### Appendix A.2 Exponents and Radicals

#### What you should learn
- Use properties of exponents.
- Use scientific notation to represent real numbers.
- Use properties of radicals.
- Simplify and combine radicals.
- Rationalize denominators and numerators.
- Use properties of rational exponents.

#### Why you should learn it
Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 105 on page A22, you will use an expression involving rational exponents to find the time required for a funnel to empty for different water heights.

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#### Integer Exponents

Repeated multiplication can be written in **exponential form**.

<table>
<thead>
<tr>
<th>Repeated Multiplication</th>
<th>Exponential Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-4)((-4))((-4))</td>
<td>((-4)^3)</td>
</tr>
<tr>
<td>((2x)(2x)(2x))</td>
<td>((2x)^3)</td>
</tr>
</tbody>
</table>

#### Exponential Notation

If \(a\) is a real number and \(n\) is a positive integer, then

\[
a^n = a \cdot a \cdot a \cdot \ldots \cdot a
\]

\(n\) factors

where \(n\) is the exponent and \(a\) is the base. The expression \(a^n\) is read “\(a\) to the \(n\)th power.”

An exponent can also be negative. In Property 3 below, be sure you see how to use a negative exponent.

#### Properties of Exponents

Let \(a\) and \(b\) be real numbers, variables, or algebraic expressions, and let \(m\) and \(n\) be integers. (All denominators and bases are nonzero.)

<table>
<thead>
<tr>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (a^m a^n = a^{m+n})</td>
<td>(3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729)</td>
</tr>
<tr>
<td>2. (a^m/a^n = a^{m-n})</td>
<td>(x^7/x^3 = x^{7-3} = x^4)</td>
</tr>
<tr>
<td>3. (a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n)</td>
<td>(y^{-4} = \frac{1}{y^4} = \left(\frac{1}{y}\right)^4)</td>
</tr>
<tr>
<td>4. (a^0 = 1, \quad a \neq 0)</td>
<td>((x^2 + 1)^0 = 1)</td>
</tr>
<tr>
<td>5. ((ab)^m = a^n b^n)</td>
<td>((5x)^3 = 5^3x^3 = 125x^3)</td>
</tr>
<tr>
<td>6. ((a)^m = a^{mn})</td>
<td>((y^3)^{-4} = y^{3(-4)} = y^{-12} = \frac{1}{y^{12}})</td>
</tr>
<tr>
<td>7. (\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m})</td>
<td>(\left(\frac{2}{x}\right)^3 = \frac{2^3}{x^3} = \frac{8}{x^3})</td>
</tr>
<tr>
<td>8. (</td>
<td>a^2</td>
</tr>
</tbody>
</table>
Appendix A  Review of Fundamental Concepts of Algebra

It is important to recognize the difference between expressions such as \((-2)^4\) and \(-2^4\). In \((-2)^4\), the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in \(-2^4 = -(2^4)\), the exponent applies only to the 2. So, \((-2)^4 = 16\) and \(-2^4 = -16\).

The properties of exponents listed on the preceding page apply to all integers \(m\) and \(n\), not just to positive integers as shown in the examples in this section.

**Example 1**  Using Properties of Exponents

Use the properties of exponents to simplify each expression.

\[\begin{align*}
\text{a.} & \quad (-3ab^3)(4ab^{-3}) & \text{b.} & \quad (2xy^3)^3 & \text{c.} & \quad 3a(-4a^2)^0 & \text{d.} & \quad \left(\frac{5x^3}{y}\right)^2
\end{align*}\]

**Solution**

\[\begin{align*}
\text{a.} & \quad (-3ab^3)(4ab^{-3}) = (-3)(4)(a)(a)(b^3)(b^{-3}) = -12a^2b \\
\text{b.} & \quad (2xy^3)^3 = 2^3(x)^3(y^3)^3 = 8x^3y^9 \\
\text{c.} & \quad 3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0 \\
\text{d.} & \quad \left(\frac{5x^3}{y}\right)^2 = \frac{5^2(x)^2}{y^2} = \frac{25x^6}{y^2}
\end{align*}\]

Now try Exercise 25.

**Example 2**  Rewriting with Positive Exponents

Rewrite each expression with positive exponents.

\[\begin{align*}
\text{a.} & \quad x^{-1} & \text{b.} & \quad \frac{1}{3x^{-2}} & \text{c.} & \quad \frac{12a^3b^{-4}}{4a^{-2}b} & \text{d.} & \quad \left(\frac{3x^2}{y}\right)^{-2}
\end{align*}\]

**Solution**

\[\begin{align*}
\text{a.} & \quad x^{-1} = \frac{1}{x} & \quad \text{Property 3} \\
\text{b.} & \quad \frac{1}{3x^{-2}} = \frac{1(x^2)}{3} = \frac{x^2}{3} & \quad \text{The exponent} -2 \text{ does not apply to 3.} \\
\text{c.} & \quad \frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4} = \frac{3a^5}{b^5} & \quad \text{Properties 3 and 1} \\
\text{d.} & \quad \left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}} & \quad \text{Properties 5 and 7} \\
& \quad = \frac{3^{-2}x^{-4}}{y^{-2}} & \quad \text{Property 6} \\
& \quad = \frac{y^2}{3^2x^4} & \quad \text{Property 3} \\
& \quad = \frac{y^2}{9x^4} & \quad \text{Simplify.}
\end{align*}\]

Now try Exercise 33.
Appendix A.2 Exponents and Radicals

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

359,000,000,000,000,000

It is convenient to write such numbers in scientific notation. This notation has the form $\pm c \times 10^n$, where $1 \leq c < 10$ and $n$ is an integer. So, the number of gallons of water on Earth can be written in scientific notation as

$3.59 \times 10^{20}$.

The positive exponent 20 indicates that the number is large (10 or more) and that the decimal point has been moved 20 places. A negative exponent indicates that the number is small (less than 1). For instance, the mass (in grams) of one electron is approximately

$9.0 \times 10^{-28} = 0.0000000000000000000000000009$.

Example 3 Scientific Notation

Write each number in scientific notation.

a. 0.0000782  

b. 836,100,000

Solution

a. $0.0000782 = 7.82 \times 10^{-5}$  
b. $836,100,000 = 8.361 \times 10^8$

Now try Exercise 37.

Example 4 Decimal Notation

Write each number in decimal notation.

a. $9.36 \times 10^{-6}$  
b. $1.345 \times 10^2$

Solution

a. $9.36 \times 10^{-6} = 0.00000936$  
b. $1.345 \times 10^2 = 134.5$

Now try Exercise 41.

Technology

Most calculators automatically switch to scientific notation when they are showing large (or small) numbers that exceed the display range.

To enter numbers in scientific notation, your calculator should have an exponential entry key labeled

$\text{EE}$ or $\text{EXP}$.

Consult the user’s guide for your calculator for instructions on keystrokes and how numbers in scientific notation are displayed.
Radicals and Their Properties

A square root of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a cube root of a number is one of its three equal factors, as in $125 = 5^3$.

**Definition of $n$th Root of a Number**

Let $a$ and $b$ be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then $b$ is an $n$th root of $a$. If $n = 2$, the root is a square root. If $n = 3$, the root is a cube root.

Some numbers have more than one $n$th root. For example, both 5 and $-5$ are square roots of 25. The principal square root of 25, written as $\sqrt{25}$, is the positive root, 5. The principal $n$th root of a number is defined as follows.

**Principal $n$th Root of a Number**

Let $a$ be a real number that has at least one $n$th root. The principal $n$th root of $a$ is the $n$th root that has the same sign as $a$. It is denoted by a radical symbol

$$\sqrt[n]{a}.$$  

Principal $n$th root

The positive integer $n$ is the index of the radical, and the number $a$ is the radicand. If $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is indices.)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: $\sqrt{-4} = \pm 2$  
Correct: $-\sqrt{4} = -2$  
and  
$\sqrt{4} = 2$

**Example 5**  
Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-\sqrt{36} = -(\sqrt{36}) = -(6) = -6$.

c. $\sqrt[3]{125} = \frac{5}{4}$ because $\left(\frac{5}{4}\right)^3 = \frac{5^3}{4^3} = \frac{125}{64}$.

d. $\sqrt[3]{-32} = -2$ because $(-2)^3 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce $-81$.

**Checkpoint**  
Now try Exercise 51.
Appendix A.2  Exponents and Radicals

Here are some generalizations about the \( n \)th roots of real numbers.

### Generalizations About \( n \)th Roots of Real Numbers

<table>
<thead>
<tr>
<th>Real Number ( a )</th>
<th>Integer ( n )</th>
<th>Root(s) of ( a )</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a &gt; 0 )</td>
<td>( n &gt; 0 ), is even.</td>
<td>( \sqrt[n]{a}, -\sqrt[n]{a} )</td>
<td>( \sqrt[3]{81} = 3, -\frac{3}{3} = -3 )</td>
</tr>
<tr>
<td>( a &gt; 0 ) or ( a &lt; 0 )</td>
<td>( n ) is odd.</td>
<td>( \sqrt[n]{a} )</td>
<td>( \sqrt[5]{-8} = -2 )</td>
</tr>
<tr>
<td>( a &lt; 0 )</td>
<td>( n ) is even.</td>
<td>No real roots</td>
<td>( \sqrt[4]{-4} ) is not a real number.</td>
</tr>
<tr>
<td>( a = 0 )</td>
<td>( n ) is even or odd.</td>
<td>( \sqrt[6]{0} = 0 )</td>
<td>( \sqrt[6]{0} = 0 )</td>
</tr>
</tbody>
</table>

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

### Properties of Radicals

Let \( a \) and \( b \) be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let \( m \) and \( n \) be positive integers.

**Property** | **Example**
---|---
1. \( \sqrt[n]{a^n} = (\sqrt[n]{a})^n \) | \( \sqrt[3]{8}^2 = (\sqrt[3]{8})^2 = 2^2 = 4 \)
2. \( \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \) | \( \sqrt[6]{9} \cdot \sqrt[6]{3} = \sqrt[6]{27} = \sqrt[6]{3^3} = 3 \)
3. \( \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad b \neq 0 \) | \( \frac{\sqrt[3]{81}}{\sqrt[3]{3}} = \sqrt[3]{\frac{81}{3}} = \sqrt[3]{27} = 3 \)
4. \( \sqrt[\mu]{\sqrt[n]{a}} = \sqrt[\mu n]{a} \) | \( \sqrt[\mu]{\sqrt{10}} = \sqrt[\mu n]{10} \)
5. \( (\sqrt[n]{a})^m = a \) | \( (\sqrt[4]{3})^2 = 3 \)
6. For \( n \) even, \( \sqrt[n]{a^n} = |a| \). For \( n \) odd, \( \sqrt[n]{a^n} = a \). | \( \sqrt[4]{(-12)^2} = |-12| = 12 \) \( \sqrt[3]{(-12)^3} = -12 \)

A common special case of Property 6 is \( \sqrt[2]{a^2} = |a| \).

**Example 6**  **Using Properties of Radicals**

Use the properties of radicals to simplify each expression.

**a.** \( \sqrt[4]{8} \cdot \sqrt[2]{2} \)  **b.** \( (\sqrt[3]{5})^3 \)  **c.** \( \sqrt[3]{x^3} \)  **d.** \( \sqrt[2]{y^2} \)

**Solution**

**a.** \( \sqrt[4]{8} \cdot \sqrt[2]{2} = \sqrt[4]{16} = 4 \)

**b.** \( (\sqrt[3]{5})^3 = 5 \)

**c.** \( \sqrt[3]{x^3} = x \)

**d.** \( \sqrt[2]{y^2} = |y| \)

**CHECKPOINT**  Now try Exercise 61.
Simplifying Radicals

An expression involving radicals is in simplest form when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called rationalizing the denominator).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the “leftover” factors make up the new radicand.

Example 7  Simplifying Even Roots

<table>
<thead>
<tr>
<th>Perfect 4th power</th>
<th>Leftover factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{2^4 \cdot 3} = 2\sqrt{3}$</td>
<td></td>
</tr>
<tr>
<td>Find largest square factor.</td>
<td></td>
</tr>
<tr>
<td>b. $\sqrt[4]{75x^3} = \sqrt[4]{25x^2 \cdot 3x} = \sqrt[4]{(5x)^2 \cdot 3x}$</td>
<td></td>
</tr>
<tr>
<td>Find root of perfect square.</td>
<td></td>
</tr>
<tr>
<td>c. $\sqrt[4]{(5x)^3} =</td>
<td>5x</td>
</tr>
</tbody>
</table>

Now try Exercise 63(a).

Example 8  Simplifying Odd Roots

<table>
<thead>
<tr>
<th>Perfect cube</th>
<th>Leftover factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$</td>
<td></td>
</tr>
<tr>
<td>Find largest cube factor.</td>
<td></td>
</tr>
<tr>
<td>b. $\sqrt[3]{24a^4} = \sqrt[3]{8a^3 \cdot 3a} = \sqrt[3]{(2a)^3 \cdot 3a} = 2a\sqrt[3]{3a}$</td>
<td></td>
</tr>
<tr>
<td>Find root of perfect cube.</td>
<td></td>
</tr>
<tr>
<td>c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5} = \sqrt[3]{(-2x^2)^3 \cdot 5} = -2x^2\sqrt[3]{5}$</td>
<td></td>
</tr>
<tr>
<td>Find root of perfect cube.</td>
<td></td>
</tr>
</tbody>
</table>

Now try Exercise 63(b).
Appendix A.2  Exponents and Radicals

Radical expressions can be combined (added or subtracted) if they are like radicals—that is, if they have the same index and radicand. For instance, \( \sqrt{2} \), \( 3\sqrt{2} \), and \( \frac{1}{2}\sqrt{2} \) are like radicals, but \( \sqrt{3} \) and \( \sqrt{2} \) are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

### Combining Radicals

1. Find square factors.
2. Find square roots and multiply by coefficients.
3. Combine like terms.
4. Simplify.

### Example 9

#### Combining Radicals

a. \( 2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3} \)
   \[
   = 8\sqrt{3} - 9\sqrt{3} \\
   = (8 - 9)\sqrt{3} \\
   = -\sqrt{3}
   \]

b. \( \sqrt{16x} - \frac{1}{2}\sqrt{54x^3} = \frac{\sqrt{8 \cdot 2x} - \sqrt{27 \cdot x^3 \cdot 2x}}{2} \)
   \[
   = 2\sqrt{2x} - 3x\sqrt{2x} \\
   = (2 - 3x)\sqrt{2x}
   \]

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### Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form \( a - b\sqrt{m} \) or \( a + b\sqrt{m} \), multiply both numerator and denominator by a conjugate: \( a + b\sqrt{m} \) and \( a - b\sqrt{m} \) are conjugates of each other. If \( a = 0 \), then the rationalizing factor for \( \sqrt{m} \) is itself, \( \sqrt{m} \). For cube roots, choose a rationalizing factor that generates a perfect cube.

#### Example 10

#### Rationalizing Single-Term Denominators

Rationalize the denominator of each expression.

a. \( \frac{5}{2\sqrt{3}} \)

**Solution**

\[
\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{6}
\]

b. \( \frac{2}{\sqrt{5}} \)

**Solution**

\[
\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}
\]

---

Now try Exercise 71. Now try Exercise 79.
Additional Examples

a. \( \frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7} \)

b. \( \frac{2}{\sqrt{4}} = \frac{2}{\sqrt{4}} \cdot \frac{\sqrt{4}}{\sqrt{4}} = \frac{2\sqrt{4}}{4} = \frac{2}{2} = 1 \)

c. \( \sqrt{2} + \sqrt{3} \)

\[ = \frac{6}{\sqrt{2} + \sqrt{3}} \cdot \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}} \]

\[ = \frac{6(\sqrt{2} - \sqrt{3})}{(\sqrt{2})^2 - (\sqrt{3})^2} \]

\[ = -6\sqrt{2} + 6\sqrt{3} \]

**STUDY TIP**

Do not confuse the expression \( \sqrt{2} + \sqrt{3} \) with the expression \( \sqrt{2} + \sqrt{7} \). In general, \( \sqrt{x} + \sqrt{y} \) does not equal \( \sqrt{x + y} \). Similarly, \( \sqrt{x^2} + \sqrt{y^2} \) does not equal \( x + y \).

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**Example 11** **Rationalizing a Denominator with Two Terms**

\[ \frac{2}{3 + \sqrt{7}} = \frac{2}{3 + \sqrt{7}} \cdot \frac{3 - \sqrt{7}}{3 - \sqrt{7}} \]

Multiply numerator and denominator by conjugate of denominator.

\[ = \frac{2(3 - \sqrt{7})}{3(3 + 3(\sqrt{7}) + \sqrt{7})(3 - \sqrt{7})} \]

Use Distributive Property.

\[ = \frac{2(3 - \sqrt{7})}{(3)^2 - (\sqrt{7})^2} \]

Simplify.

\[ = \frac{2(3 - \sqrt{7})}{9 - 7} \]

Square terms of denominator.

\[ = \frac{2(3 - \sqrt{7})}{2} \]

Simplify.

\[ = 3 - \sqrt{7} \]

**CHECKPOINT** Now try Exercise 81.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Appendix A.4 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

**Example 12** **Rationalizing a Numerator**

\[ \frac{\sqrt{3} - \sqrt{7}}{2} \]

Multiply numerator and denominator by conjugate of numerator.

\[ = \frac{\sqrt{3} - \sqrt{7}}{2} \cdot \frac{\sqrt{3} + \sqrt{7}}{\sqrt{3} + \sqrt{7}} \]

Simplify.

\[ = \frac{(\sqrt{3})^2 - (\sqrt{7})^2}{2(\sqrt{3} + \sqrt{7})} \]

Square terms of numerator.

\[ = \frac{5 - 7}{2(\sqrt{3} + \sqrt{7})} \]

Simplify.

\[ = \frac{-2}{2(\sqrt{3} + \sqrt{7})} \]

\[ = \frac{-1}{\sqrt{3} + \sqrt{7}} \]

**CHECKPOINT** Now try Exercise 85.

**Rational Exponents**

**Definition of Rational Exponents**

If \( a \) is a real number and \( n \) is a positive integer such that the principal \( n \)th root of \( a \) exists, then \( a^{1/n} \) is defined as

\[ a^{1/n} = \sqrt[n]{a} \text{, where } 1/n \text{ is the rational exponent of } a. \]

Moreover, if \( m \) is a positive integer that has no common factor with \( n \), then

\[ a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}. \]

The symbol \( \blacksquare \) indicates an example or exercise that highlights algebraic techniques specifically used in calculus.
Appendix A.2  Exponents and Radicals

The numerator of a rational exponent denotes the power to which the base is raised, and the denominator denotes the index or the root to be taken.

\[ b^{m/n} = (\sqrt[n]{b})^m = \sqrt[n]{b^m} \]

When you are working with rational exponents, the properties of integer exponents still apply. For instance, 
\[ 2^{1/2} \cdot 2^{1/3} = 2^{(1/2) + (1/3)} = 2^{5/6}. \]

**Example 13**  Changing from Radical to Exponential Form

a. \( \sqrt[3]{3} = 3^{1/3} \)

b. \( \sqrt[3]{(3xy)^3} = \frac{\sqrt[3]{3xy}}{(3xy)^3} = (3xy)^{5/3} \)

c. \( 2x \sqrt[3]{x^2} = (2x)(x^{3/4}) = 2x^{1+3/4} = 2x^{7/4} \)

**CHECKPOINT**  Now try Exercise 87.

**Example 14**  Changing from Exponential to Radical Form

a. \( (x^2 + y^3)^{1/2} = \sqrt{x^2 + y^3} \)

b. \( 2y^{3/4}z^{3/4} = 2(y^{3/2})^{1/4} = 2y^{3/8}z^{3/8} \)

c. \( a^{-3/2} = \frac{1}{a^{3/2}} = \frac{1}{\sqrt{a^3}} \)

d. \( x^{0.2} = x^{1/5} = \sqrt[5]{x} \)

**CHECKPOINT**  Now try Exercise 89.

Rational exponents are useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions in calculus.

**Example 15**  Simplifying with Rational Exponents

a. \( (-32)^{-4/5} = (\frac{1}{\sqrt[5]{32}})^4 = (-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16} \)

b. \( (-5x^{5/3})(3x^{-3/4}) = -15x^{5/3-3/4} = -15x^{11/12}, \quad x \neq 0 \)

c. \( \sqrt[3]{a^2} = a^{2/3} = \frac{1}{\sqrt[3]{a^2}} \)  

Reduce index.

d. \( \sqrt[3]{125} = \sqrt[3]{5^3} = 5^{1/3} = 5^{1/3} = \sqrt[3]{5} \)

e. \( (2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{4/3-1/3} = (2x - 1)^{1/3} = 2x - 1, \quad x \neq \frac{1}{2} \)

f. \( \frac{x - 1}{(x - 1)^{-1/2}} = \frac{x - 1}{(x - 1)^{-1/2}} \cdot \frac{x - 1)^{1/2}}{(x - 1)^{1/2}} = \frac{(x - 1)^0}{(x - 1)^{1/2}} = (x - 1)^{3/2}, \quad x \neq 1 \)

**CHECKPOINT**  Now try Exercise 99.
A.2 Exercises

**Vocabulary Check:** Fill in the blanks.

1. In the exponential form \(a^n\), \(n\) is the _____ and \(a\) is the _____.
2. A convenient way of writing very large or very small numbers is called ________ ________.
3. One of the two equal factors of a number is called a __________ __________ of the number.
4. The ________ ________ ________ of a number is the \(n\)th root that has the same sign as \(a\), and is denoted by \(\sqrt[n]{a}\).
5. In the radical form, \(\sqrt[n]{a}\) the positive integer \(n\) is called the ________ of the radical and the number \(a\) is called the ________.
6. When an expression involving radicals has all possible factors removed, radical-free denominators, and a reduced index, it is in ________ ________.
7. The expressions \(a + b\sqrt{m}\) and \(a - b\sqrt{m}\) are ________ of each other.
8. The process used to create a radical-free denominator is know as ________ the denominator.
9. In the expression \(b^{m/n}\), \(m\) denotes the ________ to which the base is raised and \(n\) denotes the ________ or root to be taken.

In Exercises 1 and 2, write the expression as a repeated multiplication problem.

1. \(8^5\)
2. \((-2)^7\)

In Exercises 3 and 4, write the expression using exponential notation.

3. \((4.9)(4.9)(4.9)(4.9)(4.9)\)
4. \((-10)(-10)(-10)(-10)(-10)\)

In Exercises 5–12, evaluate each expression.

5. (a) \(3^3 \cdot 3\) 
   (b) \(3 \cdot 3^3\)
6. (a) \(\frac{5^2}{5^2}\) 
   (b) \(\frac{2^3}{3^2}\)
7. (a) \((3^3)^0\) 
   (b) \(-3^2\)
8. (a) \((2^3 \cdot 3^2)^2\) 
   (b) \((-\frac{1}{3})^3\) \((\frac{1}{3})^2\)
9. (a) \(3 \cdot 4^{-4} \cdot 4^{-1}\) 
   (b) \(32(-2)^{-5}\)
10. (a) \(\frac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}\) 
    (b) \((-2)^0\)
11. (a) \(2^{-1} + 3^{-1}\) 
    (b) \((2^{-1})^{-2}\)
12. (a) \(3^{-1} + 2^{-2}\) 
    (b) \((3^{-2})^2\)

In Exercises 13–16, use a calculator to evaluate the expression. (If necessary, round your answer to three decimal places.)

13. \((-4)^3(5^2)\)
14. \((8^{-4})(10^3)\)
15. \(\frac{3^6}{7^1}\)
16. \(\frac{4^3}{3^{-1}}\)

In Exercises 17–24, evaluate the expression for the given value of \(x\).

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>17. (-3x^3)</td>
<td>(x = 2)</td>
</tr>
<tr>
<td>18. (7x^{-2})</td>
<td>(x = 4)</td>
</tr>
<tr>
<td>19. (6x^0)</td>
<td>(x = 10)</td>
</tr>
<tr>
<td>20. (5(-x)^3)</td>
<td>(x = 3)</td>
</tr>
<tr>
<td>21. (2x^3)</td>
<td>(x = -3)</td>
</tr>
<tr>
<td>22. (-3x^4)</td>
<td>(x = -2)</td>
</tr>
<tr>
<td>23. (4x^2)</td>
<td>(x = \frac{1}{2})</td>
</tr>
<tr>
<td>24. (5(-x)^3)</td>
<td>(x = \frac{1}{3})</td>
</tr>
</tbody>
</table>

In Exercises 25–30, simplify each expression.

25. (a) \((-5x)^3\) 
   (b) \(5x^4(2x)^2\)
26. (a) \((3x)^2\) 
   (b) \((4x^3)^0\)
27. (a) \(6y^7(2y^m)^2\) 
   (b) \(\frac{3y^5}{x^3}\)
28. (a) \((-z)^3(3z^6)\) 
   (b) \(\frac{25y^8}{10y^4}\)
29. (a) \(\frac{7x^2}{x^3}\) 
   (b) \(\frac{12(x + y)^3}{9(x + y)}\)
30. (a) \(\frac{r^4}{r^6}\) 
   (b) \(\frac{\left(\frac{1}{3}\right)^3}{\left(\frac{1}{3}\right)^4}\)

In Exercises 31–36, rewrite each expression with positive exponents and simplify.

31. (a) \((x + 5)^0\), \(x \neq -5\) 
    (b) \((2x^2)^{-2}\)
32. (a) \((2x^3)^0\), \(x \neq 0\) 
    (b) \((z + 2)^{-3}(z + 2)^{-1}\)
In Exercises 37–40, write the number in scientific notation.
37. Land area of Earth: 57,300,000 square miles
38. Light year: 9,460,000,000,000 kilometers
39. Relative density of hydrogen: 0.0000899 gram per cubic centimeter
40. One micron (millionth of a meter): 0.00003937 inch

In Exercises 41–44, write the number in decimal notation.
41. Worldwide daily consumption of Coca-Cola: 4.568 × 10^9 ounces
   (Source: The Coca-Cola Company)
42. Interior temperature of the sun: 1.5 × 10^6 degrees Celsius
43. Charge of an electron: 1.6022 × 10^-19 coulomb
44. Width of a human hair: 9.0 × 10^-5 meter

In Exercises 45 and 46, evaluate each expression without using a calculator.
45. (a) \(\sqrt{25} \times 10^8\)  (b) \(\sqrt[3]{8} \times 10^{15}\)
46. (a) \((1.2 \times 10^7)(5 \times 10^{-5})\)  (b) \((6.0 \times 10^8)/(3.0 \times 10^{-3})\)

In Exercises 47–50, use a calculator to evaluate each expression. (Round your answer to three decimal places.)
47. (a) \(750(1 + \frac{0.11}{365})^{100}\)  (b) \(67,000,000 + 93,000,000 \div 0.0052\)
48. (a) \((9.3 \times 10^9)(6.1 \times 10^{-4})\)  (b) \(\frac{2.414 \times 10^4}{1.68 \times 10^5}\)
49. (a) \(\sqrt{4.5 \times 10^7}\)  (b) \(\sqrt[3]{6.3 \times 10^7}\)
50. (a) \(2.65 \times 10^{-4}\)^{1/3}  (b) \(\sqrt[3]{9 \times 10^{-4}}\)

In Exercises 51–56, evaluate each expression without using a calculator.
51. (a) \(\sqrt{5}\)  (b) \(\sqrt[3]{2}\)
52. (a) \(27^{1/3}\)  (b) \(36^{3/2}\)
53. (a) \(32^{-3/5}\)  (b) \(\sqrt[3]{16}/17\)
54. (a) \(100^{-3/2}\)  (b) \(\frac{27}{8}\)^{-1/2}
55. (a) \((-\frac{1}{64})^{-1/3}\)  (b) \(\frac{1}{\sqrt[5]{32}}\)^{-2/5}
56. (a) \((-\frac{125}{27})^{-1/3}\)  (b) \(-\frac{1}{125}^{-4/3}\)

In Exercises 57–60, use a calculator to approximate the number. (Round your answer to three decimal places.)
57. (a) \(\sqrt[3]{57}\)  (b) \(\sqrt[5]{27}\)
58. (a) \(\sqrt[3]{45}\)  (b) \(\sqrt[5]{125}\)
59. (a) \((-12.4)^{-1.8}\)  (b) \((5 \sqrt[3]{3})^{-2.5}\)
60. (a) \(\frac{7 - (4.1)^{-3.2}}{2}\)  (b) \(\frac{13}{3}^{-3/2} - \left(-\frac{3}{2}\right)^{1/3}\)

In Exercises 61 and 62, use the properties of radicals to simplify each expression.
61. (a) \(\sqrt[3]{4}\)  (b) \(\sqrt[5]{65}\)
62. (a) \(\sqrt{12} \cdot \sqrt[3]{3}\)  (b) \(\sqrt[3]{3x}^3\)

In Exercises 63–74, simplify each radical expression.
63. (a) \(\sqrt[3]{8}\)  (b) \(\sqrt[5]{54}\)
64. (a) \(\sqrt[6]{8}\)  (b) \(\sqrt[5]{4}\)
65. (a) \(\sqrt[7]{2x^3}\)  (b) \(\sqrt[3]{18x^3}\)
66. (a) \(\sqrt[3]{4x^3}\)  (b) \(\sqrt[3]{32x^3}\)
67. (a) \(\sqrt[6]{16x^5}\)  (b) \(\sqrt[5]{75x^2y}\)
68. (a) \(\sqrt[3]{3x^2y^2}\)  (b) \(\sqrt[5]{160x^2z}\)
69. (a) \(2\sqrt{50} + 12\sqrt{8}\)  (b) \(10\sqrt{32} - 6\sqrt{18}\)
70. (a) \(4\sqrt{27} - \sqrt{75}\)  (b) \(\sqrt{16} + 3\sqrt{54}\)
71. (a) \(5\sqrt{5} - 3\sqrt{3}\)  (b) \(-2\sqrt[5]{9y} + 10\sqrt{5}\)
72. (a) \(8\sqrt{49x} - 14\sqrt{100x}\)  (b) \(-3\sqrt{48x^2} + 7\sqrt{75x^2}\)
73. (a) \(3\sqrt{x} + 1 + 10\sqrt{x} + 1\)  (b) \(7\sqrt{80x} - 2\sqrt{125x}\)
74. (a) \(-\sqrt{3x} - 7 + 5\sqrt{x^2} - 7\)  (b) \(11\sqrt{245x^3} - 9\sqrt{45x^3}\)

In Exercises 75–78, complete the statement with <, =, or >.
75. \(\sqrt[3]{5} + \sqrt[3]{3}\)  \(\sqrt[5]{5 + 3}\)
76. \(\frac{3}{10}\)  \(\frac{\sqrt{3}}{\sqrt{11}}\)
77. \(5\sqrt[3]{3} + 2\sqrt[3]{2}\)
78. \(5\sqrt[3]{3}^2 + 4\sqrt[5]{2}\)

In Exercises 79–82, rationalize the denominator of the expression. Then simplify your answer.
79. \(\frac{1}{\sqrt[3]{3}}\)
80. \(\frac{5}{\sqrt[10]{10}}\)
81. \( \frac{2}{5 - \sqrt{3}} \)  
82. \( \frac{3}{\sqrt{5} + \sqrt{6}} \)

In Exercises 83–86, rationalize the numerator of the expression. Then simplify your answer.

83. \( \frac{\sqrt{2}}{2} \)  
84. \( \frac{\sqrt{2}}{3} \)  
85. \( \frac{\sqrt{4} + \sqrt{12}}{3} \)  
86. \( \frac{\sqrt{7} - 3}{4} \)

In Exercises 87–94, fill in the missing form of the expression.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>Rational Exponent Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{9} )</td>
<td>3</td>
</tr>
<tr>
<td>( \sqrt{64} )</td>
<td>4</td>
</tr>
<tr>
<td>( \sqrt{-2} )</td>
<td>-1</td>
</tr>
<tr>
<td>( \sqrt{10} )</td>
<td>( \frac{3}{2} )</td>
</tr>
<tr>
<td>( \sqrt{18} )</td>
<td>1.5</td>
</tr>
<tr>
<td>( \sqrt{50} )</td>
<td>2.5</td>
</tr>
<tr>
<td>( \sqrt{72} )</td>
<td>3.2</td>
</tr>
<tr>
<td>( \sqrt{128} )</td>
<td>4.2</td>
</tr>
</tbody>
</table>

In Exercises 95–98, perform the operations and simplify.

95. \( \frac{2x^{3/2}}{x^{1/2}} \)  
96. \( \frac{x^{4/3}y^{2/3}}{(xy)^{1/3}} \)  
97. \( \frac{x^{-3} \cdot x^{1/2}}{x^{3/2} \cdot x^{-1}} \)  
98. \( \frac{5^{-1/2} \cdot 5^{3/2}}{(5x)^{1/2}} \)

In Exercises 99 and 100, reduce the index of each radical.

99. (a) \( \sqrt[3]{x} \)  
   (b) \( \sqrt[4]{(x + 1)^2} \)  
100. (a) \( \sqrt[3]{x} \)  
    (b) \( \sqrt[5]{(x + 1)^3} \)

In Exercises 101 and 102, write each expression as a single radical. Then simplify your answer.

101. (a) \( \sqrt{32} \)  
    (b) \( \sqrt{243(x + 1)} \)  
102. (a) \( \sqrt{243(x + 1)} \)  
    (b) \( \sqrt{10a^2b} \)

103. Period of a Pendulum

The period \( T \) (in seconds) of a pendulum is

\[ T = 2\pi \sqrt{\frac{L}{32}} \]

where \( L \) is the length of the pendulum (in feet). Find the period of a pendulum whose length is 2 feet.

104. Erosion

A stream of water moving at the rate of \( v \) feet per second can carry particles of size \( 0.03\sqrt{v} \) inches. Find the size of the largest particle that can be carried by a stream flowing at the rate of \( \frac{1}{4} \) foot per second.

105. Mathematical Modeling

A funnel is filled with water to a height of \( h \) centimeters. The formula

\[ t = 0.03[(12h/5) - (12 - h)^{5/2}] \quad 0 \leq h \leq 12 \]

represents the amount of time \( t \) (in seconds) that it will take for the funnel to empty.

(a) Use the table feature of a graphing utility to find the times required for the funnel to empty for water heights of \( h = 0, \ h = 1, \ h = 2, \ldots, h = 12 \) centimeters.

(b) What value does \( t \) appear to be approaching as the height of the water becomes closer and closer to 12 centimeters?

106. Speed of Light

The speed of light is approximately \( 11,180,000 \) miles per minute. The distance from the sun to Earth is approximately \( 93,000,000 \) miles. Find the time for light to travel from the sun to Earth.

**Synthesis**

True or False? In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.

107. \( \frac{a^{-1}}{x} = x^a \)
108. \( (a^n)^k = a^{nk} \)

109. Verify that \( a^0 = 1, \ a \neq 0 \). (Hint: Use the property of exponents \( a^m/a^n = a^{m-n} \).)

110. Explain why each of the following pairs is not equal.

(a) \( (3x)^{-1} \neq \frac{3}{x} \)
(b) \( y^2 \cdot y^3 \neq y^6 \)
(c) \( (a^3b^2)^{1/2} \neq a^3b^7 \)
(d) \( (a + b)^2 \neq a^2 + b^2 \)
(e) \( \sqrt{4x^2} \neq 2x \)
(f) \( \sqrt{3} + \sqrt{5} \neq \sqrt{8} \)

111. Exploration

List all possible digits that occur in the units place of the square of a positive integer. Use that list to determine whether \( \sqrt{5233} \) is an integer.

112. Think About It

Square the real number \( 2/\sqrt{3} \) and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?