Chapter 7 Repetitive Change: Cyclic Functions

7.1 Cycles and Sine Functions

Data that is periodic may often be modeled by trigonometric functions. This chapter will help you use Excel to deal with periodic phenomena.

7.1.1 FINDING OUTPUTS OF TRIG FUNCTIONS WITH RADIAN INPUTS The Excel functions **SIN** and **COS** are used to calculate the sine and cosine of an angle, respectively. The units of the input to these functions are radians. If you want to use degrees, you must numerically convert the degrees into radians before calculating the function. We show how to evaluate trig functions as with the following example.

Find $\sin\frac{9\pi}{8}$ and $\cos\frac{9\pi}{8}$.

Enter "=9/8*PI()" in cell A2. The function for pi is **PI().** The parentheses are mandatory.

A2 = =9/8*PI()

	A	B	C	D
1	theta	sine	cosine	sin^2+cos^2
2	3.5343	-0.382683432	-0.9238795	1.0000

Enter "=SIN(A2)" in cell B2 and "=COS(A2)" in cell C2. We see $\sin\frac{9\pi}{8} \approx -0.3827$ and $\cos\frac{9\pi}{8} \approx -0.9239$. The range of both sine and cosine is the closed interval [-1, 1].

In cell D2, enter "=B2^2 + C2^2". Notice that $\sin^2\theta + \cos^2\theta = 1$. This equation is valid for any value of θ.

7.2 Cyclic Functions as Models

We now introduce another model – the sine model. As you might expect, this function should be fit to data that repeatedly varies between alternate extremes. The form of the sine model is given by $f(x) = a \sin(bx + h) + k$ where $|a|$ is the amplitude, b is the frequency (where $b > 0$), $2\pi/b$ is the period, $|h|/b$ is the horizontal shift (to the right if $h < 0$ and to the left if $h > 0$), and k is the vertical shift (up if $k > 0$ and down if $k < 0$). Unlike many other functions, Excel does not have a built-in modeling tool for sine functions.

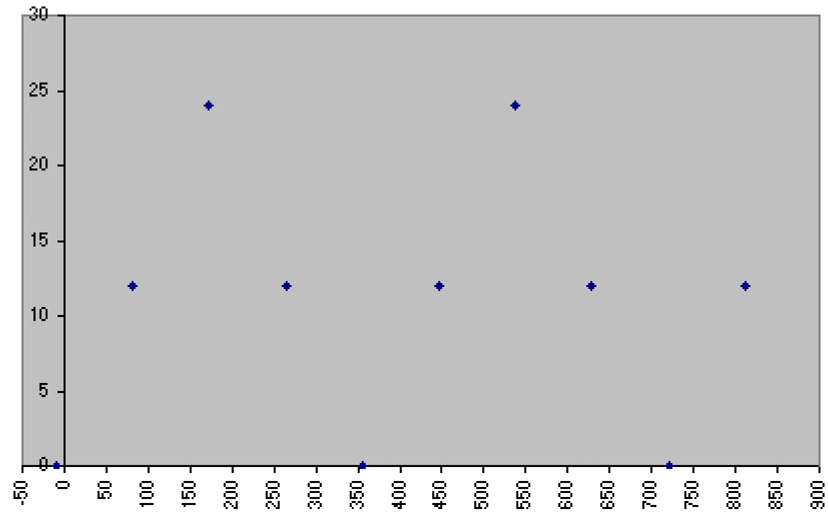
7.2.1 FITTING A SINE MODEL TO DATA Before fitting any model to data, remember that you should construct a scatter plot of the data and observe what pattern the data appear to follow. Example 2 in Section 7.2 asks you to find a sine model for cyclic data with the hours of daylight on the Arctic Circle as a function of the day of the year on which the hours of daylight are measured. (January 1 is day 1.) These data appear in Table 7.2 of *Calculus Concepts*.

Day of the year	−10	81.5	173	264	355	446.5	538	629	720	811.5
Hours of daylight	0	12	24	12	0	12	24	12	0	12

Enter the data into a table and draw the scatter plot. The graph looks periodic. One cycle of data appears to be about 538-173 = 365 days. A sine function may fit the data well.

The parameters for our function are *a*, *b*, *h*, and *k*.

We will use the Solver and the least-squares method.



Create a worksheet like the one shown. Save the worksheet because you will use it each time you do sine regression. If you have over ten data points, you will need to duplicate the formulas in columns C and H and change the formula in column I to include the extra cells.

	A	B	C	D	E	F	G	H	I
1	**x**	**y**	**Model y**	**a**	**b**	**h**	**k**	**squared errors**	**sum of squared errors**
2	-10	0	=$D$2*SIN($E$2*A2+$F$2) +$G$2					=(B2-C2)^2	=SUM(H1:H11)
3	81.5	12	=$D$2*SIN($E$2*A3+$F$2) +$G$2					=(B3-C3)^2	
4	173	24	=$D$2*SIN($E$2*A4+$F$2) +$G$2					=(B4-C4)^2	
5	264	12	=$D$2*SIN($E$2*A5+$F$2) +$G$2					=(B5-C5)^2	
6	355	0	=$D$2*SIN($E$2*A6+$F$2) +$G$2					=(B6-C6)^2	
7	446.5	12	=$D$2*SIN($E$2*A7+$F$2) +$G$2					=(B7-C7)^2	
8	538	24	=$D$2*SIN($E$2*A8+$F$2) +$G$2					=(B8-C8)^2	
9	629	12	=$D$2*SIN($E$2*A9+$F$2) +$G$2					=(B9-C9)^2	
10	720	0	=$D$2*SIN($E$2*A10+$F$2) +$G$2					=(B10-C10)^2	
11	811.5	12	=$D$2*SIN($E$2*A11+$F$2) +$G$2					=(B11-C11)^2	

Select **Tools: Solver** from the menu bar.

Adjust the settings to minimize cell I2 by changing cells D2, E2, F2, and G2.

	A	B	C	D	E	F	G	H	I
1	**x**	**y**	**Model y**	**a**	**b**	**h**	**k**	**squared errors**	**sum of squared errors**
2	-10.0	0	2.8E-09	12.009	0.15493	-0.06025	12	0.00000000	0.0000000
3	81.5	12	12					0.00000000	
4	173.0	24	24					0.00000000	
5	264.0	12	12					0.00000000	
6	355.0	0	1.2E-09					0.00000000	
7	446.5	12	12					0.00000000	
8	538.0	24	24					0.00000000	
9	629.0	12	12					0.00000000	
10	720.0	0	-3.6E-10					0.00000000	
11	811.5	12	12					0.00000000	

Solver will adjust the values of *a*, *b*, *h*, and *k* as it minimizes the sum of the squared errors. You can reduce the solution time by setting the adjustable cells to values that you suspect are close to optimal. To estimate the amplitude, *a*, divide the difference between the maximum and minimum outputs by 2. To estimate *k*, subtract *a* from the maximum output.

CAUTION: If Solver has found a solution that is significantly different from what you expected, it may be helpful to try different starting values for the adjustable cells. Be aware that you may have to run the Solver more than once to find the optimal solution. This is because Solver goes through a finite number of iterations (typically 100) in finding the solution. You may also adjust the tolerance, convergence, and extrapolation method in the Solver Options dialog box to improve accuracy. Access the Excel help file to get more details about these features. Also be aware that you will not get the same model every time. You can visually determine the accuracy of your model by comparing the entries is column B to those in column C. The model estimates should be relatively close to the actual output values.

We find the model is $y = 12.009\sin(1.5493x - 0.06025) + 12$. By looking at the model outputs in column C and the sum of the squared errors in column I, we see that the model fits the data well.

7.3 Rates of Change and Derivatives

All the previous techniques given for other functions also hold for the sine model. You can find intersections, maxima, minima, inflection points, derivatives, integrals, and so forth.

7.3.1 DERIVATIVES OF SINE AND COSINE FUNCTIONS We illustrate how to find the value of the derivative of the sine function with Example 1 in Section 7.3 of *Calculus Concepts*:

> The calls for service made to a county sheriff's department in a certain rural/suburban county can be modeled as $c(t) = 2.8\sin(0.262t + 2.5) + 5.38$ calls during the tth hour after midnight.

Part *c* of Example 1 asks how quickly the number of calls received each hour is changing at noon and at midnight. Use the techniques initially covered in Section 4.1.1 to find the rate of change in calls per hour.

At noon, the calls were increasing by 0.589 calls per hours. It is unclear whether midnight is when $t = 0$ or $t = 24$. In either case, the calls were decreasing by about 0.59 calls per hour.

	A	B	C
1	**t hours after midnight**	**Service calls**	**Rate of change (calls/hour)**
2	0	7.055722003	-0.5877189
3	6	3.13478248	-0.4383314
4	12	3.709683012	0.58877416
5	18	7.62923855	0.43691404
6	24	7.044902293	-0.589826

We now return to Example 1. Part *a* asks for the average number of calls the county sheriff's department receives each hour. The easiest way to obtain this answer is to remember that the parameter *k* in the sine function is the average value. So on average, the sheriff's department receives 5.38 calls per hour.

You can also find the average value over one period of the function using the methods of Section 5.5.1a. The average value is given by $\dfrac{\int_0^{2\pi/0.262} c(t)\,dt}{2\pi/0.262 - 0}$.

Calculate the definite integral and divide by the length of the interval to find the average value.

The average value is about 5.38 hours.

	A	B	C	D	E	F
1	**a**	**b**	**n**	**delta**		
2	0	23.98162331	1000	0.023982		
3						
4	**Hours after midnight**	**Number of calls**	**Left**	**Right**	**Trapezoid**	**Average Value**
5	0	7.055722003	129.0211	129.0211	129.02109	5.3799984
6	0.023982	7.041594578				
7	0.047963	7.027401556				
8	0.071945	7.013143497				

7.5 Accumulation in Cycles

As with the other functions we have studied, applications of accumulated change with the sine and cosine functions involve numerical integration.

7.5.1 INTEGRALS OF SINE AND COSINE FUNCTIONS We illustrate with Example 1 in Section 7.5 of *Calculus Concepts*.

Enter the rate of change of temperature in Philadelphia on August 27, 1993. Find the accumulated change in the temperature between 9 a.m. and 3 p.m.

The accumulated change in the temperature is 12.77 degrees.

	A	B	C	D	E
1	**a**	**b**	**n**	**delta**	
2	9	15	1000	0.006	
3					
4	**hours after midnight**	**rate of change in temperature**	**Left**	**Right**	**Trapezoid**
5	9	2.552960239	12.7748	12.7632	12.769009