8.5 UNITARY AND HERMITIAN MATRICES

Problems involving diagonalization of complex matrices and the associated eigenvalue problems require the concept of **unitary** and **Hermitian** matrices. These matrices roughly correspond to orthogonal and symmetric real matrices. In order to define unitary and Hermitian matrices, the concept of the **conjugate transpose** of a complex matrix must first be introduced.

Definition of the Conjugate Transpose of a Complex Matrix

The **conjugate transpose** of a complex matrix A, denoted by A^*, is given by

$$A^* = \overline{A}^T$$

where the entries of $\overline{A}$ are the complex conjugates of the corresponding entries of A.

Note that if A is a matrix with real entries, then $A^* = A^T$. To find the conjugate transpose of a matrix, first calculate the complex conjugate of each entry and then take the transpose of the matrix, as shown in the following example.

EXAMPLE 1 ***Finding the Conjugate Transpose of a Complex Matrix***

Determine A^* for the matrix

$$A = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix}.$$

Solution

$$\overline{A} = \begin{bmatrix} \overline{3+7i} & \overline{0} \\ \overline{2i} & \overline{4-i} \end{bmatrix} = \begin{bmatrix} 3-7i & 0 \\ -2i & 4+i \end{bmatrix}$$

$$A^* = \overline{A}^T = \begin{bmatrix} 3-7i & -2i \\ 0 & 4+i \end{bmatrix}$$

Several properties of the conjugate transpose of a matrix are listed in the following theorem. The proofs of these properties are straightforward and are left for you to supply in Exercises 49–52.

Theorem 8.8

Properties of Conjugate Transpose

If A and B are complex matrices and k is a complex number, then the following properties are true.

1. $(A^*)^* = A$
2. $(A + B)^* = A^* + B^*$
3. $(kA)^* = \overline{k}A^*$
4. $(AB)^* = B^*A^*$

Unitary Matrices

Recall that a real matrix A is *orthogonal* if and only if $A^{-1} = A^T$. In the complex system, matrices having the property that $A^{-1} = A^*$ are more useful and such matrices are called **unitary.**

Definition of a Unitary Matrix

A complex matrix A is **unitary** if

$$A^{-1} = A^*.$$

EXAMPLE 2 ***A Unitary Matrix***

Show that the matrix is unitary.

$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+1 \end{bmatrix}$$

Solution Because

$$AA^* = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I_2,$$

you can conclude that $A^* = A^{-1}$. So, A is a unitary matrix.

In Section 7.3, you saw that a real matrix is orthogonal if and only if its row (or column) vectors form an orthonormal set. For complex matrices, this property characterizes matrices that are unitary. Note that a set of vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$$

in C^n (complex Euclidean space) is called orthonormal if the following are true.

1. $\|\mathbf{v}_i\| = 1,\ i = 1, 2, \ldots, m$
2. $\mathbf{v}_i \cdot \mathbf{v}_j = 0,\ \ i \neq j$

The proof of the following theorem is similar to the proof of Theorem 7.8 given in Section 7.3.

Theorem 8.9
Unitary Matrices

An $n \times n$ complex matrix A is unitary if and only if its row (or column) vectors form an orthonormal set in C^n.

EXAMPLE 3 ***The Row Vectors of a Unitary Matrix***

Show that the complex matrix is unitary by showing that its set of row vectors form an orthonormal set in C^3.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1+i}{2} & -\frac{1}{2} \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{5i}{2\sqrt{15}} & \frac{3+i}{2\sqrt{15}} & \frac{4+3i}{2\sqrt{15}} \end{bmatrix}$$

Solution Let $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ be defined as follows.

$$\mathbf{r}_1 = \left(\frac{1}{2}, \frac{1+i}{2}, -\frac{1}{2}\right)$$

$$\mathbf{r}_2 = \left(-\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\mathbf{r}_3 = \left(\frac{5i}{2\sqrt{15}}, \frac{3+i}{2\sqrt{15}}, \frac{4+3i}{2\sqrt{15}}\right)$$

The length of $\mathbf{r}_1$ is

$$\begin{aligned}\|\mathbf{r}_1\| &= (\mathbf{r}_1 \cdot \mathbf{r}_1)^{1/2} \\ &= \left[\left(\frac{1}{2}\right)\overline{\left(\frac{1}{2}\right)} + \left(\frac{1+i}{2}\right)\overline{\left(\frac{1+i}{2}\right)} + \left(\frac{-1}{2}\right)\overline{\left(\frac{-1}{2}\right)}\right]^{1/2} \\ &= \left[\frac{1}{4} + \frac{2}{4} + \frac{1}{4}\right]^{1/2} = 1.\end{aligned}$$

The vectors $\mathbf{r}_2$ and $\mathbf{r}_3$ can also be shown to be unit vectors. The inner product of $\mathbf{r}_1$ and $\mathbf{r}_2$ is given by

$$\begin{aligned}\mathbf{r}_1 \cdot \mathbf{r}_2 &= \left(\frac{1}{2}\right)\overline{\left(\frac{-i}{\sqrt{3}}\right)} + \left(\frac{1+i}{2}\right)\overline{\left(\frac{i}{\sqrt{3}}\right)} + \left(\frac{-1}{2}\right)\overline{\left(\frac{1}{\sqrt{3}}\right)}\\ &= \left(\frac{1}{2}\right)\left(\frac{i}{\sqrt{3}}\right) + \left(\frac{1+i}{2}\right)\left(\frac{-i}{\sqrt{3}}\right) + \left(\frac{-1}{2}\right)\left(\frac{1}{\sqrt{3}}\right)\\ &= \frac{i}{2\sqrt{3}} - \frac{i}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}}\\ &= 0.\end{aligned}$$

Similarly, $\mathbf{r}_1 \cdot \mathbf{r}_3 = 0$ and $\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$. So, you can conclude that $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is an orthonormal set. (Try showing that the column vectors of A also form an orthonormal set in C^3.)

Hermitian Matrices

A real matrix is called symmetric if it is equal to its own transpose. In the complex system, the more useful type of matrix is one that is equal to its own *conjugate* transpose. Such a matrix is called **Hermitian** after the French mathematician Charles Hermite (1822–1901).

Definition of a Hermitian Matrix

A square matrix A is **Hermitian** if

$$A = A^*.$$

As with symmetric matrices, you can easily recognize Hermitian matrices by inspection. To see this, consider the 2×2 matrix A.

$$A = \begin{bmatrix} a_1 + a_2i & b_1 + b_2i \\ c_1 + c_2i & d_1 + d_2i \end{bmatrix}$$

The conjugate transpose of A has the form

$$\begin{aligned}A^* &= \bar{A}^T\\ &= \begin{bmatrix} \overline{a_1 + a_2i} & \overline{c_1 + c_2i} \\ \overline{b_1 + b_2i} & \overline{d_1 + d_2i} \end{bmatrix}\\ &= \begin{bmatrix} a_1 - a_2i & c_1 - c_2i \\ b_1 - b_2i & d_1 - d_2i \end{bmatrix}.\end{aligned}$$

If A is Hermitian, then $A = A^*$. So, you can conclude that A must be of the form

$$A = \begin{bmatrix} a_1 & b_1 + b_2i \\ b_1 - b_2i & d_1 \end{bmatrix}.$$

Similar results can be obtained for Hermitian matrices of order $n \times n$. In other words, a square matrix A is Hermitian if and only if the following two conditions are met.

1. The entries on the main diagonal of A are real.
2. The entry a_{ij} in the ith row and the jth column is the complex conjugate of the entry a_{ji} in the jth row and ith column.

EXAMPLE 4 ***Hermitian Matrices***

Which matrices are Hermitian?

(a) $\begin{bmatrix} 1 & 3-i \\ 3+i & i \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3-2i \\ 3-2i & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$

Solution (a) This matrix is not Hermitian because it has an imaginary entry on its main diagonal.

(b) This matrix is symmetric but not Hermitian because the entry in the first row and second column is not the complex conjugate of the entry in the second row and first column.

(c) This matrix is Hermitian.

(d) This matrix is Hermitian, because all real symmetric matrices are Hermitian.

One of the most important characteristics of Hermitian matrices is that their eigenvalues are real. This is formally stated in the next theorem.

Theorem 8.10
The Eigenvalues of a Hermitian Matrix

If A is a Hermitian matrix, then its eigenvalues are real numbers.

Proof Let λ be an eigenvalue of A and

$$\mathbf{v} = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$$

be its corresponding eigenvector. If both sides of the equation $A\mathbf{v} = \lambda\mathbf{v}$ are multiplied by the row vector $\mathbf{v}^*$, then

$$\mathbf{v}^*A\mathbf{v} = \mathbf{v}^*(\lambda\mathbf{v}) = \lambda(\mathbf{v}^*\mathbf{v}) = \lambda(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots + a_n^2 + b_n^2).$$

Furthermore, because

$$(\mathbf{v}^*A\mathbf{v})^* = \mathbf{v}^*A^*(\mathbf{v}^*)^* = \mathbf{v}^*A\mathbf{v},$$

it follows that $\mathbf{v}^*A\mathbf{v}$ is a Hermitian 1×1 matrix. This implies that $\mathbf{v}^*A\mathbf{v}$ is a real number, thus λ is real.

REMARK: Note that this theorem implies that the eigenvalues of a *real symmetric matrix* are real, as stated in Theorem 7.7.

To find the eigenvalues of complex matrices, follow the same procedure as for real matrices.

EXAMPLE 5 *Finding the Eigenvalues of a Hermitian Matrix*

Find the eigenvalues of the matrix A.

$$A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{bmatrix} \lambda - 3 & -2+i & 3i \\ -2-i & \lambda & -1+i \\ -3i & -1-i & \lambda \end{bmatrix} \\ &= (\lambda - 3)(\lambda^2 - 2) - (-2+i)[(-2-i)\lambda - (3i+3)] \\ &\quad + 3i[(1+3i) + 3\lambda i] \\ &= (\lambda^3 - 3\lambda^2 - 2\lambda + 6) - (5\lambda + 9 + 3i) + (3i - 9 - 9\lambda) \\ &= \lambda^3 - 3\lambda^2 - 16\lambda - 12 \\ &= (\lambda + 1)(\lambda - 6)(\lambda + 2). \end{aligned}$$

This implies that the eigenvalues of A are -1, 6, and -2.

To find the eigenvectors of a complex matrix, use a similar procedure to that used for a real matrix. For instance, in Example 5, the eigenvector corresponding to the eigenvalue $\lambda = -1$ is obtained by solving the following equation.

$$\begin{bmatrix} \lambda - 3 & -2+i & 3i \\ -2-i & \lambda & -1+i \\ -3i & -1-i & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2+i & 3i \\ -2-i & -1 & -1+i \\ -3i & -1-i & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Jordan elimination, or a computer or calculator, obtain the following eigenvector corresponding to $\lambda_1 = -1$.

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 + 2i \\ 1 \end{bmatrix} (\lambda_1 = -1)$$

Eigenvectors for $\lambda_2 = 6$ and $\lambda_3 = -2$ can be found in a similar manner. They are

$$\begin{bmatrix} 1 - 21i \\ 6 - 9i \\ 13 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 + 3i \\ -2 - i \\ 5 \end{bmatrix}, \quad \text{respectively.}$$

TECHNOLOGY NOTE

Some computers and calculators have built-in programs for finding the eigenvalues and corresponding eigenvectors of complex matrices. For example, on the *TI-86*, the *eigVl* key on the *matrix math* menu calculates the eigenvalues of the matrix *A*, and the *eigVc* key gives the corresponding eigenvectors.

Just as you saw in Section 7.3 that real symmetric matrices were orthogonally diagonalizable, you will see now that Hermitian matrices are **unitarily diagonalizable.** A square matrix A is unitarily diagonalizable if there exists a unitary matrix P such that

$$P^{-1}AP$$

is a diagonal matrix. Because P is unitary, $P^{-1} = P^*$, so an equivalent statement is that A is unitarily diagonalizable if there exists a unitary matrix P such that P^*AP is a diagonal matrix. The next theorem states that Hermitian matrices are unitarily diagonalizable.

Theorem 8.11 Hermitian Matrices and Diagonalization

If A is an $n \times n$ Hermitian matrix, then

1. eigenvectors corresponding to distinct eigenvalues are orthogonal.
2. A is unitarily diagonalizable.

Proof To prove part 1, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two eigenvectors corresponding to the distinct (and real) eigenvalues λ_1 and λ_2. Because $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, you have the following equations for the matrix product $(A\mathbf{v}_1)^*\mathbf{v}_2$.

$$(A\mathbf{v}_1)^*\mathbf{v}_2 = \mathbf{v}_1^*A^*\mathbf{v}_2 = \mathbf{v}_1^*A\mathbf{v}_2 = \mathbf{v}_1^*\lambda_2\mathbf{v}_2 = \lambda_2\mathbf{v}_1^*\mathbf{v}_2$$
$$(A\mathbf{v}_1)^*\mathbf{v}_2 = (\lambda_1\mathbf{v}_1)^*\mathbf{v}_2 = \mathbf{v}_1^*\lambda_1\mathbf{v}_2 = \lambda_1\mathbf{v}_1^*\mathbf{v}_2$$

So,

$$\begin{aligned} \lambda_2\mathbf{v}_1^*\mathbf{v}_2 - \lambda_1\mathbf{v}_1^*\mathbf{v}_2 &= 0 \\ (\lambda_2 - \lambda_1)\mathbf{v}_1^*\mathbf{v}_2 &= 0 \\ \mathbf{v}_1^*\mathbf{v}_2 &= 0 \qquad \text{because } \lambda_1 \neq \lambda_2, \end{aligned}$$

and this shows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Part 2 of Theorem 8.11 is often called the **Spectral Theorem,** and its proof is left to you.

EXAMPLE 6 *The Eigenvectors of a Hermitian Matrix*

The eigenvectors of the Hermitian matrix given in Example 5 are mutually orthogonal because the eigenvalues are distinct. You can verify this by calculating the Euclidean inner products $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_1 \cdot \mathbf{v}_3$, and $\mathbf{v}_2 \cdot \mathbf{v}_3$. For example,

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= (-1)\overline{(1 - 21i)} + (1 + 2i)\overline{(6 - 9i)} + (1)\overline{(13)}\\ &= (-1)(1 + 21i) + (1 + 2i)(6 + 9i) + 13\\ &= -1 - 21i + 6 + 12i + 9i - 18 + 13\\ &= 0.\end{aligned}$$

The other two inner products $\mathbf{v}_1 \cdot \mathbf{v}_3$ and $\mathbf{v}_2 \cdot \mathbf{v}_3$ can be shown to equal zero in a similar manner.

The three eigenvectors in Example 6 are mutually orthogonal because they correspond to distinct eigenvalues of the Hermitian matrix A. Two or more eigenvectors corresponding to the same eigenvector may not be orthogonal. However, once any set of linearly independent eigenvectors is obtained for a given eigenvalue, the Gram-Schmidt orthonormalization process can be used to find an orthogonal set.

EXAMPLE 7 *Diagonalization of a Hermitian Matrix*

Find a unitary matrix P such that P^*AP is a diagonal matrix where

$$A = \begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}.$$

Solution The eigenvectors of A are given after Example 5. Form the matrix P by normalizing these three eigenvectors and using the results to create the columns of P. So, because

$$\begin{aligned}\|\mathbf{v}_1\| &= \|(-1, 1 + 2i, 1)\| = \sqrt{1 + 5 + 1} = \sqrt{7}\\ \|\mathbf{v}_2\| &= \|(1 - 21i, 6 - 9i, 13)\| = \sqrt{442 + 117 + 169} = \sqrt{728}\\ \|\mathbf{v}_3\| &= \|(1 + 3i, -2 - i, 5)\| = \sqrt{10 + 5 + 25} = \sqrt{40},\end{aligned}$$

the unitary matrix P is obtained.

$$P = \begin{bmatrix} -\dfrac{1}{\sqrt{7}} & \dfrac{1 - 21i}{\sqrt{728}} & \dfrac{1 + 3i}{\sqrt{40}} \\ \dfrac{1 + 2i}{\sqrt{7}} & \dfrac{6 - 9i}{\sqrt{728}} & \dfrac{-2 - i}{\sqrt{40}} \\ \dfrac{1}{\sqrt{7}} & \dfrac{13}{\sqrt{728}} & \dfrac{5}{\sqrt{40}} \end{bmatrix}$$

Try computing the product P^*AP for the matrices A and P in Example 7 to see that you obtain

$$P^*AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

where -1, 6, and -2 are the eigenvalues of A.

You have seen that Hermitian matrices are unitarily diagonalizable. However, it turns out that there is a larger class of matrices, called **normal** matrices, which are also unitarily diagonalizable. A square complex matrix A is **normal** if it commutes with its conjugate transpose: $AA^* = A^*A$. The main theorem of normal matrices states that a complex matrix A is normal if and only if it is unitarily diagonalizable. You are asked to explore normal matrices further in Exercise 59.

The properties of complex matrices described in this section are comparable to the properties of real matrices discussed in Chapter 7. The following summary indicates the correspondence between unitary and Hermitian complex matrices when compared with orthogonal and symmetric real matrices.

Comparison of Hermitian and Symmetric Matrices

A is a symmetric matrix *(Real)*	*A is a Hermitian matrix* *(Complex)*
1. Eigenvalues of A are real.	1. Eigenvalues of A are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.	2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. There exists an orthogonal matrix P such that P^TAP is diagonal.	3. There exists a unitary matrix P such that P^*AP is diagonal.

SECTION 8.5 ❑ EXERCISES

In Exercises 1–8, determine the conjugate transpose of the matrix.

1. $A = \begin{bmatrix} i & -i \\ 2 & 3i \end{bmatrix}$

2. $A = \begin{bmatrix} 1+2i & 2-i \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 4+3i & 2+i \\ 2-i & 6i \end{bmatrix}$

5. $A = \begin{bmatrix} 0 & 5+i & \sqrt{2}i \\ 5-i & 6 & 4 \\ -\sqrt{2}i & 4 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 2+i & 3-i & 4+5i \\ 3-i & 2 & 6-2i \end{bmatrix}$

7. $A = \begin{bmatrix} 7+5i \\ 2i \\ 4 \end{bmatrix}$

8. $A = \begin{bmatrix} 2 & i \\ 5 & 3i \\ 0 & 6-i \end{bmatrix}$

In Exercises 9–12, explain why the matrix is *not* unitary.

9. $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

11. $A = \begin{bmatrix} \frac{1+i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1+i}{2} \\ -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1+i}{2} \end{bmatrix}$

In Exercises 13–18, determine whether A is unitary by calculating AA^*.

13. $A = \begin{bmatrix} 1+i & 1+i \\ 1-i & 1-i \end{bmatrix}$

14. $A = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$

15. $A = I_n$

16. $A = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$

17. $A = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{6}} \end{bmatrix}$

18. $A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5}i & \frac{4}{5}i \end{bmatrix}$

In Exercises 19–22. (a) verify that A is unitary by showing that its rows are orthonormal, and (b) determine the inverse of A.

19. $A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5}i \\ \frac{3}{5} & \frac{4}{5}i \end{bmatrix}$

20. $A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

21. $A = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3}-i & 1+\sqrt{3}i \\ \sqrt{3}+i & 1-\sqrt{3}i \end{bmatrix}$

22. $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1+i}{\sqrt{6}} & 0 & \frac{1-i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$

In Exercises 23–28, determine whether the matrix A is Hermitian.

23. $A = \begin{bmatrix} 0 & 2+i & 1 \\ 2-i & i & 0 \\ 1 & 0 & 1 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

25. $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 2+i & 3-i \\ 2-i & 2 & 3+i \end{bmatrix}$

27. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & \sqrt{2}+i & 5 \\ \sqrt{2}-i & 2 & 3+i \\ 5 & 3-i & 6 \end{bmatrix}$

In Exercises 29–34, determine the eigenvalues of the matrix A.

29. $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

30. $A = \begin{bmatrix} 0 & 2+i \\ 2-i & 4 \end{bmatrix}$

31. $A = \begin{bmatrix} 3 & 1-i \\ 1+i & 2 \end{bmatrix}$

32. $A = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix}$

33. $A = \begin{bmatrix} 2 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}$

34. $A = \begin{bmatrix} 1 & 4 & 1-i \\ 0 & i & 3i \\ 0 & 0 & 2+i \end{bmatrix}$

In Exercises 35–38, determine the eigenvectors of the matrix.

35. The matrix in Exercise 29.

36. The matrix in Exercise 30.

37. The matrix in Exercise 33.

38. The matrix in Exercise 32.

In Exercises 39–43, find a unitary matrix P that diagonalizes the matrix A.

39. $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

40. $A = \begin{bmatrix} 0 & 2+i \\ 2-i & 4 \end{bmatrix}$

33. $A = \begin{bmatrix} 2 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}$

42. $A = \begin{bmatrix} 4 & 2+2i \\ 2-2i & 6 \end{bmatrix}$

43. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$

44. Let z be a complex number with modulus 1. Show that the matrix A is unitary.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{bmatrix}$$

In Exercises 45–48, use the result of Exercise 44 to determine a, b, and c so that A is unitary.

45. $A = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & a \\ b & c \end{bmatrix}$

46. $A = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{3-4i}{5} & a \\ b & c \end{bmatrix}$

47. $A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & a \\ b & c \end{bmatrix}$

48. $A = \frac{1}{\sqrt{2}} \begin{bmatrix} a & \frac{6+3i}{\sqrt{45}} \\ b & c \end{bmatrix}$

In Exercises 49–52, prove the given formula, where A and B are $n \times n$ complex matrices.

49. $(A^*)^* = A$

50. $(A + B)^* = A^* + B^*$

51. $(kA)^* = \bar{k}A^*$

52. $(AB)^* = B^*A^*$

53. Let A be a matrix such that $A^* + A = O$. Prove that iA is Hermitian.

54. Show that $\det(\bar{A}) = \overline{\det(A)}$, where A is a 2×2 matrix.

In Exercises 55–56, assume that the result of Exercise 54 is true for matrices of any size.

55. Show that $\det(A^*) = \overline{\det(A)}$.

56. Prove that if A is unitary, then $|\det(\mathrm{A})| = 1$.

57. (a) Prove that every Hermitian matrix A can be written as the sum $A = B + iC$, where B is a real symmetric matrix and C is real and skew-symmetric.

(b) Use part (a) to write the matrix

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

as a sum $A = B + iC$, where B is a real symmetric matrix and C is real and skew-symmetric.

(c) Prove that every $n \times n$ complex matrix A can be written as $A = B + iC$, where B and C are Hermitian.

(d) Use part (c) to write the complex matrix

$$A = \begin{bmatrix} i & 2 \\ 2+i & 1-2i \end{bmatrix}$$

as a sum $A = B + iC$, where B and C are Hermitian.

58. Determine which of the following sets are subspaces of the vector space of $n \times n$ complex matrices.

(a) The set of $n \times n$ Hermitian matrices.

(b) The set of $n \times n$ unitary matrices.

(c) The set of $n \times n$ normal matrices.

59. (a) Prove that every Hermitian matrix is normal.

(b) Prove that every unitary matrix is normal.

(c) Find a 2×2 matrix that is Hermitian, but not unitary.

(d) Find a 2×2 matrix that is unitary, but not Hermitian.

(e) Find a 2×2 matrix that is normal, but neither Hermitian nor unitary.

(f) Find the eigenvalues and corresponding eigenvectors of your matrix from part (e).

(g) Show that the complex matrix

$$\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$$

is not diagonalizable. Is this matrix normal?