Power Series Solution of a Differential Equation

Power series can be used to solve certain types of differential equations. This section begins with the general power series solution method.

A power series represents a function \( f \) on an interval of convergence, and you can successively differentiate the power series to obtain a series for \( f', f'', \) and so on. These properties are used in the power series solution method demonstrated in the first two examples.

**EXAMPLE 1  Power Series Solution**

Use a power series to solve the differential equation \( y' - 2y = 0. \)

**Solution** Assume that \( y = \sum a_n x^n \) is a solution. Then, \( y' = \sum n a_n x^{n-1}. \) Substituting for \( y' \) and \(-2y\), you obtain the following series form of the differential equation. (Note that, from the third step to the fourth, the index of summation is changed to ensure that \( x^n \) occurs in both sums.)

\[
y' - 2y = 0 \\
\sum_{n=1}^{\infty} na_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0 \\
\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} 2a_n x^n \\
\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} 2a_n x^n
\]

Now, by equating coefficients of like terms, you obtain the recursion formula \((n+1)a_{n+1} = 2a_n\), which implies that

\[
a_{n+1} = \frac{2a_n}{n+1}, \quad n \geq 0.
\]

This formula generates the following results.

\[
a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad \ldots \\
a_0 \quad 2a_0 \quad \frac{2^2a_0}{2} \quad \frac{2^3a_0}{3!} \quad \frac{2^4a_0}{4!} \quad \frac{2^5a_0}{5!} \quad \ldots
\]

Using these values as the coefficients for the solution series, you have

\[
y = \sum_{n=0}^{\infty} \frac{2^n a_0}{n!} x^n \\
= a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\
= a_0 e^{2x}.
\]
In Example 1, the differential equation could be solved easily without using a series. The differential equation in Example 2 cannot be solved by any of the methods discussed in previous sections.

**EXAMPLE 2  Power Series Solution**

Use a power series to solve the differential equation \( y'' + xy' + y = 0 \).

**Solution** Assume that \( \sum_{n=0}^{\infty} a_n x^n \) is a solution. Then you have

\[
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad xy' = \sum_{n=1}^{\infty} n a_n x^n, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.
\]

Substituting for \( y'' \), \( xy' \), and \( y \) in the given differential equation, you obtain the following series.

\[
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0
\]

To obtain equal powers of \( x \), adjust the summation indices by replacing \( n \) by \( n + 2 \) in the left-hand sum, to obtain

\[
\sum_{n=0}^{\infty} (n + 2)(n + 1) a_{n+2} x^n = - \sum_{n=0}^{\infty} (n + 1) a_n x^n.
\]

By equating coefficients, you have \( (n + 2)(n + 1) a_{n+2} = -(n + 1) a_n \), from which you obtain the recursion formula

\[
a_{n+2} = -\frac{(n + 1)}{(n + 2)(n + 1)} a_n = -\frac{a_n}{n + 2}, \quad n \geq 0,
\]

and the coefficients of the solution series are as follows.

\[
a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{3}, \quad a_4 = -\frac{a_2}{4} = \frac{a_0}{2 \cdot 4}, \quad a_5 = -\frac{a_3}{5} = \frac{a_1}{3 \cdot 5}, \quad a_6 = -\frac{a_4}{6} = -\frac{a_0}{2 \cdot 4 \cdot 6}, \quad a_7 = -\frac{a_5}{7} = -\frac{a_1}{3 \cdot 5 \cdot 7}, \quad \ldots
\]

\[
a_{2k} = \frac{(-1)^k a_0}{2 \cdot 4 \cdot 6 \cdots (2k)} = \frac{(-1)^k a_0}{2^k k!}, \quad a_{2k+1} = \frac{(-1)^k a_1}{3 \cdot 5 \cdot 7 \cdots (2k + 1)}
\]

So, you can represent the general solution as the sum of two series—one for the even-powered terms with coefficients in terms of \( a_0 \) and one for the odd-powered terms with coefficients in terms of \( a_1 \).

\[
y = a_0 \left( 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \cdots \right) + a_1 \left( x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \cdots \right)
\]

\[
= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{3 \cdot 5 \cdot 7 \cdots (2k + 1)}
\]

The solution has two arbitrary constants, \( a_0 \) and \( a_1 \), as you would expect in the general solution of a second-order differential equation.
Approximation by Taylor Series

A second type of series solution method involves a differential equation with initial conditions and makes use of Taylor series.

**EXAMPLE 3  Approximation by Taylor Series**

Use a Taylor series to find the series solution of

\[ y' = y^2 - x \]

given the initial condition \( y = 1 \) when \( x = 0 \). Then, use the first six terms of this series solution to approximate values of \( y \) for \( 0 \leq x \leq 1 \).

**Solution**  Recall that, for \( c = 0 \),

\[ y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots. \]

Because \( y(0) = 1 \) and \( y' = y^2 - x \), you obtain the following.

\[
\begin{align*}
  y(0) &= 1 \\
  y'(0) &= 1 \\
  y'' &= 2yy' - 1 \\
  y'' &= 2yy'' + 2(y')^2 \\
  y^{(4)} &= 2yy^{(4)} + 6y'y'' \\
  y^{(5)} &= 2yy^{(5)} + 8y'y''' + 6(y')^2
\end{align*}
\]

Because \( y''(0) = 2 - 1 = 1 \)

\[
\begin{align*}
  y'''(0) &= 2 + 2 = 4 \\
  y^{(4)}(0) &= 8 + 6 = 14 \\
  y^{(5)}(0) &= 28 + 32 + 6 = 66
\end{align*}
\]

Therefore, you can approximate the values of the solution from the series

\[
y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots
\]

\[
= 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \cdots.
\]

Using the first six terms of this series, you can compute values for \( y \) in the interval \( 0 \leq x \leq 1 \), as shown in the below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1.0000</td>
<td>1.1057</td>
<td>1.2264</td>
<td>1.3691</td>
<td>1.5432</td>
<td>1.7620</td>
<td>2.0424</td>
<td>2.4062</td>
<td>2.8805</td>
<td>3.4985</td>
<td>4.3000</td>
</tr>
</tbody>
</table>
In Exercises 1–6, verify that the power series solution of the differential equation is equivalent to the solution found using perviously learned solution techniques.

1. \( y' - y = 0 \)
2. \( y' - ky = 0 \)
3. \( y'' - 9y = 0 \)
4. \( y'' - k^2y = 0 \)
5. \( y'' + 4y = 0 \)
6. \( y'' + k^2y = 0 \)

In Exercises 7–10, use power series to solve the differential equation and find the interval of convergence of the series.

7. \( y' + 3xy = 0 \)
8. \( y' - 2xy = 0 \)
9. \( y' - xy' = 0 \)
10. \( y'' - xy' - y = 0 \)

In Exercises 11 and 12, find the first three terms of each of the power series representing independent solutions of the differential equation.

11. \((x^2 + 4)y'' + y = 0\)
12. \(y'' + x^2y = 0\)

In Exercises 13 and 14, use Taylor’s Theorem to find the series solution of the differential equation under the specified initial conditions. Use \( n \) terms of the series to approximate \( y \) for the given value of \( x \) and compare the result with the approximation given by Euler’s Method for \( \Delta x = 0.1 \).

13. \( y' + (2x - 1)y = 0 \), \( y(0) = 2 \), \( n = 5 \), \( x = \frac{1}{2} \)
14. \( y' - 2xy = 0 \), \( y(0) = 1 \), \( n = 4 \), \( x = 1 \)

15. **Investigation** Consider the differential equation

\[ y'' + 9y = 0 \]

with initial conditions \( y(0) = 2 \) and \( y'(0) = 6 \).

(a) Find the solution of the differential equation using the techniques of Section G.2.
(b) Find the series solution of the differential equation.
(c) The figure shows the graph of the solution of the differential equation and the third-degree and fifth-degree polynomial approximations of the solution. Identify each.

16. Consider the differential equation \( y'' - xy' = 0 \) with the initial conditions \( y(0) = 0 \) and \( y'(0) = 2 \). (See Exercise 9.)

(a) Find the series solution satisfying the initial conditions.
(b) Use a graphing utility to graph the third-degree and fifth-degree series approximations of the solution. Identify the approximations.
(c) Identify the symmetry of the solution.

In Exercises 17 and 18, use Taylor’s Theorem to find the series solution of the differential equation under the specified initial conditions. Use \( n \) terms of the series to approximate \( y \) for the given value of \( x \).

17. \( y'' - 2xy = 0 \), \( y(0) = 1 \), \( y'(0) = -3 \), \( n = 6 \), \( x = \frac{1}{2} \)
18. \( y'' - 2xy' + y = 0 \), \( y(0) = 1 \), \( y'(0) = 2 \), \( n = 8 \), \( x = \frac{1}{2} \)

In Exercises 19–22, verify that the series converges to the given function on the indicated interval. (Hint: Use the given differential equation.)

19. \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \), \( (-\infty, \infty) \)

Differential equation: \( y' - y = 0 \)

20. \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x \), \( (-\infty, \infty) \)

Differential equation: \( y'' + y = 0 \)

21. \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n + 1} = \arctan x \), \( (-1, 1) \)

Differential equation: \( (x^2 + 1)y'' + 2xy' = 0 \)

22. \( \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{(2n)^n (2n + 1)} = \arcsin x \), \( (-1, 1) \)

Differential equation: \( (1 - x^2)y'' - xy' = 0 \)

23. Find the first six terms in the series solution of Airy’s equation \( y'' - xy = 0 \).