Section 3.3 Vectors in the Plane

Objective: In this lesson you learned how to write the component forms of vectors, perform basic vector operations, and find the direction angles of vectors.

I. Introduction (Page 291)

A directed line segment has an initial point and a terminal point. The magnitude of the directed line segment $\overrightarrow{PQ}$, denoted by $||\overrightarrow{PQ}||$, is its length. The magnitude of a directed line segment can be found by using the distance formula.

II. Component Form of a Vector (Page 292)

A vector whose initial point is at the origin $(0, 0)$ can be uniquely represented by the coordinates of its terminal point $(v_1, v_2)$. This is the component form of a vector $v$, written $v = \langle v_1, v_2 \rangle$, where $v_1$ and $v_2$ are the components of $v$.

The component form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = v.$$
The magnitude (or length) of \( \mathbf{v} \) is:

\[
|| \mathbf{v} || = \sqrt{(q_1-p_1)^2 + (q_2-p_2)^2} = \sqrt{v_1^2 + v_2^2}
\]

**Example 1:** Find the component form and magnitude of the vector \( \mathbf{v} \) that has \((1, 7)\) as its initial point and \((4, 3)\) as its terminal point.

\[
\mathbf{v} = \langle 3, -4 \rangle; || \mathbf{v} ||= 5
\]

**III. Vector Operations (Pages 293–295)**

In operations with vectors, numbers are usually referred to as **scalars**. Geometrically, the product of a vector \( \mathbf{v} \) and a scalar \( k \) is . . . the vector that is \(|k|\) times as long as the vector \( \mathbf{v} \).

If \( k \) is positive, \( k \mathbf{v} \) has the **same** direction as \( \mathbf{v} \), and if \( k \) is negative, \( k \mathbf{v} \) has the **opposite** direction.

To add two vectors geometrically, . . . position them (without changing length or direction) so that the initial point of one coincides with the terminal point of the other.

This technique is called the **parallelogram law** for vector addition because the vector \( \mathbf{u} + \mathbf{v} \), often called the **resultant** of vector addition, is . . . the diagonal of a parallelogram having \( \mathbf{u} \) and \( \mathbf{v} \) as its adjacent sides.

Let \( \mathbf{u} = \langle u_1, u_2 \rangle \) and \( \mathbf{v} = \langle v_1, v_2 \rangle \) be vectors and let \( k \) be a scalar (a real number). Then the sum of \( \mathbf{u} \) and \( \mathbf{v} \) is the vector:

\[
\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle
\]

and the scalar multiple of \( k \) times \( \mathbf{u} \) is the vector:

\[
k \mathbf{u} = k \langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle
\]

**Example 2:** Let \( \mathbf{u} = \langle 1, 6 \rangle \) and \( \mathbf{v} = \langle -4, 2 \rangle \). Find:

(a) \( 3 \mathbf{u} \)
(b) \( \mathbf{u} + \mathbf{v} \)

(a) \( \langle 3, 18 \rangle \)
(b) \( \langle -3, 8 \rangle \)
Let \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) be vectors and \( c \) and \( d \) be scalars. Complete the following properties of vector addition and scalar multiplication:

1. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)
2. \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \)
3. \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)
4. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \)
5. \( c(d\mathbf{u}) = (cd)\mathbf{u} \)
6. \( (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} \)
7. \( c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \)
8. \( 1(\mathbf{u}) = \mathbf{u} \)
9. \( 0(\mathbf{u}) = \mathbf{0} \)
10. \( ||c\mathbf{v}|| = |c| ||\mathbf{v}|| \)

IV. Unit Vectors (Pages 295–296)

To find a unit vector \( \mathbf{u} \) that has the same direction as a given nonzero vector \( \mathbf{v} \), . . . divide \( \mathbf{v} \) by its magnitude, that is:

\[
\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}.
\]

In this case, the vector \( \mathbf{u} \) is called a ______ unit vector in the direction of \( \mathbf{v} \).__________

Example 3: Find a unit vector in the direction of \( \mathbf{v} = \langle -8, 6 \rangle \).

\[
\frac{\mathbf{v}}{||\mathbf{v}||} = \langle -8/10, 6/10 \rangle = \langle -0.8, 0.6 \rangle
\]

Let \( \mathbf{v} = \langle v_1, v_2 \rangle \). Then the standard unit vectors can be used to represent \( \mathbf{v} \) as \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} \), where the scalar \( v_1 \) is called the ______ horizontal component of \( \mathbf{v} \) ______ and the scalar \( v_2 \) is called the ______ vertical component of \( \mathbf{v} \) ______. The vector sum \( v_1 \mathbf{i} + v_2 \mathbf{j} \) is called a ______ linear combination ______ of the vectors \( \mathbf{i} \) and \( \mathbf{j} \).

Example 4: Let \( \mathbf{v} = \langle -5, 3 \rangle \). Write \( \mathbf{v} \) as a linear combination of the standard unit vectors \( \mathbf{i} \) and \( \mathbf{j} \).

\[
\mathbf{v} = -5\mathbf{i} + 3\mathbf{j}
\]

Example 5: Let \( \mathbf{v} = 3\mathbf{i} - 4\mathbf{j} \) and \( \mathbf{w} = 2\mathbf{i} + 9\mathbf{j} \). Find \( \mathbf{v} + \mathbf{w} \).

\[
\mathbf{v} + \mathbf{w} = 5\mathbf{i} + 5\mathbf{j}
\]
V. Direction Angles (Page 297)

If \( \mathbf{u} \) is a unit vector and \( \theta \) is its direction angle, the terminal point of \( \mathbf{u} \) lies on the unit circle and

\[
\mathbf{u} = \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}
\]

Now, if \( \mathbf{v} \) is any vector that makes an angle \( \theta \) with the positive \( x \)-axis, it has the same direction as \( \mathbf{u} \) and

\[
\mathbf{v} = \|\mathbf{v}\| \langle \cos \theta, \sin \theta \rangle = \|\mathbf{v}\| (\cos \theta)\mathbf{i} + \|\mathbf{v}\| (\sin \theta)\mathbf{j}
\]

If \( \mathbf{v} \) can be written as \( \mathbf{v} = a\mathbf{i} + b\mathbf{j} \), then the direction angle \( \theta \) for \( \mathbf{v} \) can be determined from \( \tan \theta = \frac{b}{a} \).

Example 6: Let \( \mathbf{v} = -4\mathbf{i} + 5\mathbf{j} \). Find the direction angle for \( \mathbf{v} \).

128.66°

VI. Applications of Vectors (Pages 298–299)

Describe several real-life applications of vectors.

Answers will vary.