Most of the functions that we have seen in this text have been motivated by real-world phenomena. In many cases, it is the underlying rate of change that determines whether the equation describing a phenomenon should be linear, quadratic, logistic, and so forth.

In this chapter we look at some applications of calculus that use information about rates of change. We consider equations that describe the forces that drive certain rates of change. Equations stating the behavior of a rate of change (that is, equations involving derivatives) are called differential equations. We look at differential equations graphically using slope fields, numerically using Euler’s method, and analytically using integration.

Concept Objectives

This chapter will help you understand the concepts of
- Constant and linear differential equations
- Direct, inverse, and joint proportionality
- Slope fields
- Separable differential equations
- Euler’s method
- Second-order differential equations

and you will learn to
- Find and graph particular solutions to differential equations
- Set up differential equations using information about proportionality
- Use slope fields to analyze a differential equation graphically
- Use Euler’s method to analyze a differential equation numerically
- Solve first- and second-order differential equations using antiderivatives
**Concept Application**

The rate of decay that a radioisotope undergoes dictates the amount of that isotope left after a certain amount of time. It is this relationship that makes it possible for archeologists, geologists, and environmental engineers to use radioisotopes to help determine the age of artifacts and geological samples and to determine how much of a threat certain radioactive substances are to the environment. Activities 26 and 27 in Section 11.2 deal with developing differential equations from information about the decay of radioisotopes.
In Section 5.5 we saw how to use a known equation to develop a related-rates equation that shows us the interconnection between different rates of change. Now we consider the opposite process: using equations that describe rates of change to develop equations that describe the situation. Equations that involve rates of change (derivatives) are called **differential equations**. The following equations are examples of differential equations:

\[
\begin{align*}
\frac{dc}{dp} &= -0.16p \\
\frac{dy}{dx} &= 5.9x - 3.2y \\
\frac{d^2s}{dt^2} &= 2.1s \;
\end{align*}
\]

**Differential Equations**

Consider, for instance, the differential equation

\[
\frac{dy}{dx} = 63 \text{ mph}
\]

that describes the speed of a vehicle after \(x\) hours of traveling west on a straight stretch of interstate in North Dakota. Suppose we want an equation for the distance between the vehicle and the eastern border of the state after \(x\) hours. Because we are given a derivative \(\frac{dy}{dx}\) and wish to find a function \(y\), we simply write the general anti-derivative:

\[
y(x) = 63x + C \text{ miles}
\]

after \(x\) hours, where \(C\) represents the distance from the border when \(x = 0\). This equation is the solution to the differential equation given above.

It is important to note that the solution of a differential equation is a function with derivatives that satisfy the differential equation. The solution is not a numerical value. We define the function \(f\) to be a **general solution** to a differential equation \(\frac{dy}{dx} = g(x, y)\) if the substitution of \(f(x)\) for \(y\) gives an identity (that is, a true statement).

**Differential Equation and General Solution**

A differential equation involves one or more derivatives. A general solution for a differential equation is a function that has derivatives that satisfy the differential equation.
In the velocity example, we found \( y(x) = 63x + C \) miles to be a solution to \( \frac{dy}{dx} = 63 \) mph. To verify that this is indeed a general solution, we substitute \( y(x) \) into the differential equation and obtain

\[
\frac{d}{dx}(63x + C) = 63
\]

\[
63 = 63
\]

The statement \( 63 = 63 \) is called an identity. This identity confirms that we have a solution. The \( C \) parameter in the solution is an arbitrary constant; it is used to designate the family of functions that when differentiated, gives the differential equation. In practice, unique values for \( C \) will be determined by the initial conditions stated in the specific problem that we are solving. In those cases, we call the solution a particular solution.

The velocity example is a specific example of the simplest differential equation—a constant differential equation that has the form \( \frac{dy}{dx} = k \), where \( k \) is a constant. To solve this differential equation, we used our understanding of the relationship between a rate-of-change function, the accumulation function of that rate of change, and the quantity function. That is, we found the solution to the differential equation by determining the general antiderivative of the rate-of-change function. Another simple differential equation, called a linear differential equation, has the form \( \frac{dy}{dx} = ax + b \) and is given in Example 1.

**EXAMPLE 1** Finding Solutions for a Linear Differential Equation

**Population** Between 1900 and 2000 the population* of the United States was growing at rate of \( 0.0181x + 1.129 \) million people per year where \( x \) is the number of years since 1900. In 1990 the population of the United States was 249.9 million people.

a. Write a differential equation expressing the growth rate of the population with respect to time.

b. Find a general solution for this differential equation.

c. Find a particular solution for this differential equation. Graph the differential equation and the particular solution.

d. Use the particular solution to estimate the population in the year 2005.

**Solution**

a. Let \( P \) represent the population of the United States \( x \) years after 1900. The growth rate of the population is expressed by the differential equation

\[
\frac{dP}{dx} = 0.0181x + 1.129 \text{ million people per year }
\]

\( x \) years after 1900.

b. The general solution is a function $P$ describing the population. Because the differential equation $\frac{dP}{dx} = 0.0181x + 1.129$ is simply the derivative of the population function $P$ written in terms of the input variable $x$, a general antiderivative of $\frac{dP}{dx}$ is a general solution for this differential equation. Thus we have as a general solution

$$P(x) = 0.00905x^2 + 1.129x + C \text{ million people}$$

$x$ years after 1900.

c. We know the population in 1990, so we use the fact that $P(90) = 249.9$ to solve for $C$ in the general solution. Thus we obtain the particular solution

$$P(x) = 0.00905x^2 + 1.129x + 74.985 \text{ million people}$$

$x$ years after 1900. Figure 11.1 shows graphs of the differential equation and the particular solution.

d. We estimate the population of the United States to be $P(105) \approx 293.3$ million people in the year 2005.

In order for us to use antiderivatives as solutions, the differential equation must satisfy two criteria:

1. The rate of change must be in terms of the input variable only.
2. We must know an antiderivative formula for the given rate-of-change function.

Many differential equations satisfy these criteria. Some of these differential equations are based on the ideas of proportionality.

**Proportionality**

We have seen the two differential equations $\frac{dy}{dx} = k$ and $\frac{dy}{dx} = kx$, where $k$ is a constant. The first differential equation states that the rate of change of $y$ with respect to $x$ is
constant. The second differential equation, \( \frac{dy}{ds} = kx \), states that the rate of change of \( y \) with respect to \( x \) is proportional to the input \( x \).

The idea of proportionality is one that is often used in setting up differential equations. We say that a variable \( y \) is directly proportional to another variable \( x \) if there is a constant \( k \) such that \( y = kx \). We call \( k \) the constant of proportionality. We use the terms proportional and directly proportional interchangeably.

**Direct Proportionality**

For input \( x \) and output \( y \), \( y \) is directly proportional to \( x \) if there exists some constant \( k \) such that \( y = kx \). The constant \( k \) is called the constant of proportionality.

For example, if \( A(t) = 23.50t \) dollars represents the amount it costs to purchase \( t \) tickets to a concert, then we say that the cost is directly proportional to the number of tickets purchased. In this case, 23.50 is the constant of proportionality.

Another type of proportionality occurs when a quantity \( y \) is related to a quantity \( x \) by the equation \( y = \frac{k}{x} \), where \( k \) is a constant. In this case, we say that \( y \) is inversely proportional to \( x \).

**Inverse Proportionality**

For input \( x \) and output \( y \), \( y \) is inversely proportional to \( x \) if there exists some constant of proportionality \( k \) such that \( y = \frac{k}{x} \).

The German physiologist Gustav Fechner* said that the rate of change of the intensity of a response \( R \) with respect to the intensity of a stimulus \( s \) is inversely proportional to the intensity of the stimulus. That is, there is some constant \( k \) such that

\[
\frac{dR}{ds} = \frac{k}{s}
\]

This differential equation, known as Fechner’s Law, says that if you are in a quiet environment and a small bell rings, you will perceive the sound from the bell as being rather loud, whereas if you are in a noisy environment and the same small bell rings, you will perceive the sound as being almost inaudible. Consider the following response differential equation:

\[
\frac{dR}{ds} = \frac{2.94}{s}
\]

where the stimulus (input \( s \)) is measured in decibels and the response (output \( R \)) is measured on a scale of sound intensity where 0 represents no sound and 10

represents unbearably loud sound. A general solution of this differential equation is simply a general antiderivative:

\[ R(s) = 2.94 \ln s + C \]

where \( s \) is measured in decibels and is always positive.

If the smallest sound that can be detected is 10 decibels, then we can use the initial condition \( R(10) = 0 \) to find a particular solution to the differential equation. In this case, the particular solution is

\[ R(s) = 2.94 \ln s - 6.77 \]

where \( s \) is measured in decibels.

**EXAMPLE 2  Evaluating the Constant of Proportionality**

**Sales** Suppose that the total sales (in billions of dollars) of a computer product are growing in inverse proportion to \( \ln(t + 1.2) \), where \( t \) is the number of years since the product was introduced. Sales totaled $53.2 billion by the end of the first year.

a. Write a differential equation representing the rate of change of sales with respect to time.

b. At the end of the first year, total sales were growing by 8.3 billion dollars per year. Find the constant of proportionality.

c. Can we write an explicit formula for the general solution of this differential equation?

**Solution**

a. Let \( S(t) \) represent the total sales of the computer product \( t \) years after the product was introduced. A differential equation representing the information given is

\[ \frac{dS}{dt} = \frac{k}{\ln(t + 1.2)} \text{ billion dollars per year} \]

where \( t \) is the number of years after the product was introduced.

b. We are told that \( \frac{dS}{dt} = 8.3 \) billion dollars per year when \( t = 1 \). Using this fact, we have

\[ 8.3 = \frac{k}{\ln(1 + 1.2)} \]

Thus the constant of proportionality is \( k \approx 6.544 \). The differential equation is

\[ \frac{dS}{dt} \approx \frac{6.544}{\ln(t + 1.2)} \text{ billion dollars per year} \]

where \( t \) is the number of years after the product was introduced.

c. This differential equation expresses the derivative of \( S \) as a function of the input variable \( t \). If we knew a formula for a general antiderivative of this function, then we could write a general solution of the differential equation. However, we do not know such a formula.
As in Example 2, even when the differential equation appears to be in a simple form, we may not be able to find an explicit formula for its general solution. However, we still can analyze such a differential equation graphically and numerically to develop useful insight into the nature of the function and to estimate the solution.

**Slope Fields**

One way to obtain a graphical representation of a solution to a differential equation is to draw a slope field. A slope field is constructed by placing a grid on a portion of the Cartesian plane and, at each point on the grid, drawing a short line segment whose slope is determined by the differential equation. For example, consider the differential equation $\frac{dy}{dx} = 2x$. At the point $(1, 1)$, the slope of a solution to this equation is $2x = 2(1) = 2$, and at $(-0.5, 2)$, the slope is $2x = 2(-0.5) = -1$. Using the differential equation to determine the slopes at points on a grid on a plane where $-3 \leq x \leq 3$ and $-6 \leq y \leq 6$ and then sketching short line segments with those slopes at the appropriate points gives the slope field shown in Figure 11.2. (This construction is a tedious process and is usually done with computer software.)

Particular solutions for the differential equation can be sketched by following the line segments in such a way that the solution curves are tangent to each of the segments they meet. Figure 11.3 shows the graph of a particular solution for $\frac{dy}{dx} = 2x$. This particular solution goes through the point, or initial condition, $(0, -1)$. We use the term initial condition to refer to a known point on the graph of a particular solution. Knowing an initial condition allows us to find a particular, rather than a general, solution.

The slope field indicates that at the point $(0, -1)$, the derivative of the solution is zero. Thus the solution has a maximum, a minimum, or an inflection point at $(0, -1)$. The slopes to the left of $x = 0$ indicate that the solution is decreasing toward $(0, -1)$, and the slopes to the right of $x = 0$ indicate that the solution is increasing after it reaches $(0, -1)$. Therefore $(0, -1)$ is a minimum point. We form the solution graph by following the general direction indicated by the slopes. You may find it helpful to consider this as a more-sophisticated version of “connecting the dots.” Remember, however, that slopes in a slope field graph are plotted at only some, not all, points on the plane.
Figure 11.4 shows the graphs of several particular solutions for \( \frac{dy}{dx} = 2x \) drawn on the slope field. You should recognize the curves in Figure 11.4 as graphs of solutions of the form \( y(x) = x^2 + C \), which we know is the general solution of \( \frac{dy}{dx} = 2x \). The particular solution in Figure 11.4 with initial condition \((0, -1)\) is shown in teal. In this case, the particular solution is \( y(x) = x^2 - 1 \). The other particular solutions shown correspond to \( C = -3, -2, 0, 1, 2, 3 \).

Although this example of a slope field is for a differential equation with a known solution, slope fields are particularly helpful when we are graphing solutions for a differential equation for which we do not know solution formulas. A little practice looking at slope fields will assist you in sketching graphs of solutions and will help you understand how the forces that drive the rate of change of a quantity determine the shape of the graph of the quantity function.

Each of Figures 11.5 through 11.8 shows slope fields with three particular solutions sketched. On each figure the particular solutions for the initial conditions \((2, 2), (1, 1), \) and \((1, -1)\) are shown for the quantity function \( y \) with input \( x \).

Now that we have seen a few examples of solution graphs drawn on slope fields, we use slope fields to help us determine a solution graph in an applied situation.
EXAMPLE 3  **Sketching Graphs of Particular Solutions on a Slope Field**

**Sales**  In Example 2 we derived a differential equation representing the rate of change of computer sales with respect to time:

\[
\frac{dS}{dt} \approx \frac{6.544}{\ln(t + 1.2)} \text{ billion dollars per year}
\]

where \( t \) is the number of years after the product was introduced.

a. Examine the slope field of this differential equation shown in Figure 11.9, and sketch three particular solutions on the slope field.

b. In Example 2 we were told that sales were \$53.2 billion at the end of the first year. Sketch the particular solution corresponding to this initial condition, and use it to estimate sales after 3 years.

**Solution**

a. Figure 11.10 shows three particular solutions graphed on the slope field. You may have sketched different solutions, but they should all have the same general shape.

b. Figure 11.11 shows the particular solution when \( S(t) = 53.2 \) at \( t = 1 \). The point on the solution graph at \( t = 3 \) appears to be \((3, 60)\). Thus we estimate sales after 3 years as approximately \$60 billion.  

So far we have seen only a few simple differential equations in which the derivative of some function \( f \) is given in terms of only the input variable \( x \). In these cases, a general solution is the general antiderivative of the differential equation when one is known. Graphing the differential equation using slope fields can be helpful in obtaining insight into the nature of the function.
11.1 Concept Inventory

- Solutions to differential equations of the form \( \frac{dy}{dx} = k \)
- Solutions to differential equations of the form \( \frac{dy}{dx} = f(x) \)
- Direct proportionality
- Inverse proportionality
- Slope fields

No Instructor Activity Included:
All activities can be performed with the aid of technology.

11.1 Activities

For Activities 1 through 4, write an equation or differential equation for the given information.

1. The cost \( c \) to fill your gas tank is directly proportional to the number of gallons \( g \) your tank will hold.

2. The marginal cost of producing window panes (that is, the rate of change of cost \( c \) with respect to the number of units produced) is inversely proportional to the number of panes \( p \) produced.

3. Barometric pressure \( p \) is changing with respect to altitude \( a \) at a rate that is proportional to the altitude.

4. The rate of change of the cost \( c \) of mailing a first-class letter with respect to the weight of the letter is constant.

For each of the differential equations in Activities 5 through 11,

a. Use the corresponding slope field to sketch the graphs of three particular solutions.

b. Describe how the graphs of the solutions compare with each other.

c. Write a general solution for the differential equation.
11.1 Differential Equations and Slope Fields

9. \( \frac{dy}{dx} = \frac{1}{2^x} \)

10. \( \frac{dy}{dx} = 2x \)

11. \( \frac{dy}{dx} = -x \)

12. For the differential equations in Activities 5 through 8, compare and contrast the particular solutions with initial condition \((0, 0)\).

13. For the differential equations in Activities 9 through 11, compare and contrast the particular solutions with initial condition \((0, 0)\).

14. Consider a function \( y = f(x) \) whose rate of change with respect to \( x \) is constant.
   a. Write a differential equation describing the rate of change of this function.
   b. Write a general solution for the differential equation.
   c. Verify that the general solution you gave in part \( b \) is indeed a solution by substituting it into the differential equation and obtaining an identity.

15. **Energy** Between 1975 and 1980, energy production in the United States was increasing at an approximately constant rate of 0.98 quadrillion Btu per year. In 1980 the United States produced 64.8 quadrillion Btu.
   (Source: Based on data from Statistical Abstract, 1994.)
   a. Write a differential equation for the rate of change of energy production.
   b. Write a general solution for the differential equation.
   c. Using the initial condition, determine the particular solution for energy production.
   d. Estimate the energy production in 1975 as well as the rate at which energy production was changing at that time.
   e. Use the slope field to sketch the graph of the particular solution indicated by the initial condition, and use this graph to estimate energy production in 1975. How close is your graphical estimate to that in part \( d \)?

<table>
<thead>
<tr>
<th>Energy production (quadrillion Btu)</th>
<th>Years since 1975</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>5</td>
</tr>
<tr>
<td>65</td>
<td>4</td>
</tr>
<tr>
<td>65.5</td>
<td>3</td>
</tr>
<tr>
<td>66</td>
<td>2</td>
</tr>
<tr>
<td>66.0</td>
<td>1</td>
</tr>
<tr>
<td>58</td>
<td>0</td>
</tr>
</tbody>
</table>

16. **Energy** Between 1975 and 1980, energy consumption in the United States was increasing at an approximately constant rate of 1.08 quadrillion Btu per year. In 1980 the United States consumed 76.0 quadrillion Btu.
   (Source: Based on data from Statistical Abstract, 1994.)
   a. Write a differential equation for the rate of change of energy consumption.
   b. Write a general solution for the differential equation.
   c. Determine the particular solution for energy consumption.
   d. Estimate the energy consumption in 1975 as well as the rate at which energy consumption was changing at that time.
e. Use the slope field to sketch the graph of the particular solution indicated by the initial condition, and use this graph to estimate energy consumption in 1975. How close is your graphical estimate to that in part d?

17. **Cropland**  The amount of arable and permanent cropland worldwide has been increasing at a slow but relatively steady rate of 0.0342 million square kilometers per year over the past two decades (from 1970 to 1990). In 1980 there were 14.17 million square kilometers of cropland. (Source: Ronald Bailey, ed., The True State of the Planet. New York: The Free Press for the Competitive Enterprise Institute, 1995.)

a. Write a differential equation representing the growth of cropland.
b. Write a general solution for the differential equation in part a.
c. Write the particular solution for the amount of cropland.
d. Use the equations to estimate the rate of change of cropland in 1970 and in 1990. Also estimate the amount of cropland in those years.

18. **Air Pressure**  Barometric pressure $p$ (measured in inches of mercury) decreases with respect to altitude $a$ (measured in feet) at a rate that is directly proportional to the altitude. The constant of proportionality equals $3.7 \cdot 10^{-5}$.

a. Write a differential equation representing the rate of change of barometric pressure.
b. Write a general solution for this differential equation.
c. Assume that barometric pressure at sea level is 30 inches of mercury. Find a particular solution for the differential equation.
d. Use the slope field to sketch the particular solution for the differential equation and initial condition given in part c.

19. **Falling Object**  An object that has been dropped falls at a velocity $v$ (in feet per second) that is proportional to the number of seconds $t$ after it has been dropped. The constant of proportionality depends on the force of gravity. On the earth, the proportionality constant is $-32$ feet per second squared.

a. Write an equation giving velocity as a function of time.
b. Rewrite the equation in part a as a differential equation giving the rate of change of distance as a function of time.
c. Find a general solution to the differential equation.
d. After how many seconds will the object hit the ground if it were dropped from a height of 35 feet? What is the terminal velocity of this object?

20. **Falling Object**  Refer to Activity 19. Use the slope field graph for $\frac{ds}{dt} = -32t$ that is given to answer the following questions.
11.1 Differential Equations and Slope Fields

a. Sketch a graph of the particular solution for the initial condition \( s = 35 \) when \( t = 0 \).

b. Sketch a graph of the particular solution for the initial condition \( s = 10 \) when \( t = 0 \).

c. If \( v \) represents the velocity of a falling (or thrown) object in feet per second \( t \) seconds after the object is thrown or dropped, discuss what information is given by the graphs of the particular solutions in parts \( a \) and \( b \).

21. Consider a function \( y = f(x) \) whose rate of change with respect to \( x \) is directly proportional to the input.

a. Write a differential equation describing the rate of change of this function.

b. Write a general solution for the differential equation.

c. Verify that the general solution you gave in part \( b \) is indeed a solution by substituting it into the differential equation and simplifying to obtain an identity.

22. Weight For the first 9 months of life, the average weight \( w \), in pounds, of a certain breed of dog increases at a rate that is inversely proportional to time \( t \), in years. The height of the tree is 4 feet at the end of 2 years and reaches 30 feet at the end of 7 years.

a. Write a differential equation describing the rate of change of the height of the tree.

b. Give a particular solution for this differential equation.

c. How tall will the tree be in 15 years? What will happen to the height of this tree over time?

24. Weight Refer to Activity 22. A slope field for the differential equation in part \( a \) of Activity 22 is shown.

b. Use the graph to estimate the weight of the puppy at 3 months and at 6 months.

d. Why does this differential equation describe weight gain for only 8 months instead of for the life span of the dog?

23. Height The height \( h \), in feet, of a certain tree increases at a rate that is inversely proportional to time \( t \), in years. The height of the tree is 4 feet at the end of 2 years and reaches 30 feet at the end of 7 years.

a. Write a differential equation describing the rate of change of the height of the tree.

b. Give a particular solution for this differential equation.

c. How tall will the tree be in 15 years? What will happen to the height of this tree over time?

25. Height Refer to Activity 23. A slope field for the differential equation in part \( a \) of Activity 23 is shown.
a. Sketch the graph of the particular solution in part b of Activity 23.

b. Use the graph to estimate the height of the tree after 15 years of growth.

26. Consider a function \( y = f(x) \) whose rate of change with respect to \( x \) is inversely proportional to the input.

   a. Write a differential equation describing the rate of change of this function.
   
   b. Write a general solution for the differential equation.
   
   c. Verify that the general solution you gave in part b is indeed a solution by substituting it into the differential equation and simplifying to obtain an identity.

27. For each of the following differential equations and their slope fields,

   i. Sketch the graphs of three particular solutions.
   
   ii. Describe how the graphs of the solutions behave.
   
   iii. Compare and contrast the family of solutions for each of the differential equations.

   a. \( \frac{dy}{dx} = \frac{1}{x} \)

   b. \( \frac{dy}{dx} = \frac{10}{x} \)

   c. \( \frac{dy}{dx} = \frac{-1}{x} \)

   d. \( \frac{dy}{dx} = \frac{1}{10x} \)

In Activities 28 through 31, use the slope fields given for the differential equations to sketch the particular solution for each given initial condition.

28. \( \frac{dy}{dx} = 2x + 1 \)

   a. \( x = 4, y = 11 \)
   
   b. \( x = 2, y = -4 \)
11.2 Separable Differential Equations

In Section 11.1 we considered differential equations of the form \( \frac{dy}{dx} = f(x) \).

In this section we consider differential equations of the forms \( \frac{dy}{dx} = f(y) \) and \( \frac{dy}{dx} = f(x, y) \). Because differential equations of these types give slope in terms of \( x \) and \( y \), slope fields are useful in giving graphical insight into the behavior of the underlying function. For instance, a slope field for \( \frac{dy}{dx} = 3x^2 + 2x \) (see Figure 11.12) shows that the general solution for this differential equation appears to behave in a parabolic manner, with the parabola lying on its side.

**Separation of Variables**

Because differential equations of the forms \( \frac{dy}{dx} = f(y) \) and \( \frac{dy}{dx} = f(x, y) \) are not in terms of only the input variable \( x \), they cannot be solved directly by writing an antiderivative function. Instead, we use a technique known as **separation of variables** to solve such equations.

Let us again consider the differential equation \( \frac{dy}{dx} = \frac{32}{y} \). Because we do not know what it means to find a general antiderivative of \( \frac{32}{y} \) with respect to the variable \( x \), we move all symbols containing \( y \) to one side of the equation and all symbols involving

29. \( \frac{dy}{dx} = -\sin x \)
   a. \( x = 3, y = 4 \)
   b. \( x = 0, y = 0 \)
   c. \( x = 2, y = -1 \)

30. \( \frac{dy}{dx} = \cos x \)
   a. \( x = 0, y = 0 \)
   b. \( x = 2, y = 5 \)
   c. \( x = -1, y = 2 \)

31. \( \frac{dy}{dx} = 3x^2 + 2x \)
   a. \( x = 2, y = 5 \)
   b. \( x = -3, y = 0 \)
Until now, we have considered $\frac{dy}{dx}$ as a single symbol denoting the rate of change of $y$ with respect to $x$. In separating variables, we consider $dy$ and $dx$ to be two separate symbols, sometimes referred to as differentials.

x to the other side of the equation. This procedure is known as separating the variables. In this case, we have

$$y \, dy = 32 \, dx$$

Now that the equation has the variables separated, we take antiderivatives of both sides of the equation.

$$\int y \, dy = \int 32 \, dx$$

$$\frac{1}{2}y^2 + c_1 = 32x + c_2$$

where $c_1$ and $c_2$ are both constants. Combining the constants and solving for $y^2$ yields

$$y^2 = 64x + C$$

Finally, we write the solution equation giving $y$ in terms of $x$ by taking the square root of both sides of the equation.

$$y = \pm \sqrt{64x + C}$$

This equation does indeed yield the two sides of a horizontal parabola, as the slope field indicates. In this case, $y$ is not a function of $x$ because a single value of $x$ could yield two different values of $y$.

**Differential Equations Modeling Constant Percentage Change**

Any time that percentage growth is constant, the situation can be described by the differential equation

$$\frac{dy}{dx} = ky,$$

where $k$ is the constant percentage rate of change. Because constant percentage growth characterizes exponential functions, the solution to this differential equation is $y = ae^{kt}$. Example 1 illustrates the solution to differential equations of this form.

**EXAMPLE 1 Solving Differential Equations Using Separation of Variables**

**Bank Account** Consider an account for which interest is compounded continuously at an annual interest rate of 7%.

- a. Write a differential equation expressing the rate of change of the amount in the account with respect to time.
- b. Examine the slope field of this differential equation shown in Figure 11.13, and make a conjecture about the behavior of the function.
- c. Find a general solution for this differential equation.
- d. If the amount after 3 years is $1000, find the particular solution.

**Solution**

- a. The rate of change of $A$, the amount in dollars in the account, can be expressed as the differential equation

$$\frac{dA}{dt} = 0.07A$$

dollars per year after $t$ years.
b. This slope field indicates a horizontal asymptote at \( A = 0 \) and growth that increases as \( t \) increases. This behavior is indicative of an exponential model.

c. To solve this equation, we use the technique of separating variables:

\[
\frac{1}{A} \, dA = 0.07 \, dt
\]

Determining general antiderivatives of both sides, we have

\[
\int \frac{1}{A} \, dA = \int 0.07 \, dt
\]

which gives

\[
\ln A + c_1 = 0.07t + c_2
\]

Combining the constants yields

\[
\ln A = 0.07t + C
\]

Recalling from algebra that \( \ln x = y \) is equivalent to \( x = e^y \), we have

\[
A = e^{(0.07t+C)} = e^{0.07t} \, e^C
\]

Replacing the constant \( e^C \) with the constant \( a \) gives the general solution

\[
A = ae^{0.07t} \text{ dollars}
\]

after \( t \) years.

d. Because the amount after 3 years is $1000, we find the particular solution by substituting \( t = 3 \) and \( A = 1000 \) into the general solution and solving for \( a \).

\[
1000 = ae^{0.07(3)}
\]

\[
a \approx 810.584
\]

The particular solution is

\[
A \approx 810.584e^{0.07t} \text{ dollars}
\]

after \( t \) years. This solution can also be written in the form

\[
A \approx 810.584(1.0725^t) \text{ dollars}
\]

after \( t \) years. As seen in Figure 11.14, this particular solution fits the description in part \( b \).

Example 2 illustrates the use of two different differential equations, one that requires separation of variables and one that can be solved by determining an antiderivative.
the growth in the rate of deficit spending proportional to the GNP. Last year the GNP was $3 billion, and the country’s national debt was $2.3 billion. The government has mandated that this year’s national debt be held at $2.4 billion dollars.

a. Express the country’s GNP growth as a differential equation, and find the solution.

b. Express the rate of change of the country’s national debt as a differential equation, and find the solution.

c. Evaluate the solutions for $t = 2$, and interpret the answers.

**Solution**

a. Let $G$ represent the GNP in billions of dollars. Then $\frac{dG}{dt} = 0.05G$ billion dollars per year represents the rate of change of the GNP after $t$ years. The solution is found by separating variables.

$$\frac{1}{G} dG = 0.05 \, dt$$

yields

$$\ln G = 0.05t + C$$

which can be rewritten as

$$G = ae^{0.05t}$$

billion dollars

If we consider $t$ to be the number of years since last year, then $a = 3$ since the GNP in year 0 (last year) was $3$ billion. Thus the solution to the differential equation is

$$G = 3e^{0.05t}$$
billion dollars

after $t$ years.

b. Let $D$ be the national debt in billions of dollars. The statement “the growth rate of deficit spending is proportional to the GNP” can be translated into mathematical symbols as

$$\frac{dD}{dt} = kG$$
billion dollars per year

after $t$ years, where $k$ is the constant of proportionality. Note that at this point we do not have the information needed to solve for $k$. However, we have the information to solve for $k$ once we find a general solution. Knowing a function for $G$, we substitute this into the differential equation:

$$\frac{dD}{dt} = k(3e^{0.05t})$$

Because this differential equation gives the derivative of $D$ in terms of only the input $t$, we do not need to use separation of variables but can proceed by writing a general antiderivative:

$$D(t) = \frac{3ke^{0.05t}}{0.05} + C = 60ke^{0.05t} + C$$
billion dollars
after \( t \) years. We now have two constants, \( k \) and \( C \), to determine, so we must create a system of two equations that we can solve simultaneously. Because the national debt last year (when \( t = 0 \)) was $2.3\text{ billion}$, we substitute this information into \( D \):

\[
2.3 = 60ke^0 + C = 60k + C \quad (1)
\]

Also, the national debt is $2.4\text{ billion}$ when \( t = 1 \), so

\[
2.4 = 60ke^{0.05} + C \quad (2)
\]

Solving equation 1 for \( C \) and substituting into equation 2 give the equation

\[
2.4 = 60ke^{0.05} + (2.3 - 60k)
\]

or

\[
0.1 = (60e^{0.05} - 60)k
\]

\[
k = \frac{0.1}{60e^{0.05} - 60}
\]

Thus \( k \approx 0.0325 \). Using this value of \( k \) in equation 1 gives

\[
C \approx 2.3 - 60(0.0325) = 0.3496
\]

Thus we have the particular solution

\[
D(t) \approx 1.9504e^{0.05t} + 0.3496 \text{ billion dollars}
\]

after \( t \) years.

c. When \( t = 2 \), \( G(2) \approx $3.3 \text{ billion} \) and \( D(2) \approx $2.5 \text{ billion} \). In 2 years, the GNP will be approximately $3.3 \text{ billion} \), and the national debt should be held to approximately $2.5\text{ billion}. ●

**Joint Proportionality**

Section 11.1 introduced two forms of proportionality: direct and inverse. Now we consider a third form of proportionality. When a quantity \( y \) is proportional to the product of two other quantities \( x \) and \( z \), that is, when there is some constant \( k \) such that \( y = kxz \), we say the quantity \( y \) is **jointly proportional** to the quantities \( x \) and \( z \).

**Joint Proportionality**

For inputs \( x \) and \( z \) and output \( y \), \( y \) is jointly proportional to \( x \) and \( z \) if there exists some constant of proportionality \( k \) such that \( y = kxz \).

Example 3 illustrates the concept of joint proportionality as it is used in psychology.
EXAMPLE 3  

Solving an Equation Involving Joint Proportionality

Stimulus Response  The Fechner Law relates response to stimulus. A different model that is used to describe this relationship is the Brentano-Stevens Law. This law says that the level of response $R$ changes according to a joint proportionality between the level of the response and the inverse of $s$, the level of the stimulus. That is, \[
\frac{dR}{ds} = k \frac{R}{s}
\] for some constant $k$.

Consider the following differential equation that describes a person’s perception of the intensity of sound:

\[
\frac{dR}{ds} = \frac{2.94R}{s}
\]

where the sound $s$ is measured in decibels and intensity $R$ is measured on a scale from 0 to 10, with 0 representing inaudible sound and 10 representing painfully intense sound.

a. Examine the slope field graph in Figure 11.15, and comment on the behavior of the graphs of the solutions to this differential equation.

b. Write an equation giving the response $R$ in terms of stimulus $s$.

Solution

a. The slope field of \[
\frac{dR}{ds} = \frac{2.94R}{s}
\] that is shown in Figure 11.15 suggests that the solutions are exponential.

b. Because the differential equation \[
\frac{dR}{ds} = \frac{2.94R}{s}
\] gives the rate of change of $R$ with respect to $s$ in terms of both $s$ and $R$, we use the method of separation of variables rather than simply writing an antiderivative formula. Separating the variables yields

\[
\frac{1}{R} dR = \frac{2.94}{s} ds
\]

Taking antiderivatives of both sides of the equation and realizing that $R > 0$ and $s > 0$ yield the equation $\ln R + c_1 = 2.94 \ln s + c_2$. Combining the constants $c_1$ and $c_2$ gives the equation $\ln R = 2.94 \ln s + C$. This equation is equivalent to

\[
e^{\ln R} = e^{2.94 \ln s + C}
\]

which we simplify as follows:

\[
e^{\ln R} = e^{2.94 \ln s + C}
\]
11.2 Separable Differential Equations

\[ e^{\ln R} = (e^{\ln s})^{2.94} e^C \]

\[ R = s^{2.94} e^C \]

Replacing \( e^C \) with the constant \( a \) gives the general solution as

\[ R = a s^{2.94} \]

where \( s \) is measured in decibels and \( a \) is a constant.

---

Logistic Models and Their Differential Equations

We have seen differential equations that lead to linear, quadratic, logarithmic, and exponential models. What does a differential equation that yields a logistic model look like? To help us better understand the underlying differential equation, recall the role of the limiting value \( L \) in a logistic model

\[ y(x) = \frac{L}{1 + A e^{-Bx}} \]

As we saw in Chapter 2, the logistic model may be used to study the spread of a virus in a network of computers. The virus initially spreads exponentially because the first computer infects another, then these two computers infect two others, then the four of them infect four others, and so on. At some point, the computers that have the virus contact only others that are also infected instead of uninfected computers, and the exponential sequence is broken. Eventually, the number of infected computers is close to the limiting value \( L \).

The rate of change of the number of infected computers depends on two things: the number \( y \) of computers already infected and the number \( L - y \) of computers not yet infected. Thus the rate of change is jointly proportional to the number of computers already infected and the number of uninfected computers remaining. We can describe the spread of a virus with the differential equation

\[ \frac{dy}{dx} = ky(L - y) \]

where \( L \) is the number of computers in the group and \( k \) is the constant of proportionality. It is also common to refer to the limiting value \( L \) as the carrying capacity of the system or as the saturation level.

A slope graph of a differential equation of the form \( \frac{dy}{dx} = ky(L - y) \) gives us insight into the behavior of the function \( y \). Figure 11.16 shows slope fields for differential equations of this form with different values of \( k \) and \( L \).

In each case, the slope field of the differential equation appears to have an upper and a lower horizontal asymptote, and a solution between these limiting values.
appears to be logistic (either increasing or decreasing). In fact, it can be shown that the solution for \( \frac{dy}{dx} = ky(L - y) \) is the logistic function

\[
y = \frac{L}{1 + Ae^{Bx}}
\]

Even though the logistic equation \( y(x) = \frac{L}{1 + Ae^{Bx}} \) (where \( A \) and \( L \) are constant and \( B = Lk \)) can be derived from the differential equation \( \frac{dy}{dx} = ky(L - y) \) using separation of variables, partial fractions, and a lot of algebra, it is not within the scope of this text to go into this detail. However, you will be asked in the activities to verify that \( y(x) = \frac{L}{1 + Ae^{Bx}} \), with \( B = Lk \), is a general solution for \( \frac{dy}{dx} = ky(L - y) \).

**EXAMPLE 4  Solving an Equation That Results in a Logistic Function**

**Epidemic**  In 1949 the United States experienced the second worst polio epidemic* in its history. (The worst was in 1952.) In January, 494 cases of polio were diagnosed,

and by December, a total of 42,375 cases had been diagnosed. Assume that the spread of polio followed the general principle that the rate of spread was jointly proportional to the number of infected people and the number of uninfected people. Also assume that the carrying capacity for polio in the United States in 1949 was approximately 43,000 people.

a. Write a differential equation describing the spread of polio.

b. Determine a particular solution for this differential equation.

**Solution**

a. Let \( P(m) \) be the number of polio cases diagnosed by the end of the \( m \)th month of 1949, and let \( k \) be the constant of proportionality. A differential equation describing the spread of polio is

\[
\frac{dP}{dm} = kP(43,000 - P) \quad \text{cases per month}
\]

Because we are given information about the number of cases, not information concerning the rate of change of the number of cases, we cannot find the constant of proportionality at this point.

b. A general solution for this differential equation is the logistic equation

\[
P(m) = \frac{43,000}{1 + Ae^{43,000km}} \quad \text{cases}
\]

diagnosed by the end of the \( m \)th month of 1949. This equation contains two constants, \( A \) and \( k \), for which we must solve. We are given the two points \((1, 494)\) and \((12, 42,375)\). Substituting these into the logistic equation, we obtain a system of two equations that we can solve simultaneously for the two constants, \( A \) and \( k \).

The two equations are

\[
494 = \frac{43,000}{1 + Ae^{43,000}} \quad (3)
\]

\[
42,375 = \frac{43,000}{1 + Ae^{516,000k}} \quad (4)
\]

Solving equation 3 for \( A \) and substituting into equation 4 yield

\[
42,375 = \frac{43,000}{1 + \left(\frac{42,506e^{43,000k}}{494}\right)e^{-516,000k}}
\]

Solving this equation for \( k \) yields \( k \approx 1.83328 \times 10^{-5} \), or \( B = Lk \approx 0.788312 \). Substituting \( k \) into equation 3 and solving for \( A \) yield \( A \approx 189.2704 \). Thus we have the particular solution

\[
P(m) \approx \frac{43,000}{1 + 189.2704e^{0.788312m}} \quad \text{cases}
\]

diagnosed by the \( m \)th month of 1949.
Separation of variables often yields a solution when we are considering differential equations of the form \(\frac{dy}{dx} = f(y)\) or \(\frac{dy}{dx} = f(x, y)\). Slope fields usually give graphical insight into the behavior of the underlying function, and in the case where \(\frac{dy}{dx} = ky(L - y)\), we have a specific equation that gives the solution. However, there are many differential equations for which graphical and algebraic methods of determining particular solutions are beyond the scope of this book. In fact, there are many differential equations for which algebraic solution methods fail. In such cases, we rely on numerical techniques, one of which is discussed in the next section.

**11.2 Concept Inventory**

- Solutions to differential equations of the form \(\frac{dy}{dx} = f(x, y)\)
- Solutions to differential equations of the form \(\frac{dy}{dx} = ky\)
- Solutions to differential equations of the form \(\frac{dy}{dx} = ky(L - y)\)
- Separation of variables
- Joint proportionality

**Activities Key**

See page xvi of the preface for icon explanations.

**11.2 Activities**

For Activities 1 through 5, write a differential equation for each of the statements. When possible, find a general solution to the differential equation.

1. Ice thickens with respect to time \(t\) at a rate that is inversely proportional to its thickness \(T\).

2. The Verhulst population model assumes that a population \(P\) in a country will be increasing with respect to time \(t\) at a rate that is jointly proportional to the existing population and to the remaining amount of the carrying capacity \(C\) of that country.

3. The rate of change with respect to time \(t\) of the amount \(A\) that an investment is worth is proportional to the amount in the investment.

4. The rate of change in the height \(h\) of a tree with respect to its age \(a\) is inversely proportional to the tree’s height.

5. In a community of \(N\) farmers, the number \(x\) of farmers who own a certain tractor changes with respect to time \(t\) at a rate that is jointly proportional to the number of farmers who own the tractor and to the number of farmers who do not own the tractor.

6. In mountainous country, snow accumulates at a rate proportional to time \(t\) and is packed down at a rate proportional to the depth \(S\) of the snowpack. Write a differential equation describing the rate of change in the depth of the snowpack with respect to time.

7. Water flows into a reservoir at a rate that is inversely proportional to the square root of the depth of water in the reservoir, and water flows out of the reservoir at a rate that is proportional to the depth of the water in the reservoir. Write a differential equation describing the rate of change in the depth \(D\) of water in the reservoir with respect to time \(t\).

For Activities 8 through 17, sketch three particular solutions for each given slope field.
11.2 Separable Differential Equations

9.

10. 

11. 

12. 

13. 

14. 

15. 

16.
For Activities 18 through 24, identify the differential equation as one that can be solved using only antiderivatives or as one for which separation of variables is required. Then find a general solution for the differential equation.

18. \( \frac{dy}{dx} = kx \)

19. \( \frac{dy}{dx} = ky \)

20. \( \frac{dy}{dx} = \frac{k}{y} \)

21. \( \frac{dy}{dx} = \frac{k}{x} \)

22. \( \frac{dy}{dx} = \frac{kx}{y} \)

23. \( \frac{dy}{dx} = \frac{ky}{x} \)

24. \( \frac{dy}{dx} = kxy \)

25. **Medicine**  The rate of change with respect to time of the quantity \( q \) of pain reliever in a person's body \( t \) hours after the individual takes the medication is proportional to the quantity of medication remaining. Assume that 2 hours after a person takes 200 milligrams of a pain reliever, one-half of the original dose remains.

a. Write a differential equation for the rate of change of the quantity of pain reliever in the body.

b. Find a particular solution for this differential equation.

c. How much pain reliever will remain after 4 hours? after 8 hours?

26. **Radioisotope**  Technetium-99 is a radioisotope that has been used in humans to help doctors locate possible malignant tumors. Radioisotopes decay (over time) at a rate that is directly proportional to the amount of the radioisotope. Technetium-99 has a half-life of 210,000 years. Assume that 0.1 mg of technetium-99 is injected into a person's bloodstream.

a. Write a differential equation for the rate at which the amount of technetium-99 decays.

b. Find a particular solution for this differential equation.

27. **Radioisotope**  Radon-232 is a colorless, odorless gas that undergoes radioactive decay with a half-life of 3.824 days. It is considered a health hazard, so new homebuyers often have their property tested for the presence of radon-232. Because radon-232 is a radioisotope, it decays (over time) at a rate that is directly proportional to the amount of the radioisotope.

a. Write a differential equation for the rate at which an amount of radon-232 decays.

b. Write a general solution for this differential equation.

c. If 1 gram of radon-232 is isolated, how much of it will remain after 12 hours? after 4 days? after 9 days? after 30 days?

28. Consider a function \( y = f(x) \) whose rate of change is proportional to \( f \).

a. Write a differential equation describing the rate of change of this function.

b. Write a general solution for the differential equation.

c. Verify that the general solution you gave in part \( b \) is indeed a solution by substituting it into the differential equation and simplifying to obtain an identity.

29. **Postage**  In 1880, 37 countries issued postage stamps. The rate of change (with respect to time) of the number of countries issuing postage stamps between 1836 and 1880 was jointly proportional to the number of countries that had already issued postage stamps and to the number of countries that had not yet issued postage stamps. The constant of proportionality was approximately 0.0049. By 1855, 16 countries had issued postage stamps.


a. Write a differential equation describing the rate of change in the number of countries issuing postage stamps with respect to the number of years since 1800.

b. Write a general solution for the differential equation.
30. **Patents**  The number of patents for plow sulkies between 1865 and 1925 was increasing with respect to time at a rate jointly proportional to the number of patents already obtained and to the difference between the number of patents already obtained and the carrying capacity of the system. The carrying capacity was approximately 2700 patents, and the constant of proportionality was about \(7.52 \times 10^{-5}\). By 1883, 980 patents had been obtained.


**a.** Write a differential equation describing the rate of change in the number of patents with respect to the number of years since 1865.

**b.** Write a general solution for the differential equation.

**c.** Write the particular solution for the differential equation.

**d.** Estimate the number of patents obtained by 1900.

**e.** Using the slope field, identify the upper and lower horizontal asymptotes.

**f.** On the slope field, sketch the particular solution for the given initial condition.

31. Consider a function \(y = f(x)\) whose rate of change is jointly proportional to \(f\) and to \(L - f\).

**a.** Write a differential equation describing the rate of change of this function.

**b.** Write a general solution for the differential equation.

**c.** Verify that the general solution you gave in part b is indeed a solution by substituting it into the differential equation and simplifying to obtain an identity.
Euler’s method relies on the use of the derivative function to approximate the change in the quantity function. Recall that if we know a point \( (a, f(a)) \) on a function \( f \) and we also know \( f'(a) \), the slope at that point, then we can approximate the value of the function at a close point \( (b, f(b)) \) as \( f(b) \approx f(a) + (b - a)f'(a) \). In using this approximation, we are using a point on the line tangent to the graph of \( f \) at \( x = a \) to approximate a point on the function (see Figure 11.17).

We first illustrate the Euler method of approximating change for a differential equation we can solve. The growth rate of the population* of the United States between 1900 and 2000 can be expressed as \( \frac{dP}{dx} = 0.0181x + 1.129 \) million people per year \( x \) years after 1900. In 1990 the population was 249.9 million people.

The initial condition gives us a starting point, \((90, 249.9)\), and the differential equation gives us the slope of the line tangent to the graph of \( P \) at that point, \( P'(90) = 2.758 \) million people per year. If we use this information to estimate the population in the year 2005, then we obtain \( P(105) = 249.9 + 15(2.758) = 291.27 \) million people. Figure 11.18 illustrates the tangent-line estimate and the estimate given by the solution \( P \).

![Figure 11.17](image)

**FIGURE 11.17**

Instructor Note
Students should understand that Euler’s method estimates the output value of a particular solution at a specific input. Plotting successive estimates produces an approximation of a graph of that particular solution.

![Figure 11.18](image)

**FIGURE 11.18**

Euler’s method says that we can obtain a better estimate by using many steps rather than using just one step in the approximation process. In this case, instead of approximating over 15 years all in one step, we obtain a better estimate by approximating over 1-year intervals 15 times. Using the initial condition \((90, 249.9)\) and the differential equation evaluated at \( x = 90, P'(90) = 2.758 \), we approximate the population for the year 1991 as

\[
P(91) = 249.9 + (1)(2.758) = 252.658 \text{ million people}
\]

Now we have an estimate of a new point, \((91, 252.658)\), and we use the differential equation to find the slope of the tangent line in 1991, \( P'(91) = 2.7761 \) million people per year. Using this new point and slope, we estimate the population in 1992 as

\[
P(92) = 252.658 + (1)(2.7761) = 255.4341 \text{ million people}
\]

Similarly, we estimate the populations for each of the years from 1993 through 2005. Table 11.1 records these estimates, and Figure 11.19 illustrates the estimates

and the line segments used to create them. (Note that even though the slopes found are those of tangent lines, the actual line segments may not be lines tangent to the graph of \( P \), but lines parallel to those tangent lines.)

Applying Euler’s method using steps of size 1 year yields the estimate that the population in 2005 will be 293.17 million people. This estimate is slightly larger than our original estimate of 291.27 and is closer to the estimate of 293.3 that we obtained in Example 1 of Section 11.1 by finding a particular solution to the differential equation. (To obtain an estimate closer to that given by the particular solution, we could use Euler’s method with 100 steps of size 0.1 to improve our results.)

The graph we obtain by plotting the Euler estimates (Figure 11.19) and drawing the appropriate line segments is not a smooth curve. However, it does resemble a portion of the quadratic curve we expect to see for the solution of this differential equation and is an approximation to the graph of the population function described by the differential equation.

Euler’s method is a fairly straightforward way of numerically estimating a solution to a differential equation. However, use caution. Because each step of Euler’s method is built on an estimate made in the previous step, it can lead to very bad estimates when the step size is too large or when the slopes given by the differential equation change from very large to very small over a small interval. For these reasons, there are other methods of numerically estimating solutions that are taught in more advanced courses. However, because Euler’s method is sufficient for our uses, we will not consider these other methods.

Example 1 illustrates the application of Euler’s method to the differential equation given in Example 2 of Section 11.1.

### Example 1

**Applying Euler’s Method to an Equation in One Variable**

**Sales**  In Example 2 of Section 11.1, we saw that the rate of change of the total sales of a computer product could be represented by the differential equation

\[
\frac{dS}{dt} \approx \frac{6.544}{\ln(t + 1.2)} \text{ billion dollars per year}
\]
where \( t \) is the number of years after the product was introduced. Also, we were told that at the end of the first year, sales totaled $53.2 billion. Use Euler’s method to estimate the total sales at the end of the fifth year.

**Solution**  
Our starting point (the total sales at the end of the first year) is given as \((1, 53.2)\), we know the rate of change at the end of the first year is \( S'(1) = \frac{6.544}{\ln(1 + 1.2)} \approx 8.2998 \), and we wish to estimate \( S(5) \), the total sales at the end of the fifth year. We use Euler’s method to estimate over an interval of 4 years. In order to keep the step sizes small, we choose to take 16 steps of size 0.25. The first step gives an estimate for \( S(1.25) \).

\[
S(1.25) \approx 53.2 + (0.25)(8.2998) = 55.275
\]

The estimates (to three decimal places) from each step as well as the slopes given by the differential equation are recorded in Table 11.2.

<table>
<thead>
<tr>
<th>( t )</th>
<th>Estimate of ( S(t) ) (billion dollars)</th>
<th>Slope at ( t ) (billion dollars per year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>53.2</td>
<td>8.2998</td>
</tr>
<tr>
<td>1.25</td>
<td>55.275</td>
<td>7.3029</td>
</tr>
<tr>
<td>1.5</td>
<td>57.101</td>
<td>6.5885</td>
</tr>
<tr>
<td>1.75</td>
<td>58.748</td>
<td>6.0491</td>
</tr>
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<td>2.0</td>
<td>60.260</td>
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<td>61.667</td>
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<td>3.6694</td>
</tr>
<tr>
<td>5.0</td>
<td>73.559</td>
<td>11.32</td>
</tr>
</tbody>
</table>

**Figure 11.20**

Euler’s method with step sizes of 0.25 year yields the estimate that the total sales for this computer product will be $73.6 billion at the end of the fifth year. A graph (Figure 11.20) of the Euler estimates is an approximation of a graph of total sales and shows us that as time increases, the increase in total sales begins to slow.

Euler’s method can be used on most first-order differential equations, even when separation of variables fails.

**Example 2**  
*Using Euler’s Method with Two Variables*

Consider the differential equation \( \frac{dy}{dx} = 5.9x - 3.2y \). Given an initial condition of \( y(10) = 50 \), estimate \( y(12) \).

**Solution**  
Because we know both \( x \) and \( y \) at \( x = 10 \), we can use the differential equation to calculate the slope at \( x = 10 \) and \( y = 50 \) as \( \frac{dy}{dx} = 5.9(10) - 3.2(50) = -101 \).
We choose to use Euler’s method with 10 steps of size 0.2. The first estimate is
\[ y(10.2) = 50 + (0.2)(-101) = 29.8. \]

Because the formula for the slope \( \frac{dy}{dx} = 5.9x - 3.2y \) relies on knowing both \( x \) and \( y \) to estimate the slope at \( x = 10.2 \), we must use our estimate \( y(10.2) = 29.8 \) in the slope formula. Thus at \( x = 10.2 \),
\[ \frac{dy}{dx} = 5.9(10.2) - 3.2(29.8) = 35.18. \]

We now use the estimates \( y(10.2) = 29.8 \) and \( y' = 35.18 \) at \( x = 10.2 \) to estimate the value of \( y \) at \( x = 10.4 \):
\[ y(10.4) = 29.8 + (0.2)(35.18) = 22.764. \]

To find an estimate of \( y(10.6) \), we need the slope at \( x = 10.4 \). Again we must estimate this slope at \( x = 10.4 \) using \( y(10.4) \):
\[ \frac{dy}{dx} = 5.9(10.4) - 3.2(22.764) = -11.4848. \]

Thus the value of \( y \) at \( x = 10.6 \) is
\[ y(10.6) = 22.764 + (0.2)(-11.4848) = 20.46704. \]

Proceeding in this manner, we construct Table 11.3 of Euler estimates and find that \( y(12) \approx 21.55 \).

Graphing the Euler estimates gives us an idea of how the function \( y \) behaves (see Figure 11.21). There appears to be a minimum near \( x = 10.8 \). The minimum value is
\[ y(10.8) \approx 19.876. \]

### Table 11.3

<table>
<thead>
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<th>( x )</th>
<th>Estimate of ( y(x) )</th>
<th>Slope at ( x )</th>
</tr>
</thead>
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<td>-101</td>
</tr>
<tr>
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<td>29.8</td>
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</tr>
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<td>-11.485</td>
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<tr>
<td>12</td>
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<td></td>
</tr>
</tbody>
</table>

### Figure 11.21

11.3 Concept Inventory

- Euler’s method
- Step size versus number of steps
- Graph of Euler estimates

Activities Key

- 1–10
- 11
- 12–15

See page xvi of the preface for icon explanations.

11.3 Activities

1. Use Euler’s method and two steps to estimate the following values.
   - a. Given \( \frac{dy}{dx} = \frac{1}{2} \) with the initial condition \((0, 0)\), estimate \( y \) when \( x = 4 \).
   - b. Given \( \frac{dy}{dx} = 2x \) with the initial condition \((1, 4)\), estimate \( y \) when \( x = 7 \).

2. Are the answers to parts a and b of Activity 1 approximate or exact? In each case, explain why.

3. Use Euler’s method and two steps to estimate the following values.
   - a. Given \( \frac{dy}{dx} = \frac{5}{y} \) with initial condition \((1, 1)\), estimate \( y \) when \( x = 5 \).
   - b. Given \( \frac{dy}{dx} = \frac{5}{x} \) with initial condition \((2, 2)\), estimate \( y \) when \( x = 8 \).

4. Consider the differential equations in parts a and b of Activity 3.
   - a. Write the particular solution for the given initial condition.
7. **Production** It is estimated that for the first 10 years of production, a certain oil well can be expected to produce oil at a rate of
\[ r(t) = 3.933t^{1.35135} \]
\[ \text{thousand barrels per year} \]
t years after production begins.

a. Write a differential equation for the rate of change of the total amount of oil produced \( t \) years after production begins.

b. Use Euler’s method with 10 intervals to estimate the yield from this oil well during the first 5 years of production.

c. Graph the differential equation and the Euler estimates. Discuss how the shape of the graph of the differential equation is related to the shape of the graph of the Euler estimates.

8. **Labor** The personnel manager for a large construction company keeps records of the worker hours per week spent on typical construction jobs handled by the company. The manager has developed the following model for a worker hours curve:
\[ r(x) = \frac{6,608,830e^{-0.705989x}}{(1 + 925.466e^{-0.705989x})^2} \]
worker hours per week
the \( x \)th week of the construction job.

a. Use this model to write a differential equation giving the rate of change of the total number of worker hours used by the end of the \( x \)th week.

b. Graph this differential equation, and discuss any critical points and trends that the differential equation suggests will occur.

c. Use Euler’s method with 20 intervals to estimate the total number of worker hours used by the end of the twentieth week.

d. Graph the Euler estimates, and discuss whether you believe the estimate is good. Refer to the points you discussed in part b. How could you improve the accuracy of your estimate?

9. **Cooling** Newton’s Law of Cooling says that the rate of change (with respect to time \( t \)) of the temperature \( T \) of an object is proportional to the difference between the temperature of the object and the temperature \( A \) of the object’s surroundings.

a. Write a differential equation describing this law.
11.4 Second-Order Differential Equations

b. Consider a room that has a constant temperature of $A = 70^\circ F$. An object is placed in that room and allowed to cool. When the object is first placed in the room, the temperature of the object is $98^\circ F$, and it is cooling at a rate of $1.8^\circ F$ per minute. Determine the constant of proportionality for the differential equation.

c. Use Euler’s method and 15 steps to estimate the temperature of the object after 15 minutes.

10. Learning  A person learns a new task at a rate that is equal to the percentage of the task not yet learned. Let $p$ represent the percentage of the task already learned at time $t$ (in hours).

a. Write a differential equation describing the rate of change in the percentage of the task learned at time $t$.

b. Use Euler’s method with eight steps of size 0.25 to estimate the percentage of the task that is learned in 2 hours.

c. Graph the Euler estimates, and discuss any critical points or trends.

11. Explain why you would expect Euler’s method to have better accuracy when more steps of smaller size are used. Illustrate with graphs.

12. Postage  Refer to the information given in Activity 6 about the rate of change of the number of countries issuing postage stamps.

a. Use Euler’s method with 40 steps and with 80 steps to estimate the number of countries issuing postage stamps in 1840.

b. Graph the Euler estimates in part a.

c. Discuss your expectations for the accuracy of your answers.

13. Production  Refer to the information given in Activity 7 about the rate of production of oil from an oil well.

a. Use Euler’s method with intervals of one month, then with intervals of one week, and finally with daily intervals to estimate the yield from the well during the first 5 years of production.

b. Graph the Euler estimates in part a.

c. Discuss your expectations for the accuracy of your answers.

14. Labor  Refer to the information given in Activity 8 about the manpower on a construction job.

a. Use Euler’s method with 140 intervals to estimate the total number of worker hours used by the end of the twentieth week.

b. Graph the Euler estimates in part a.

c. Discuss your expectations for the accuracy of your answers. Does an estimate using 140 intervals have any advantage over an estimate using 20 intervals? Explain.

15. Cooling  Refer to the information given in Activity 9 about Newton’s Law of Cooling. Consider especially the specification given in part b of Activity 9.

a. Use Euler’s method with intervals of length one second to estimate the temperature of the object after 15 minutes.

b. Graph the Euler estimates in part a.

c. How does this estimate compare with that found in part c of Activity 9? Discuss the accuracy of this method.

Instructor Note
You may wish to point out to your students that this acceleration example is essentially the same as Example 5 in Section 6.4.

Not all differential equations involve only the first derivative of a function. Some tell us about the second derivative. That is, they tell us about the rate at which the rate of change is changing.
Because acceleration is the derivative of velocity $v$ with respect to time $t$ and velocity is the derivative of distance $s$ with respect to time, we can write acceleration due to gravity near the surface of the Earth as the differential equation

$$\frac{d^2s}{dt^2} = -32 \text{ feet per second squared}$$

after $t$ seconds. This differential equation is of the form $s''(t) = k$ for some constant $k$. (We saw these equations in Chapter 6 when working with antiderivatives.) This is a second derivative of a function, so a general solution can be found by finding the general antiderivative of $\frac{d^2s}{dt^2}$ twice. The general antiderivative of the differential equation for acceleration is

$$\frac{ds}{dt} = -32t + C \text{ feet per second}$$

Thus the general solution to the differential equation is

$$s(t) = -16t^2 + Ct + D \text{ feet}$$

after $t$ seconds.

To confirm that this equation is a solution to the differential equation, we substitute for $s$ in the differential equation.

$$\frac{d^2}{dt^2} (-16t^2 + Ct + D) = -32$$

Differentiating the left side of the equation twice yields the identity $-32 = -32$.

In the case where the object fell from a height of 40 feet ($s = 40$ and $v = 0$ at $t = 0$), we have the particular solution $s(t) = -16t^2 + 40$ feet after $t$ seconds.

**EXAMPLE 1**  
Solving a Second-Order Linear Differential Equation

**Purchasing Power**  
The rate of change in the purchasing power* of the U.S. dollar was changing at a linear rate between 1988 and 2000. A differential equation describing the rate at which the rate of change of the purchasing power was decreasing is

$$\frac{d^2p}{dx^2} = -0.00156x + 0.0115 \text{ dollars per year squared}$$

where $x$ is the number of years since 1988. The purchasing power is based on 1982 dollars, making a 1982 dollar worth $0.93$ in 1988 and $0.81$ in 1992.

a. Write a general solution for the purchasing power of a dollar.

b. Write a particular solution for the purchasing power of a dollar.

c. Use the particular solution to determine how much a 1982 dollar was worth in 2000.

Solution

a. By finding the antiderivative of the differential equation twice, we obtain the general solution

\[ P(x) = -0.00026x^3 + 0.00575x^2 + Cx + D \text{ dollars} \]

where \( x \) is the number of years since 1988.

b. The conditions given are \( P(4) = 0.81 \) and \( P(0) = 0.93 \). The condition \( P(0) = 0.93 \) gives us the value for \( D \) as 0.93. Using \( D = 0.93 \) and \( P(4) = 0.81 \), we solve for \( C \) and find that \( C \approx -0.0488 \). Thus we have the particular solution

\[ P(x) = -0.00026x^3 + 0.00575x^2 - 0.0488x + 0.93 \text{ dollars} \]

where \( x \) is the number of years since 1988.

c. By 2000, a 1982 dollar was worth only \( P(12) \approx 0.72 \) dollar.

Let us now consider situations in which the second derivative of an amount is proportional to the amount function; that is,

\[ \frac{d^2y}{dx^2} = ky \]

First we consider the case where the constant of proportionality is negative:

\[ \frac{d^2y}{dx^2} = -ky \quad \text{for } k > 0 \]

Consider the differential equation \( \frac{d^2y}{dx^2} = -y \). One function we have looked at that satisfies this equation is the function \( y(x) = \sin x \). We verify this by differentiating the function twice:

\[ y'(x) = \cos x \]
\[ y''(x) = -\sin x = -y(x) \]

The cosine function is another function that satisfies \( \frac{d^2y}{dx^2} = -y \).

For the more general case \( \frac{d^2y}{dx^2} = ky \), where \( k > 0 \), it can be shown that the general solution is

\[ y(x) = a \sin(\sqrt{k}x + c) \]

where \( a \) and \( c \) are constants. You will be asked to verify this in the activities.

EXAMPLE 2 Solving a Second-Order Differential Equation

Fishing An exclusive fishing club on the Restigouche River in Canada kept detailed records regarding the number of fish caught by its members.* Between 1880 and 1905, the rate of change (with respect to the year) in the number of fish per rod per day was changing at a rate proportional to the number of fish per rod per day. The constant of proportionality is approximately \( -0.455625 \).

The average number of fish per rod per day for the years between 1880 and 1905 was 1.267 fish per rod per day. In 1881, the catch was 0.9425 fish per rod per day, and in 1885, 1.5776 fish per rod per day were caught.

a. Write a differential equation expressing the rate at which the rate of change in the number of fish per rod per day was changing with respect to the number of years since 1880.

b. Write a particular solution to the differential equation.

c. Write a model for the average number of fish per rod per day given the number of years since 1880.

**Solution**

a. A differential equation expressing the rate at which the rate of change in \( g \), the number of fish per rod per day, was changing is

\[
\frac{d^2g}{dy^2} = -0.455625 \text{ fish per rod per day per year squared}
\]

where \( y \) is the number of years since 1880.

b. As previously stated, the general solution is \( g(y) = a \sin(\sqrt{k}y + c) \) where \( k = 0.455625 \). Thus the general solution to this differential equation is

\[
g(y) = a \sin(0.675y + c) \text{ fish per rod per day}
\]

where \( y \) is the number of years since 1880, and \( a \) and \( c \) are constants. In order to determine the constants \( a \) and \( c \) for this equation, we must have two conditions. First, we note that the general solution does not include the parameter \( d \) giving the expected number of fish per rod per day that is normally included in a general sine model \( f(x) = a \sin(bx + c) + d \). Thus we must write the conditions in terms of the difference from the expected value. That is, in 1881, the catch was 0.3245 fewer fish per rod per day than in an average year, and in 1885, the catch was 0.3106 more fish per rod per day than in an average year. Thus the two conditions are \( g(1) = -0.3245 \) and \( g(5) = 0.3106 \) resulting in the equations

\[
-0.3245 = a \sin[0.675(1) + c] \quad (5)
\]

\[
0.3106 = a \sin[0.675(5) + c] \quad (6)
\]

Solving for \( a \) in equation 5 and substituting in equation 6 yields

\[
0.3106 = \frac{-0.3245 \sin(3.375 + c)}{\sin(0.675 + c)}
\]

which gives \( c \approx 1.019 \). Substituting this value of \( c \) into either equation 5 or equation 6 and solving for \( a \) yields \( a \approx -0.327 \). Using these constants \( a \) and \( c \), we have the particular solution for the differential equation as

\[
g(y) = -0.327 \sin(0.675y + 1.019) \text{ fish per rod per day}
\]

where \( y \) is the number of years since 1880.

c. The general and particular solutions for the differential equation are both in terms of the difference from the expected value of fish per rod per day. To write
the general sine model for this situation, we add the parameter $d$, the expected value of fish per day:

$$g(y) = -0.327 \sin(0.675y + 1.019) + 1.267 \text{ fish per rod per day}$$

where $y$ is the number of years since 1880.

Before we leave this section, let us consider the differential equation of the form

$$\frac{d^2y}{dx^2} = ky$$

where $k > 0$.

In this case, the constant of proportionality is positive (rather than negative as in Example 2) and the general solution is

$$y(x) = ae^{\sqrt{k}x} + be^{-\sqrt{k}x}$$

where $a$ and $b$ are constants. Note that the general solution to this differential equation appears to be very different from the general solution in the negative case. The negative case leads to a model with which we are familiar; the positive case does not.

11.4 Concept Inventory

- Second-order differential equations
- Solutions to differential equations of the form $\frac{d^2y}{dx^2} = k$
- Solutions to differential equations of the form $\frac{d^2y}{dx^2} = f(x)$
- Solutions to differential equations of the form $\frac{d^2y}{dx^2} = ky$

Activities Key

5–11

See page xvi of the preface for icon explanations.

11.4 Activities

For Activities 1 through 4, write a differential equation expressing the information given and, when possible, find a general solution for the differential equation.

1. The Rowan-Robinson model of the universe assumes that the universe is expanding with respect to time $t$ at a rate that is decreasing in inverse proportion to the square of its current size $S$.

2. The rate of change with respect to time $t$ of the demand $D$ for a product is decreasing proportionately to the demand at time $t$.

3. The rate of change in year $y$ of the population $P$ of the United States is increasing with respect to the year at a constant rate.

4. The rate of growth of the height $h$ of a young child with respect to the age $y$ of the child decreases in inverse proportion to the age of the child.

5. Jobs The rate of change in the number of jobs for a Michigan roofing company is increasing by approximately $6.14$ jobs per month squared. The number of jobs in January is decreasing at the rate of $0.87$ job per month, and company records indicate that the company had $14$ roofing jobs in February.
   a. Write a differential equation for the rate at which the rate of change in the number of roofing jobs for this company is changing.
   b. Find a particular solution to the differential equation in part a.
   c. Use the result of part b to estimate the number of roofing jobs in August and the number in November.

6. Marriage Between 1950 and 2000, the rate of change in the rate at which the median age of first marriage of females in the United States was changing was constant at $0.0042$ year of age per year
squared. The median age of first marriage for these females was increasing at the rate of 0.1713 year of age per year in 1991, and females were first married at a median age of 25.1 in 2000.

a. Write a differential equation for the rate at which the rate of the median age of first marriage for United States females is changing.
b. Find a particular solution to the differential equation in part a.
c. Use the result of part b to estimate the median age of first marriage of United States females in the current year.

7. AIDS Records of the number of AIDS cases diagnosed in the United States between 1988 and 1991 indicate that the rate at which the rate of change in the number of cases was changing was constant at -2099 cases per year squared. The number of AIDS cases diagnosed in 1988 was 33,590, and the number of cases was increasing at the rate of 5988.7 cases per year in 1988.
(Source: Based on data appearing in the HIV/AIDS Surveillance 1992 Year End Edition.)

a. Write a differential equation for the rate of change of the rate of change of the number of AIDS cases diagnosed in year t, where t is the number of years after 1988.
b. Find both a general and a particular solution to the differential equation.
c. Express your answer from part b to estimate how rapidly the number of AIDS cases was changing in the current year and the number of AIDS cases that was diagnosed in that year.

8. Postage Between 1919 and 1995, the rate of change of the rate of change of the postage required to mail a first-class 1-ounce letter was approximately 0.022 cent per year squared. The postage was 2 cents in 1919, and it was increasing at the rate of approximately 0.393 cent per year in 1958.
(Source: Based on data from the United States Postal Service.)

a. Write a differential equation for the rate of change of the rate of change of the first-class postage for a 1-ounce letter in year t, where t is the number of years after 1900.
b. Find both a general and a particular solution to the differential equation in part a.
c. Use your previous results to estimate the rate at which the number of AIDS cases is changing in the current year and the rate at which the number of AIDS cases that is diagnosed in that year.

9. Consider a function y = f(x) whose rate of change with respect to x is changing at a constant rate.
a. Write a differential equation describing the rate of change in the rate of change for this function.
b. Write a general solution for the differential equation.
c. Verify that the general solution you gave in part b is indeed a solution by substituting it into the differential equation and simplifying to obtain an identity.

10. Motion Laws When a spring is stretched and then released, it oscillates according to two laws of physics: Hooke's Law and Newton's Second Law. These two laws combine to form the following differential equation in the case of free, undamped oscillation:

\[ m \frac{d^2x}{dt^2} + kx = 0 \]

where m is the mass of an object attached to the spring, x is the distance the spring is stretched beyond its standard length with the object attached (its equilibrium point), t is time, and k is a constant associated with the strength of the spring. Consider a spring with k = 15 from which is hung a 30-pound weight. The spring with the weight attached stretches to its equilibrium point. The spring is then pulled 2 feet farther than its equilibrium and released.

a. Write a differential equation describing the acceleration of the spring with respect to time t measured in seconds. Use the fact that mass = \( \frac{\text{weight}}{g} \) where g is the gravitational constant 32 feet per second per second.
b. Write a particular solution for this differential equation. Use the fact that when the spring is first released, its velocity is 0.
c. Graph this solution over several periods, and explain how to interpret the graph.

d. How quickly is the mass moving when it passes its equilibrium point?

11. Radiation  The rate of change in the rate at which the average amount of extraterrestrial radiation in Amarillo, Texas, for each month of the year is changing is proportional to the amount of extraterrestrial radiation received. The constant of proportionality is $k = -0.212531$. In any given month, the expected value of radiation is 12.5 mm per day. This expected value is actually obtained in March and September.

(Source: Based on data from A. A. Hanson, ed., Practical Handbook of Agricultural Science. Boca Raton, FL: CRC Press, 1990.)

a. Write a differential equation for the information given.

b. In June, the amount of radiation received is approximately 17.0 mm per day, and in December, the amount of radiation received is approximately 7.8 mm per day. Write a particular solution for this differential equation.

c. Change the particular solution into a function giving the average amount of extraterrestrial radiation in Amarillo.

d. How well does your model estimate the amounts of extraterrestrial radiation in March and September?

12. Consider a function $y = f(x)$ whose rate of change with respect to $x$ is changing in proportion to $f$ where the constant of proportionality is negative.

a. Write a differential equation describing the rate of change in the rate of change for this function.

b. Write a general solution for the differential equation.

c. Verify that the general solution you gave in part b is indeed a solution by substituting it into the differential equation and obtaining an identity.

SUMMARY

In many situations (physical, sociological, psychological, and economic), there are certain laws that govern the dynamics of change. In these cases, differential equations often can be used to describe the change that occurs.

Differential Equations

Differential equations are equations involving derivatives. A solution for a differential equation is a function that, when substituted into the differential equation, results in an identity. A general solution contains arbitrary constants that can be determined if initial conditions are known. When the arbitrary constants are replaced by known constants, the solution is called a particular solution.

Some differential equations are of the simple form

$$\frac{dy}{dx} = f(x)$$

In this case, a general solution can be obtained by writing a general antiderivative for the given rate of change if an antiderivative is known.

Slope Fields

It is often helpful to view a graphical representation of a differential equation. This can be done in the form of a slope field. A slope field consists of line segments that represent the slopes given by the differential equation at different points on a grid placed on the coordinate plane.
**Separable Differential Equations**

Some differential equations are given as functions not only of the input variable but also of the output variable, so they are of the form

\[ \frac{dy}{dx} = f(x, y) \]

In this case, a general solution might be obtained by using the method of separation of variables.

**Euler’s Method**

Often, neither simple antidifferentiation nor separation of variables is sufficient for determining an explicit formula for a general solution of a differential equation. In these cases, we can numerically estimate a solution using Euler’s method. Euler’s method not only gives us a numerical method for estimating solutions but also, when the Euler estimates are graphed, can help us visualize how the solution function behaves.

**Second-Order Differential Equations**

Second-order differential equations involve the second derivative of a function. The simple cases we considered were the second-order differential equations that give rise to the quadratic, cubic, and sine models.

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**Instructor Note**

The Concept Check provides students a quick way of checking on which topics they need to spend more time. The activities listed here are generally simple ones, and the students should be warned that being able to do these activities does not guarantee that they fully understand all the concepts—especially not if they only read through the activity and look at the answer in the back of the book without actually working through the solution themselves.

---

**CONCEPT CHECK**

**Can you**

- Solve constant and linear differential equations?
- Set up an equation involving direct or inverse proportionality?
- Sketch a particular solution to a differential equation on a slope field?
- Solve separable differential equations?
- Set up an equation involving joint proportionality?
- Estimate a solution to a differential equation using Euler’s method?
- Set up and solve second-order differential equations using antiderivatives?

**To practice, try**

- Section 11.1 Activities 15, 19
- Section 11.1 Activities 19, 23
- Section 11.1 Activities 7, 9
- Section 11.2 Activities 9, 13, 15
- Section 11.2 Activity 25
- Section 11.2 Activity 29
- Section 11.3 Activity 7
- Section 11.4 Activities 7, 11
1. **Accident Risk**  The relationship between the relative risk \( R \) (expressed as a percent) of having a car accident and the blood alcohol level of 100\( b \)% of the driver is

\[
\frac{dR}{db} = kR
\]

a. Write a statement interpreting this differential equation.
b. Write a general solution for the differential equation.
c. The risk of an accident is 1% when there is no alcohol in the blood and 20% when the blood alcohol level is 14%. Write a particular solution for the differential equation.
d. At what blood alcohol level is a crash certain to occur?
e. Use the slope field to sketch the particular solution found in part c.

2. **Famine**  In Ireland in the 1840s, potato famines occurred. These famines are correlated with a drastic change in the dynamics of population growth and decline. The Verhulst model for population growth says that the rate of change in a population is jointly proportional to the current population and to the capacity remaining in the system.

\[
\frac{dR}{db} = kR
\]

a. From population data for 1780 through 1840, the carrying capacity of Ireland’s population appears to have been 16.396 million people. The constant of proportionality for growth during this time was 0.001175. Write a differential equation expressing the rate of change of Ireland’s population. Let \( x \) represent the number of years since 1800.
b. Write a general solution for the differential equation in part a.
c. Write a particular solution for the differential equation in part a. In 1780, there were 4.0 million people in Ireland.
d. Use the particular solution from part c to estimate the Irish population in 1840 and 1850.

3. **Famine**  Consider the differential equation from part a of Question 2 and the initial condition that in 1780 there were 4.0 million people in Ireland.

a. Use Euler’s method and 10-year steps to estimate the population of Ireland in 1840 and 1850.
b. Use Euler’s method and 5-year steps to estimate the population of Ireland in 1840 and 1850.
c. How do your answers to parts a and b compare to those from part d of Question 2?

4. **Population**  Population data for 1850 through 1920 show that Ireland’s population was decreasing according to the Verhulst model with a constant of proportionality of 0.008307. Ireland reached a population of 7.154 million people in 1910 and 1920. To use the Verhulst model for population change, we assume that the carrying capacity of Ireland was 4.4 million people above its 1910–1920 level.

a. Write a differential equation expressing the rate of change of Ireland’s population between 1850 and 1920. Let \( x \) represent the number of years since 1800.
b. Using the fact that in 1900, there were 4.5 million people in Ireland (that is 0.1 million people above its 1910–1920 level), write a particular solution to the differential equation from part a.

c. Adjust the model for 1850 through 1920 by adding the base population level of 4.4 million people, and write a piecewise model for the population of Ireland from 1780 through 1920.

d. Use the piecewise model from part c to estimate the population of Ireland in 1850. How does this differ from the estimate you found in part d of Question 2?