

A.7 Errors and the Algebra of Calculus

What you should learn

- Avoid common algebraic errors.
- Recognize and use algebraic techniques that are common in calculus.

Why you should learn it

An efficient command of algebra is critical in mastering this course and in the study of calculus.

Algebraic Errors to Avoid

This section contains five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving dividing out. Many of these errors are made because they seem to be the *easiest* things to do. For instance, the operations of subtraction and division are often believed to be commutative and associative. The following examples illustrate the fact that subtraction and division are neither commutative nor associative.

Not commutative

$$4 - 3 \neq 3 - 4$$

$$15 \div 5 \neq 5 \div 15$$

Not associative

$$8 - (6 - 2) \neq (8 - 6) - 2$$

$$20 \div (4 \div 2) \neq (20 \div 4) \div 2$$

Errors Involving Parentheses

Potential Error	Correct Form	Comment
$a - (x - b) = a - x - b$	$a - (x - b) = a - x + b$	Change all signs when distributing minus sign.
$(a + b)^2 = a^2 + b^2$	$(a + b)^2 = a^2 + 2ab + b^2$	Remember the middle term when squaring binomials.
$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{2}(ab)$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{4}(ab) = \frac{ab}{4}$	$\frac{1}{2}$ occurs twice as a factor.
$(3x + 6)^2 = 3(x + 2)^2$	$(3x + 6)^2 = [3(x + 2)]^2 = 3^2(x + 2)^2$	When factoring, apply exponents to all factors.

Errors Involving Fractions

Potential Error	Correct Form	Comment
$\frac{a}{x + b} = \frac{a}{x} + \frac{a}{b}$	Leave as $\frac{a}{x + b}$.	Do not add denominators when adding fractions.
$\frac{\left(\frac{x}{a}\right)}{b} = \frac{bx}{a}$	$\frac{\left(\frac{x}{a}\right)}{b} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$	Multiply by the reciprocal when dividing fractions.
$\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b}$	$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$	Use the property for adding fractions.
$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	Use the property for multiplying fractions.
$(1/3)x = \frac{1}{3x}$	$(1/3)x = \frac{1}{3} \cdot x = \frac{x}{3}$	Be careful when using a slash to denote division.
$(1/x) + 2 = \frac{1}{x + 2}$	$(1/x) + 2 = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$	Be careful when using a slash to denote division and be sure to find a common denominator before you add fractions.

Errors Involving Exponents

Potential Error	Correct Form	Comment
$(x^2)^3 = x^5$	$(x^2)^3 = x^{2 \cdot 3} = x^6$	Multiply exponents when raising a power to a power.
$x^2 \cdot x^3 = x^6$	$x^2 \cdot x^3 = x^{2+3} = x^5$	Add exponents when multiplying powers with like bases.
$2x^3 = (2x)^3$	$2x^3 = 2(x^3)$	Exponents have priority over coefficients.
$\frac{1}{x^2 \cdot x^3} = x^{-2} - x^{-3}$	Leave as $\frac{1}{x^2 \cdot x^3}$.	Do not move term-by-term from denominator to numerator.

Errors Involving Radicals

Potential Error	Correct Form	Comment
$\sqrt{5x} = 5\sqrt{x}$	$\sqrt{5x} = \sqrt{5}\sqrt{x}$	Radicals apply to every factor inside the radical.
$\sqrt{x^2 + a^2} = x + a$	Leave as $\sqrt{x^2 + a^2}$.	Do not apply radicals term-by-term.
$\sqrt{-x + a} = -\sqrt{x - a}$	Leave as $\sqrt{-x + a}$.	Do not factor minus signs out of square roots.

Errors Involving Dividing Out

Potential Error	Correct Form	Comment
$\frac{a + bx}{a} = 1 + bx$	$\frac{a + bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{b}{a}x$	Divide out common factors, not common terms.
$\frac{a + ax}{a} = a + x$	$\frac{a + ax}{a} = \frac{a(1 + x)}{a} = 1 + x$	Factor before dividing out.
$1 + \frac{x}{2x} = 1 + \frac{1}{x}$	$1 + \frac{x}{2x} = 1 + \frac{1}{2} = \frac{3}{2}$	Divide out common factors.

A good way to avoid errors is to *work slowly*, *write neatly*, and *talk to yourself*. Each time you write a step, ask yourself why the step is algebraically legitimate. You can justify the step below because *dividing the numerator and denominator by the same nonzero number produces an equivalent fraction*.

$$\frac{2x}{6} = \frac{2 \cdot x}{2 \cdot 3} = \frac{x}{3}$$

Example 1 Using the Property for Adding Fractions

Describe and correct the error. ~~$\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}$~~

Solution

When adding fractions, use the property for adding fractions: $\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$.

$$\frac{1}{2x} + \frac{1}{3x} = \frac{3x + 2x}{6x^2} = \frac{5x}{6x^2} = \frac{5}{6x}$$

 **CHECKPOINT** Now try Exercise 17.

Some Algebra of Calculus

In calculus it is often necessary to take a simplified algebraic expression and “unsimplify” it. See the following lists, taken from a standard calculus text.

Unusual Factoring

<i>Expression</i>	<i>Useful Calculus Form</i>	<i>Comment</i>
$\frac{5x^4}{8}$	$\frac{5}{8}x^4$	Write with fractional coefficient.
$\frac{x^2 + 3x}{-6}$	$-\frac{1}{6}(x^2 + 3x)$	Write with fractional coefficient.
$2x^2 - x - 3$	$2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right)$	Factor out the leading coefficient.
$\frac{x}{2}(x + 1)^{-1/2} + (x + 1)^{1/2}$	$\frac{(x + 1)^{-1/2}}{2}[x + 2(x + 1)]$	Factor out factor with lowest power.

Writing with Negative Exponents

<i>Expression</i>	<i>Useful Calculus Form</i>	<i>Comment</i>
$\frac{9}{5x^3}$	$\frac{9}{5}x^{-3}$	Move the factor to the numerator and change the sign of the exponent.
$\frac{7}{\sqrt{2x - 3}}$	$7(2x - 3)^{-1/2}$	Move the factor to the numerator and change the sign of the exponent.

Writing a Fraction as a Sum

<i>Expression</i>	<i>Useful Calculus Form</i>	<i>Comment</i>
$\frac{x + 2x^2 + 1}{\sqrt{x}}$	$x^{1/2} + 2x^{3/2} + x^{-1/2}$	Divide each term by $x^{1/2}$.
$\frac{1 + x}{x^2 + 1}$	$\frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$	Rewrite the fraction as the sum of fractions.
$\frac{2x}{x^2 + 2x + 1}$	$\frac{2x + 2 - 2}{x^2 + 2x + 1}$ $= \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$	Add and subtract the same term. Rewrite the fraction as the difference of fractions.
$\frac{x^2 - 2}{x + 1}$	$x - 1 - \frac{1}{x + 1}$	Use long division. (See Section 2.3.)
$\frac{x + 7}{x^2 - x - 6}$	$\frac{2}{x - 3} - \frac{1}{x + 2}$	Use the method of partial fractions. (See Section 7.4.)

Inserting Factors and Terms

Expression	Useful Calculus Form	Comment
$(2x - 1)^3$	$\frac{1}{2}(2x - 1)^3(2)$	Multiply and divide by 2.
$7x^2(4x^3 - 5)^{1/2}$	$\frac{7}{12}(4x^3 - 5)^{1/2}(12x^2)$	Multiply and divide by 12.
$\frac{4x^2}{9} - 4y^2 = 1$	$\frac{x^2}{9/4} - \frac{y^2}{1/4} = 1$	Write with fractional denominators.
$\frac{x}{x + 1}$	$\frac{x + 1 - 1}{x + 1} = 1 - \frac{1}{x + 1}$	Add and subtract the same term.

The next five examples demonstrate many of the steps in the preceding lists.

Example 2 Factors Involving Negative Exponents



Factor $x(x + 1)^{-1/2} + (x + 1)^{1/2}$.

Solution

When multiplying factors with like bases, you add exponents. When factoring, you are undoing multiplication, and so you *subtract* exponents.

$$\begin{aligned} x(x + 1)^{-1/2} + (x + 1)^{1/2} &= (x + 1)^{-1/2}[x(x + 1)^0 + (x + 1)^1] \\ &= (x + 1)^{-1/2}[x + (x + 1)] \\ &= (x + 1)^{-1/2}(2x + 1) \end{aligned}$$

 **CHECKPOINT** Now try Exercise 23.

Another way to simplify the expression in Example 2 is to multiply the expression by a fractional form of 1 and then use the Distributive Property.

$$\begin{aligned} x(x + 1)^{-1/2} + (x + 1)^{1/2} &= [x(x + 1)^{-1/2} + (x + 1)^{1/2}] \cdot \frac{(x + 1)^{1/2}}{(x + 1)^{1/2}} \\ &= \frac{x(x + 1)^0 + (x + 1)^1}{(x + 1)^{1/2}} = \frac{2x + 1}{\sqrt{x + 1}} \end{aligned}$$

Example 3 Inserting Factors in an Expression



Insert the required factor: $\frac{x + 2}{(x^2 + 4x - 3)^2} = (\quad) \frac{1}{(x^2 + 4x - 3)^2} (2x + 4)$.

Solution

The expression on the right side of the equation is twice the expression on the left side. To make both sides equal, insert a factor of $\frac{1}{2}$.

$$\frac{x + 2}{(x^2 + 4x - 3)^2} = \left(\frac{1}{2}\right) \frac{1}{(x^2 + 4x - 3)^2} (2x + 4)$$

Right side is multiplied and divided by 2.

 **CHECKPOINT** Now try Exercise 25.

Example 4 Rewriting Fractions

Explain why the two expressions are equivalent.

$$\frac{4x^2}{9} - 4y^2 = \frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{1}{4}}$$

Solution

To write the expression on the left side of the equation in the form given on the right side, multiply the numerators and denominators of both terms by $\frac{1}{4}$.

$$\frac{4x^2}{9} - 4y^2 = \frac{4x^2}{9} \left(\frac{\frac{1}{4}}{\frac{1}{4}} \right) - 4y^2 \left(\frac{\frac{1}{4}}{\frac{1}{4}} \right) = \frac{x^2}{\frac{9}{4}} - \frac{y^2}{\frac{1}{4}}$$



CHECKPOINT Now try Exercise 29.

Example 5 Rewriting with Negative Exponents

Rewrite each expression using negative exponents.

a. $\frac{-4x}{(1 - 2x^2)^2}$ b. $\frac{2}{5x^3} - \frac{1}{\sqrt{x}} + \frac{3}{5(4x)^2}$

Solution

a. $\frac{-4x}{(1 - 2x^2)^2} = -4x(1 - 2x^2)^{-2}$

b. Begin by writing the second term in exponential form.

$$\begin{aligned} \frac{2}{5x^3} - \frac{1}{\sqrt{x}} + \frac{3}{5(4x)^2} &= \frac{2}{5x^3} - \frac{1}{x^{1/2}} + \frac{3}{5(4x)^2} \\ &= \frac{2}{5}x^{-3} - x^{-1/2} + \frac{3}{5}(4x)^{-2} \end{aligned}$$



CHECKPOINT Now try Exercise 39.

Example 6 Writing a Fraction as a Sum of Terms

Rewrite each fraction as the sum of three terms.

a. $\frac{x^2 - 4x + 8}{2x}$ b. $\frac{x + 2x^2 + 1}{\sqrt{x}}$

Solution

$$\begin{aligned} \text{a. } \frac{x^2 - 4x + 8}{2x} &= \frac{x^2}{2x} - \frac{4x}{2x} + \frac{8}{2x} \\ &= \frac{x}{2} - 2 + \frac{4}{x} \end{aligned} \quad \begin{aligned} \text{b. } \frac{x + 2x^2 + 1}{\sqrt{x}} &= \frac{x}{x^{1/2}} + \frac{2x^2}{x^{1/2}} + \frac{1}{x^{1/2}} \\ &= x^{1/2} + 2x^{3/2} + x^{-1/2} \end{aligned}$$



CHECKPOINT Now try Exercise 43.

A.7 Exercises

VOCABULARY CHECK: Fill in the blanks.

- To write the expression $\frac{2}{\sqrt{x}}$ with negative exponents, move \sqrt{x} to the _____ and change the sign of the exponent.
- When dividing fractions, multiply by the _____.

In Exercises 1–18, describe and correct the error.

- ~~$2x - (3y + 4) = 2x - 3y + 4$~~
- ~~$5z + 3(x - 2) = 5z + 3x - 2$~~
- ~~$\frac{4}{16x - (2x + 1)} = \frac{4}{14x + 1}$~~
- ~~$\frac{1 - x}{(5 - x)(-x)} = \frac{x - 1}{x(x - 5)}$~~
- ~~$(5z)(6z) = 30z$~~
- ~~$x(yz) = (xy)(xz)$~~
- ~~$a\left(\frac{x}{y}\right) = \frac{ax}{ay}$~~
- ~~$(4x)^2 = 4x^2$~~
- ~~$\sqrt{x + 9} = \sqrt{x} + 3$~~
- ~~$\sqrt{25 - x^2} = 5 - x$~~
- ~~$\frac{2x^2 + 1}{5x} = \frac{2x + 1}{5}$~~
- ~~$\frac{6x + y}{6x - y} = \frac{x + y}{x - y}$~~
- ~~$\frac{1}{a^{-1} + b^{-1}} = \left(\frac{1}{a + b}\right)^{-1}$~~
- ~~$\frac{1}{x + y^{-1}} = \frac{y}{x + 1}$~~
- ~~$(x^2 + 5x)^{1/2} = x(x + 5)^{1/2}$~~
- ~~$x(2x - 1)^2 = (2x^2 - x)^2$~~
- ~~$\frac{3}{x} + \frac{4}{y} = \frac{7}{x + y}$~~
- ~~$\frac{1}{2y} = (1/2)y$~~

In Exercises 19–38, insert the required factor in the parentheses.

- $\frac{3x + 2}{5} = \frac{1}{5}(\quad)$
- $\frac{7x^2}{10} = \frac{7}{10}(\quad)$
- $\frac{2}{3}x^2 + \frac{1}{3}x + 5 = \frac{1}{3}(\quad)$
- $\frac{3}{4}x + \frac{1}{2} = \frac{1}{4}(\quad)$
- $x^2(x^3 - 1)^4 = (\quad)(x^3 - 1)^4(3x^2)$
- $x(1 - 2x^2)^3 = (\quad)(1 - 2x^2)^3(-4x)$
- $\frac{4x + 6}{(x^2 + 3x + 7)^3} = (\quad)\frac{1}{(x^2 + 3x + 7)^3}(2x + 3)$
- $\frac{x + 1}{(x^2 + 2x - 3)^2} = (\quad)\frac{1}{(x^2 + 2x - 3)^2}(2x + 2)$
- $\frac{3}{x} + \frac{5}{2x^2} - \frac{3}{2}x = (\quad)(6x + 5 - 3x^3)$
- $\frac{(x - 1)^2}{169} + (y + 5)^2 = \frac{(x - 1)^3}{169(\quad)} + (y + 5)^2$
- $\frac{9x^2}{25} + \frac{16y^2}{49} = \frac{x^2}{(\quad)} + \frac{y^2}{(\quad)}$
- $\frac{3x^2}{4} - \frac{9y^2}{16} = \frac{x^2}{(\quad)} - \frac{y^2}{(\quad)}$

- $\frac{x^2}{1/12} - \frac{y^2}{2/3} = \frac{12x^2}{(\quad)} - \frac{3y^2}{(\quad)}$
- $\frac{x^2}{4/9} + \frac{y^2}{7/8} = \frac{9x^2}{(\quad)} + \frac{8y^2}{(\quad)}$
- $x^{1/3} - 5x^{4/3} = x^{1/3}(\quad)$
- $3(2x + 1)x^{1/2} + 4x^{3/2} = x^{1/2}(\quad)$
- $(1 - 3x)^{4/3} - 4x(1 - 3x)^{1/3} = (1 - 3x)^{1/3}(\quad)$
- $\frac{1}{2\sqrt{x}} + 5x^{3/2} - 10x^{5/2} = \frac{1}{2\sqrt{x}}(\quad)$
- $\frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} = \frac{(2x + 1)^{3/2}}{15}(\quad)$
- $\frac{3}{7}(t + 1)^{7/3} - \frac{3}{4}(t + 1)^{4/3} = \frac{3(t + 1)^{4/3}}{28}(\quad)$

In Exercises 39–42, write the expression using negative exponents.

- $\frac{3x^2}{(2x - 1)^3}$
- $\frac{x + 1}{x(6 - x)^{1/2}}$
- $\frac{4}{3x} + \frac{4}{x^4} - \frac{7x}{\sqrt[3]{2x}}$
- $\frac{x}{x - 2} + \frac{1}{x^2} + \frac{8}{3(9x)^3}$

In Exercises 43–48, write the fraction as a sum of two or more terms.

- $\frac{16 - 5x - x^2}{x}$
- $\frac{x^3 - 5x^2 + 4}{x^2}$
- $\frac{4x^3 - 7x^2 + 1}{x^{1/3}}$
- $\frac{2x^5 - 3x^3 + 5x - 1}{x^{3/2}}$
- $\frac{3 - 5x^2 - x^4}{\sqrt{x}}$
- $\frac{x^3 - 5x^4}{3x^2}$

In Exercises 49–60, simplify the expression.

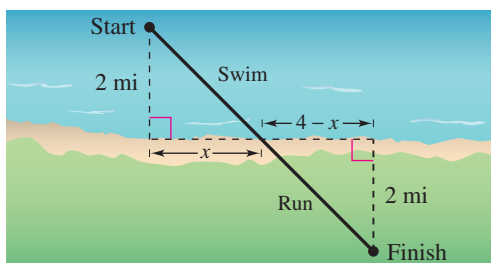
- $\frac{-2(x^2 - 3)^{-3}(2x)(x + 1)^3 - 3(x + 1)^2(x^2 - 3)^{-2}}{[(x + 1)^3]^2}$
- $\frac{x^5(-3)(x^2 + 1)^{-4}(2x) - (x^2 + 1)^{-3}(5)x^4}{(x^5)^2}$
- $\frac{(6x + 1)^3(27x^2 + 2) - (9x^3 + 2x)(3)(6x + 1)^2(6)}{[(6x + 1)^3]^2}$

52. $\frac{(4x^2 + 9)^{1/2}(2) - (2x + 3)(\frac{1}{2})(4x^2 + 9)^{-1/2}(8x)}{[(4x^2 + 9)^{1/2}]^2}$
53. $\frac{(x + 2)^{3/4}(x + 3)^{-2/3} - (x + 3)^{1/3}(x + 2)^{-1/4}}{[(x + 2)^{3/4}]^2}$
54. $(2x - 1)^{1/2} - (x + 2)(2x - 1)^{-1/2}$
55. $\frac{2(3x - 1)^{1/3} - (2x + 1)(\frac{1}{3})(3x - 1)^{-2/3}(3)}{(3x - 1)^{2/3}}$
56. $\frac{(x + 1)(\frac{1}{2})(2x - 3x^2)^{-1/2}(2 - 6x) - (2x - 3x^2)^{1/2}}{(x + 1)^2}$
57. $\frac{1}{(x^2 + 4)^{1/2}} \cdot \frac{1}{2}(x^2 + 4)^{-1/2}(2x)$
58. $\frac{1}{x^2 - 6}(2x) + \frac{1}{2x + 5}(2)$
59. $(x^2 + 5)^{1/2}(\frac{3}{2})(3x - 2)^{1/2}(3) + (3x - 2)^{3/2}(\frac{1}{2})(x^2 + 5)^{-1/2}(2x)$
60. $(3x + 2)^{-1/2}(3)(x - 6)^{1/2}(1) + (x - 6)^3(-\frac{1}{2})(3x + 2)^{-3/2}(3)$

61. Athletics An athlete has set up a course for training as part of her regimen in preparation for an upcoming triathlon. She is dropped off by a boat 2 miles from the nearest point on shore. The finish line is 4 miles down the coast and 2 miles inland (see figure). She can swim 2 miles per hour and run 6 miles per hour. The time t (in hours) required for her to reach the finish line can be approximated by the model

$$t = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{(4 - x)^2 + 4}}{6}$$

where x is the distance down the coast (in miles) to which she swims and then leaves the water to start her run.



- (a) Find the times required for the triathlete to finish when she swims to the points $x = 0.5$, $x = 1.0$, . . . , $x = 3.5$, and $x = 4.0$ miles down the coast.
- (b) Use your results from part (a) to determine the distance down the coast that will yield the minimum amount of time required for the triathlete to reach the finish line.

(c) The expression below was obtained using calculus. It can be used to find the minimum amount of time required for the triathlete to reach the finish line. Simplify the expression.

$$\frac{1}{2}x(x^2 + 4)^{-1/2} + \frac{1}{6}(x - 4)(x^2 - 8x + 20)^{-1/2}$$

62. (a) Verify that $y_1 = y_2$ analytically.

$$y_1 = x^2\left(\frac{1}{3}\right)(x^2 + 1)^{-2/3}(2x) + (x^2 + 1)^{1/3}(2x)$$

$$y_2 = \frac{2x(4x^2 + 3)}{3(x^2 + 1)^{2/3}}$$

(b) Complete the table and demonstrate the equality in part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{5}{2}$
y_1							
y_2							

Synthesis

True or False? In Exercises 63–66, determine whether the statement is true or false. Justify your answer.

63. $x^{-1} + y^{-2} = \frac{y^2 + x}{xy^2}$

64. $\frac{1}{x^{-2} + y^{-1}} = x^2 + y$

65. $\frac{1}{\sqrt{x} + 4} = \frac{\sqrt{x} - 4}{x - 16}$

66. $\frac{x^2 - 9}{\sqrt{x} - 3} = \sqrt{x} + 3$

In Exercises 67–70, find and correct any errors. If the problem is correct, state that it is correct.

67. $x^n \cdot x^{3n} = x^{3n^2}$

68. $(x^n)^{2n} + (x^{2n})^n = 2x^{2n^2}$

69. $x^{2n} + y^{2n} = (x^n + y^n)^2$

70. $\frac{x^{2n} \cdot x^{3n}}{x^{3n} + x^2} = \frac{x^{5n}}{x^{3n} + x^2}$

71. Think About It You are taking a course in calculus, and for one of the homework problems you obtain the following answer.

$$\frac{1}{10}(2x - 1)^{5/2} + \frac{1}{6}(2x - 1)^{3/2}$$

The answer in the back of the book is $\frac{1}{15}(2x - 1)^{3/2}(3x + 1)$. Show how the second answer can be obtained from the first. Then use the same technique to simplify each of the following expressions.

(a) $\frac{2}{3}x(2x - 3)^{3/2} - \frac{2}{15}(2x - 3)^{5/2}$

(b) $\frac{2}{3}x(4 + x)^{3/2} - \frac{2}{15}(4 + x)^{5/2}$