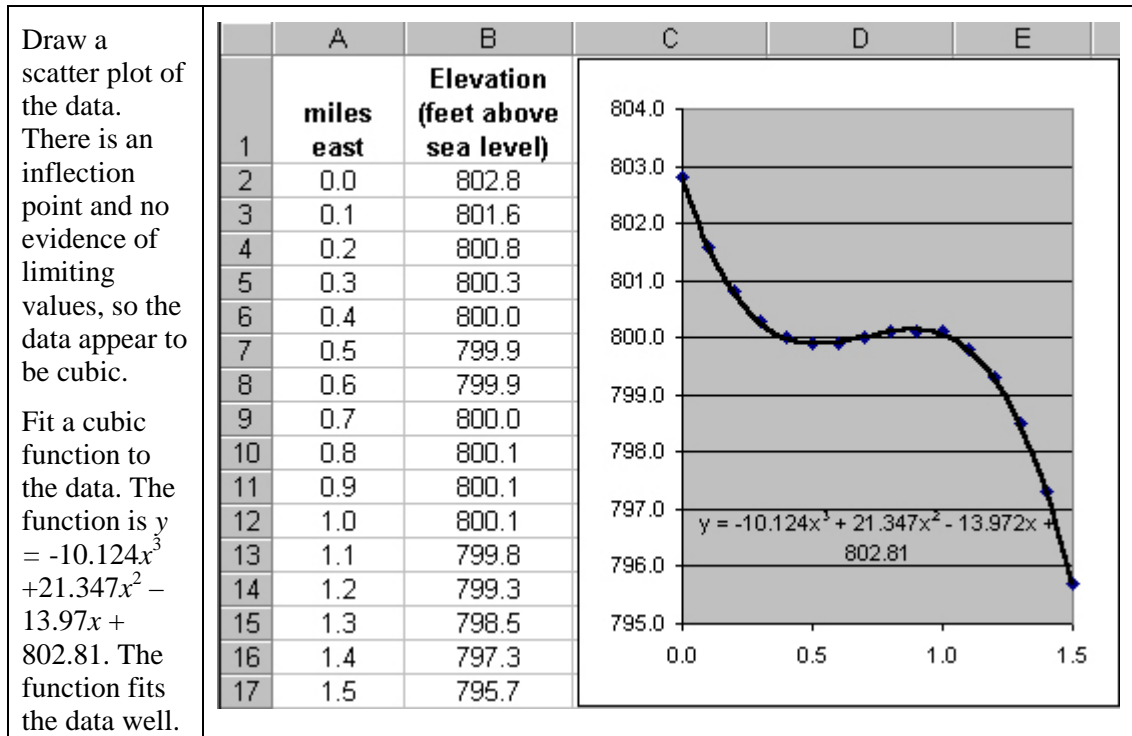
So, refer for a moment to Table 9.3. When you are asked to find $E(0.8, n)$, e is constant at 0.8 and n varies.



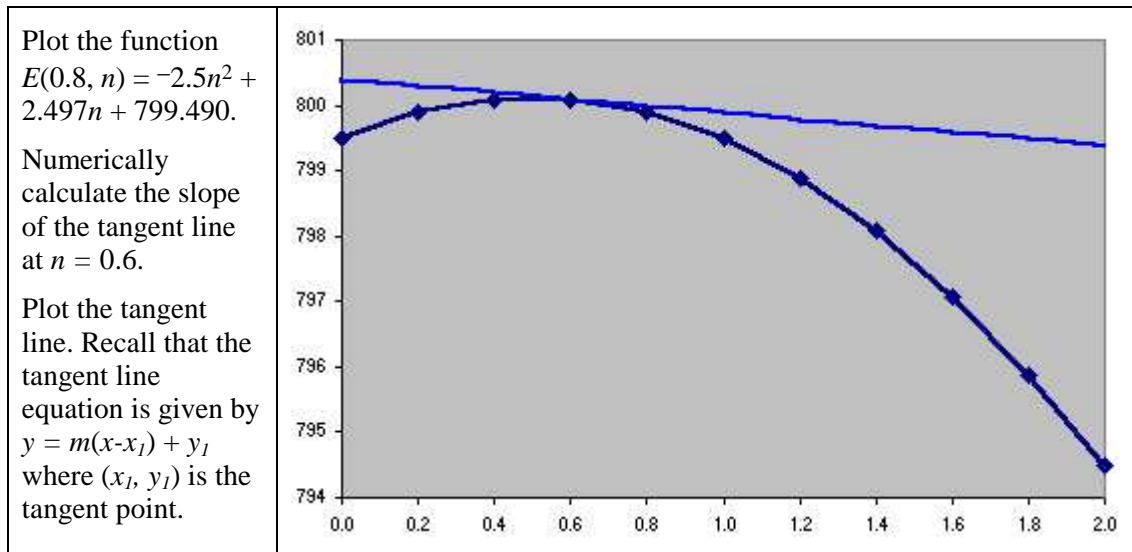
CAUTION: Because you will often be asked to find several different cross-sectional models using the same data table, calling different variables by the same names x and y would be very confusing. It is very important that you call the variables by the names that have been assigned in the problem. When working with multivariable functions, you must translate the Excel equation $y = -2.5x^2 + 2.497x + 799.490$ into the symbols that are used in the application. You should write the cross-sectional function as $E(0.8, n) = -2.5n^2 + 2.497n + 799.490$. Don't forget to completely describe (including units) all of the variables.

9.2.1b FINDING A CROSS-SECTIONAL MODEL FROM DATA (HOLDING THE SECOND INPUT VARIABLE CONSTANT) The only difference in this and the previous section of this *Guide* is that the second input, instead of the first, is held constant. We illustrate using the situation in Example 1 of Section 9.1 in *Calculus Concepts*. Refer to Table 9.3. Because we are asked to find the cross-sectional model $E(e, 0.6)$, $n = 0.6$ and the inputs are the values of e that are across the top of the table. Enter these values in the first column. The outputs are the elevations E obtained in the $n = 0.6$ mile row in Table 9.3. Enter these in the second column.

NOTE: You may find it helpful to place a piece of paper or a ruler under the row (or to the right of the column) in which the data appear to help avoid typing an incorrect value.



9.2.2 VISUALIZING AND ESTIMATING PARTIAL RATES OF CHANGE A partial rate of change of a multivariable function (evaluated at a specific point) is the slope of the line tangent to the graph of a cross-sectional model at a given location. We illustrate this concept in this section using the Missouri farmland cross-sectional equations for elevation: $E(0.8, n)$ and $E(e, 0.6)$. It would be best to use the un-rounded functions that were found in Sections 9.2.1a and 9.2.1b of this *Guide*. However, here we use the rounded functions.



The window settings can be obtained by drawing a scatter plot of the data used to obtain the function $E(0.8, n)$ or by looking at the $e = 0.8$ column in Table 9.3 in *Calculus Concepts*.

We numerically calculated that the slope of the tangent, $\frac{dE(0.8, n)}{dn}$ at $n = 0.6$, is about -0.503 foot/mile. The graph of the function and the tangent line visually confirm our calculation.



9.3 Partial Rates of Change

When you hold all but one of the input variables in a multivariable function constant, you are actually looking at a function of one input variable. Thus, every technique we discussed previously can be used. Although Excel does not give formulas for derivatives, you can use the techniques discussed in Sections 4.3.2b of this *Guide* to check your algebraic formula.

9.3.1 EVALUATING OUTPUTS OF MULTIVARIABLE FUNCTIONS As is the case with single-variable functions, outputs of multivariable functions are found by evaluating the function at the given values of the input variables. The main difference is that you usually will not be using X as the input variable symbol. We illustrate with the investment function from Example 1 of Section 9.3 in *Calculus Concepts*.

The answer to part *a* of Example 2, as derived from the compound interest formula, uses the formula for the accumulated amount of an investment of P dollars for t years in an account paying 6% interest compounded quarterly:

$$A(P, t) = P \left(1 + \frac{0.06}{4} \right)^{4t} \text{ dollars}$$

When 10 is substituted for t , the cross-sectional function becomes $A(P, 10) \approx 1.814018409P$. Part *b* of Example 1 asks for $A(5300, 10)$. Even though it is simplest here to substitute 5300 for P in $A(P, 10) \approx 1.814018409P$, we return to the original function to illustrate evaluating multivariable formulas.

<p>We're asked to find $A(5300, 10)$. Check your result with the values in Table 9.4 in the text to see if \$9614.30 is a reasonable amount.</p>	
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Even though it is not necessary in this example, you may encounter activities in this section in which you need to evaluate a multivariable function at several different inputs. You could use what is shown above, but there are easier methods than entering each one individually. You will also use the techniques shown below in later sections of this chapter. When evaluating a multivariable function at several different input values, you may find it more convenient to create a column for each input variable and use cell references in the formula instead of numeric values.

<p>Label the columns and enter the formula “=A2*(1+B2/C2)^(C2*D2)” in cell E2.</p> <p>Evaluate $A(5300, 0.06, 4, 10)$ and $A(6500, 0.06, 4, 8)$.</p>	
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We see that \$5300 invested at 6% compounded quarterly will grow to \$9614.30 in 10 years and that \$6500 invested at 6% compounded quarterly will grow to \$10467.11 in 8 years.

9.3.2 NUMERICALLY CHECKING PARTIAL DERIVATIVE FORMULAS As mentioned in Chapter 3, the basic concept in checking your algebraically-found partial derivative formula is that your formula and the numeric derivative should have the same outputs when each is evaluated at several different randomly-chosen inputs. You can use the methods in Section 9.3.1 of this *Guide* to evaluate each derivative formula at several different inputs and determine if the same numerical values are obtained from each formula. We illustrate these ideas by checking the answers for the partial derivative formulas found in parts *b* and *d* of Example 1 for this investment function:

The accumulated value of an investment of P dollars over t years at an APR of 6% compounded quarterly is $A(P, t) = P(1.061363551)^t$.

We determine algebraically that $\frac{\partial A}{\partial t} = P \ln(1.061363551) \cdot 1.061363551^t$ and

$\frac{\partial A}{\partial P} = 1.061363551^t$. We must numerically confirm that we are correct.

Create a table like the one shown to the right. Enter the algebraic formulas for accumulated value and the algebraic partial derivatives.

	A	B	C	D
1	Principal	Time	Accumulated Value	
2	\$1,000.00	5.00	\$1,346.86	
3				
4	Numeric Partial Derivative with respect to Time (\$/year)	Algebraic Partial Derivative with respect to Time	Numeric Partial Derivative with respect to Principal (\$/\$)	Algebraic Partial Derivative with respect to Principal
5	80.21	80.21	1.35	1.35

Enter “ $= (A2 * 1.061363551^{(B2+0.0001)} - A2 * 1.061363551^{(B2-0.0001)}) / 0.0002$ ” to numerically estimate $\frac{\partial A}{\partial t}$. We hold the principal constant and vary the time by 0.0001 years.

Enter “ $= ((A2+0.0001) * 1.061363551^{B2} - (A2-0.0001) * 1.061363551^{B2}) / 0.0002$ ” to numerically estimate $\frac{\partial A}{\partial P}$. We hold the time constant and vary the principal by 0.0001 dollars.

Select various values for principal and time and confirm that the numeric and algebraic partial derivatives are equivalent.



9.4 Compensating for Change

As you have just seen, Excel closely estimates numerical values of partial derivatives. The numerical derivative technique can also be very beneficial and help you eliminate many potential calculation mistakes when you find the rate of change of one input variable with

respect to another input variable (that is, the slope of the tangent line) at a point on a contour curve.

9.4.1a EVALUATING PARTIAL DERIVATIVES OF MULTIVARIABLE FUNCTIONS The last few sections of this *Guide* indicate how to estimate and evaluate partial derivatives using cross-sectional models. We illustrate using the body-mass index function that is in Example 1 of Section 9.4 of *Calculus Concepts*:

A person's body-mass index is given by $B(h, w) = \frac{0.4536w}{0.00064516h^2}$ where h is the person's height in inches and w is the person's weight in pounds. We first find B_h and B_w at a specific height and weight and then use those values in the next section of this *Guide* to find the value of the derivative $\frac{dw}{dh}$ at that particular height and weight. The person in this example is 5 feet 7 inches tall and weighs 129 pounds.

<p>Use the techniques shown in Section 9.3.2 to construct a table like the one shown. Recall</p> $\frac{\partial B}{\partial h} \approx \frac{B(h + 0.0001, w) - B(h - 0.0001, w)}{0.0002}$ <p>and</p> $\frac{\partial B}{\partial w} \approx \frac{B(h, w + 0.0001) - B(h, w - 0.0001)}{0.0002}$ <p>So $\frac{\partial B}{\partial h} \approx -0.603$ and $\frac{\partial B}{\partial w} \approx 0.1566$.</p>	<table border="1"> <tr> <td colspan="2">C2</td> <td colspan="2">=</td> <td colspan="2">=0.4536*A2/(0.00064516*B2^2)</td> </tr> <tr> <td></td> <td>A</td> <td>B</td> <td>C</td> <td></td> <td></td> </tr> <tr> <td>1</td> <td>Weight (lbs)</td> <td>Height (inches)</td> <td>Body Mass Index</td> <td></td> <td></td> </tr> <tr> <td>2</td> <td>129.00</td> <td>67.00</td> <td>20.20</td> <td></td> <td></td> </tr> <tr> <td>4</td> <td>Partial w/ respect to Weight</td> <td>Partial w/ respect to Height</td> <td></td> <td></td> <td></td> </tr> <tr> <td>5</td> <td>0.1566</td> <td>-0.6031</td> <td></td> <td></td> <td></td> </tr> </table>	C2		=		=0.4536*A2/(0.00064516*B2^2)			A	B	C			1	Weight (lbs)	Height (inches)	Body Mass Index			2	129.00	67.00	20.20			4	Partial w/ respect to Weight	Partial w/ respect to Height				5	0.1566	-0.6031			
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9.4.1b FINDING THE SLOPE OF A LINE TANGENT TO A CONTOUR CURVE We continue the previous illustration with the body-mass index function in Example 1 of Section 9.4 of *Calculus Concepts*. Part *a* of Example 1 asks for $\frac{dw}{dh}$ at the point (67, 129) on the contour curve corresponding to the person's current body-mass index. The formula is $\frac{dw}{dh} = \frac{-B_h}{B_w}$.

An easy way to remember this formula is that whatever variable is in the numerator of the derivative (in this case, w) is the same variable that appears as the changing variable in the denominator of the slope formula.

<p>Update the worksheet with the formula for $\frac{dw}{dh}$. The rate of change is about 3.85 pounds per inch for a 67-inch, 129-pound person.</p>	<table border="1"> <tr> <td colspan="2">C5</td> <td colspan="2">=</td> <td colspan="2">=-B5/A5</td> </tr> <tr> <td></td> <td>A</td> <td>B</td> <td>C</td> <td></td> <td></td> </tr> <tr> <td>1</td> <td>Weight (lbs)</td> <td>Height (inches)</td> <td>Body Mass Index</td> <td></td> <td></td> </tr> <tr> <td>2</td> <td>129.00</td> <td>67.00</td> <td>20.20</td> <td></td> <td></td> </tr> <tr> <td>4</td> <td>Partial w/ respect to Weight</td> <td>Partial w/ respect to Height</td> <td>dw/dh</td> <td></td> <td></td> </tr> <tr> <td>5</td> <td>0.1566</td> <td>-0.6031</td> <td>3.850746269</td> <td></td> <td></td> </tr> </table>	C5		=		=-B5/A5			A	B	C			1	Weight (lbs)	Height (inches)	Body Mass Index			2	129.00	67.00	20.20			4	Partial w/ respect to Weight	Partial w/ respect to Height	dw/dh			5	0.1566	-0.6031	3.850746269		
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9.4.1c COMPENSATING FOR CHANGE When one input of a two-variable multivariable function changes by a small amount, the value of the function is no longer the same as it was before the change. The methods illustrated below show how to determine the amount by which the other input must change so that the output of the function remains at the value it was before any changes were made. We again continue the previous illustration with the body-mass index function and part *b* of Example 1 of Section 9.4 of *Calculus Concepts*.

<p>To estimate the change in weight needed to compensate for growths of 0.5 inch, 1 inch, and 2 inches if the person's body-mass index is to remain constant, you need to find</p> $\Delta w \approx \frac{dw}{dh} (\Delta h)$ <p>at the given values of Δh.</p> <p>Update the table as shown. Input 0.5, 1, and 2 for Δh to calculate the corresponding Δw</p>		<p>D5 = =C5*D2</p> <table border="1"> <thead> <tr> <th></th> <th>A</th> <th>B</th> <th>C</th> <th>D</th> </tr> </thead> <tbody> <tr> <td>1</td> <td>Weight (lbs)</td> <td>Height (inches)</td> <td>Body Mass Index</td> <td>Δh</td> </tr> <tr> <td>2</td> <td>129.00</td> <td>67.00</td> <td>20.20</td> <td>0.50</td> </tr> <tr> <td>3</td> <td></td> <td></td> <td></td> <td></td> </tr> <tr> <td>4</td> <td>Partial w/ respect to Weight</td> <td>Partial w/ respect to Height</td> <td>dw/dh</td> <td>Δw</td> </tr> <tr> <td>5</td> <td>0.1566</td> <td>-0.6031</td> <td>3.850746269</td> <td>1.9253731</td> </tr> </tbody> </table>					A	B	C	D	1	Weight (lbs)	Height (inches)	Body Mass Index	Δh	2	129.00	67.00	20.20	0.50	3					4	Partial w/ respect to Weight	Partial w/ respect to Height	dw/dh	Δw	5	0.1566	-0.6031	3.850746269	1.9253731
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For a 0.5-inch increase in height, a person's weight may increase 1.925 pounds. For a 1-inch increase in height, a person may gain 3.851 pounds. For a 2-inch increase in height, a person may gain 7.701 pounds.