Chapter 8 Dynamics of Change: Differential Equations and Proportionality

8.3 Numerically Estimating by Using Differential Equations: Euler's Method

Many of the differential equations we encounter have solutions that can be found by determining an antiderivative of a given rate-of-change function. Thus, many of the techniques that we learned using numerical integration apply to this chapter. (See Chapters 6 and 7 of this *Guide*.)

8.3.1 **EULER'S METHOD FOR A DIFFERENTIAL EQUATION WITH ONE INPUT VARIABLE** You may encounter a differential equation that cannot be solved by standard methods and you may need to draw an accumulation graph for a differential equation without first finding an antiderivative. In either of these cases, numerically estimating a solution using Euler's method is helpful. This method relies on the use of the derivative of a function to approximate the change in that function. Recall that the approximate change in f at a point is the rate of change of f at that point times a small change in x. That is,

$$f(x + h) - f(x) \approx f'(x) \cdot h \quad \text{where } h \text{ represents the small change in } x.$$

Now, if we let $b = x + h$ and $x = a$, the above expression becomes

$$f(b) - f(a) \approx f'(a) \cdot (b - a) \quad \text{or} \quad f(b) \approx f(a) + (b - a) \cdot f'(a)$$

The starting values for the coordinates of the point (a, b) will be given to you and are often called the *initial condition*. The next step is to repeatedly apply the formula given above to use the slope of the tangent line at $x = a$ to approximate the change in the function between the inputs a and b. When h, the distance between a and b, is fairly small, Euler's method will often give close numerical estimates of points on the solution to the differential equation containing $f'(x)$.

WARNING: Be wary of the fact that there is some error involved in each step of the Euler approximation process that is compounded when each result is used to obtain the next result.

We illustrate Euler's method for a differential equation containing one input variable with the differential equation in Example 1 of Section 8.3. This equation gives the rate of change of the total sales of a computer product t years after the product was introduced:

$$\frac{dS}{dt} = \frac{6.544}{\ln(t + 1.2)} \text{ billion dollars per year}$$

Because Euler's method involves a repetitive process, a program that performs the calculations used to find the approximate change in the function can save you time and eliminate computational errors and some error in rounding.

Create a table with the column labels shown.

Enter the initial value for *a* in cell A2. Use the **Edit: Fill: Series** command with **Step value "**0.25" and **Stop value** "4" to create the remaining terms in the column.

Enter "=A2+0.25"in cell B2. Duplicate the formula in cell B2 for cells B3 through B14.

Enter the initial output, *f*(*a*), in cell C2. In this case, *f*(1) = 53.2.

Enter the derivative function in cell D2 using cell A2 as the input. Duplicate the formula in cell D2 for cells D3 through D14.

Enter "=C2+(B2-A2)*D2" in cell E2. This is the estimate of *f*(*a* + *h*). Duplicate the formula in cell E2 for cells E3 through E14.

E2 = =C2+(B2-A2)*D2

	A	B	C	D	E
1	a	a+h	f(a)	f'(a)	f(a+h)
2	1.00	1.25	53.200	8.300	55.27494
3	1.25	1.50	55.27494	7.303	57.10065
4	1.50	1.75	57.10065	6.588	58.74777
5	1.75	2.00	58.74777	6.049	60.26005
6	2.00	2.25	60.26005	5.626	61.66658
7	2.25	2.50	61.66658	5.284	62.98766
8	2.50	2.75	62.98766	5.002	64.23811
9	2.75	3.00	64.23811	4.764	65.42904
10	3.00	3.25	65.42904	4.560	66.56904
11	3.25	3.50	66.56904	4.383	67.6649
12	3.50	3.75	67.6649	4.229	68.72204
13	3.75	4.00	68.72204	4.092	69.74493
14	4.00	4.25	69.74493	3.969	70.73725
15	4.25	4.50	70.73725	3.859	71.7021
16	4.50	4.75	71.7021	3.760	72.64207
17	4.75	5.00	72.64207	3.669	73.55943
18	5.00	5.25	73.55943	3.587	74.45609

Enter "=E2" in cell C3 and copy the formula in cell C3 to cells C4 through C14. If you like, you may plot the points to visualize the estimate of the differential equation solution.

8.3.2 EULER'S METHOD FOR A DIFFERENTIAL EQUATION WITH TWO INPUT VARIABLES

Euler's Method can be used when the differential equation is a function of *x* and *y* with *y* = *f*(*x*). Follow the same process illustrated in the previous section of this *Guide*.

The differential equation may be given in terms of *y* only. For instance, if $\frac{dy}{dx} = k(30 - y)$ where *k* is a constant, enter the function substituting a numeric value for *k*.

We illustrate using Euler's method with two input variables by using the equation in Example 2 of Section 8.3 in *Calculus Concepts*.

We have $\frac{dy}{dx} = 5.9x - 3.2y$.

In cell D2, enter "5.9*A2-3.2*C2", the formula for $\frac{dy}{dx}$. Copy this formula to cells D3 through D12.

Edit column A to include data points beginning at 10 and ending at 12 with values spaced 0.2 units apart.

Update column B as shown.

The estimate for *y*(12) is 21.55.

D2 = =5.9*A2-3.2*C2

	A	B	C	D	E
1	a	a+h	f(a)	f'(a)	f(a+h)
2	10.00	10.20	50.000	-101.000	29.8
3	10.20	10.40	29.8	-35.180	22.764
4	10.40	10.60	22.764	-11.485	20.46704
5	10.60	10.80	20.46704	-2.955	19.87613
6	10.80	11.00	19.87613	0.116	19.89941
7	11.00	11.20	19.89941	1.222	20.14379
8	11.20	11.40	20.14379	1.620	20.46776
9	11.40	11.60	20.46776	1.763	20.82039
10	11.60	11.80	20.82039	1.815	21.18334
11	11.80	12.00	21.18334	1.833	21.55
12	12.00	12.20	21.55	1.840	21.918