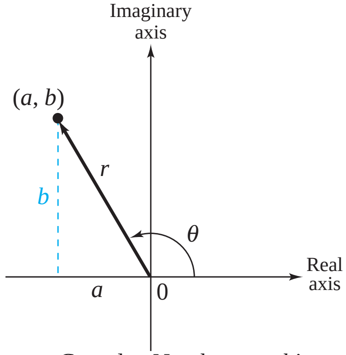
8.3 POLAR FORM AND DEMOIVRE'S THEOREM

Figure 8.6



Complex Number: $a + bi$
Rectangular Form: (a, b)
Polar Form: (r, θ)

At this point you can add, subtract, multiply, and divide complex numbers. However, there is still one basic procedure that is missing from the algebra of complex numbers. To see this, consider the problem of finding the square root of a complex number such as i. When you use the four basic operations (addition, subtraction, multiplication, and division), there seems to be no reason to guess that

$$\sqrt{i} = \frac{1 + i}{\sqrt{2}}. \quad \text{That is,} \quad \left(\frac{1 + i}{\sqrt{2}}\right)^2 = i.$$

To work effectively with *powers* and *roots* of complex numbers, it is helpful to use a polar representation for complex numbers, as shown in Figure 8.6. Specifically, if $a + bi$ is a nonzero complex number, then let θ be the angle from the positive x-axis to the radial line passing through the point (a, b) and let r be the modulus of $a + bi$. So,

$$a = r\cos\theta, \qquad b = r\sin\theta, \qquad \text{and} \qquad r = \sqrt{a^2 + b^2}$$

and you have $a + bi = (r\cos\theta) + (r\sin\theta)i$ from which the following **polar form** of a complex number is obtained.

Definition of Polar Form of a Complex Number

The **polar form** of the nonzero complex number $z = a + bi$ is given by

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = b/a$. The number r is the **modulus** of z and θ is the **argument** of z.

REMARK: The polar form of $z = 0$ is given by $z = 0(\cos\theta + i\sin\theta)$ where θ is any angle.

Because there are infinitely many choices for the argument, the polar form of a complex number is not unique. Normally, the values of θ that lie between $-\pi$ and π are used, though on occasion it is convenient to use other values. The value of θ that satisfies the inequality

$$-\pi < \theta \leq \pi \qquad \text{Principal argument}$$

is called the **principal argument** and is denoted by Arg(z). Two nonzero complex numbers in polar form are equal if and only if they have the same modulus and the same principal argument.

EXAMPLE 1 ***Finding the Polar Form of a Complex Number***

Find the polar form of each of the complex numbers. (Use the principal argument.)

(a) $1 - i$ (b) $2 + 3i$ (c) i

Solution

(a) Because $a = 1$ and $b = -1$, then $r^2 = 1^2 + (-1)^2 = 2$, which implies that $r = \sqrt{2}$. From $a = r\cos\theta$ and $b = r\sin\theta$,

$$\cos\theta = \frac{a}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\theta = \frac{b}{r} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

So, $\theta = -\pi/4$ and

$$z = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right].$$

(b) Because $a = 2$ and $b = 3$, then $r^2 = 2^2 + 3^2 = 13$, which implies that $r = \sqrt{13}$. So,

$$\cos\theta = \frac{a}{r} = \frac{2}{\sqrt{13}} \quad \text{and} \quad \sin\theta = \frac{b}{r} = \frac{3}{\sqrt{13}}$$

and it follows that $\theta = \arctan(3/2)$. So, the polar form is

$$z = \sqrt{13}\left[\cos\left(\arctan\frac{3}{2}\right) + i\sin\left(\arctan\frac{3}{2}\right)\right]$$

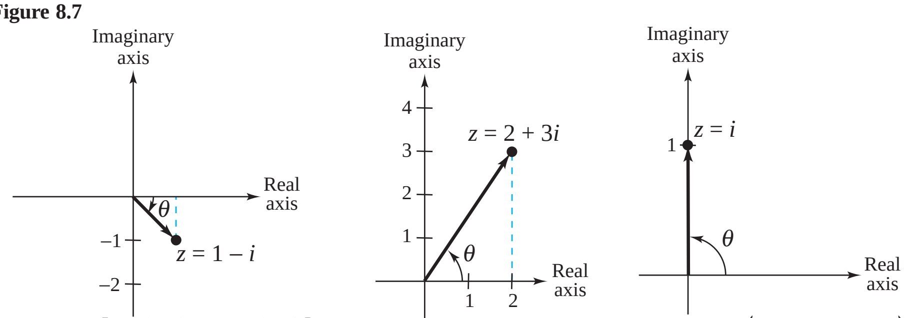
$$\approx \sqrt{13}[\cos(0.98) + i\sin(0.98)].$$

(c) Because $a = 0$ and $b = 1$, it follows that $r = 1$ and $\theta = \pi/2$, so

$$z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

The polar forms derived in parts (a), (b), and (c) are depicted graphically in Figure 8.7.

Figure 8.7



(a) $z = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ (b) $z = \sqrt{13}[\cos(0.98) + i\sin(0.98)]$ (c) $z = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

EXAMPLE 2 *Converting from Polar to Standard Form*

Express the complex number in standard form.

$$z = 8\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

Solution Because $\cos(-\pi/3) = 1/2$ and $\sin(-\pi/3) = -\sqrt{3}/2$, you can obtain the standard form

$$z = 8\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 8\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = 4 - 4\sqrt{3}i.$$

The polar form adapts nicely to multiplication and division of complex numbers. Suppose you are given two complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then the product of z_1 and z_2 is given by

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]. \end{aligned}$$

Using the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

and

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

you have

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

This establishes the first part of the following theorem. The proof of the second part is left to you. (See Exercise 63.)

Theorem 8.4

Product and Quotient of Two Complex Numbers

Given two complex numbers in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r^2(\cos\theta_2 + i\sin\theta_2)$$

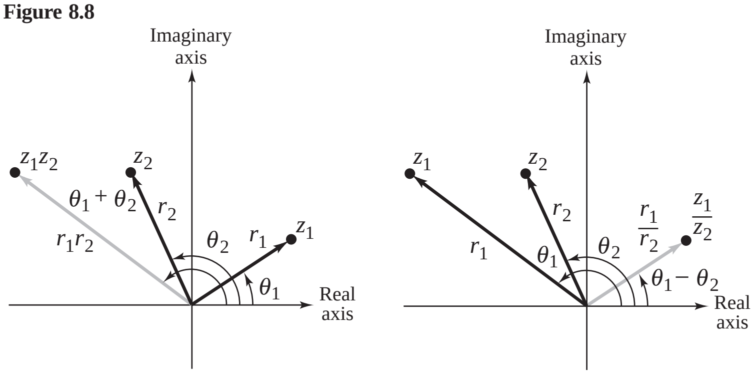
the product and quotient of the numbers are as follows.

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \textbf{Product}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0 \qquad \textbf{Quotient}$$

This theorem says that to multiply two complex numbers in polar form, multiply moduli and add arguments, and to divide two complex numbers, divide moduli and subtract arguments. (See Figure 8.8.)

Figure 8.8



To multiply z_1 and z_2:
Multiply moduli and add arguments.

To divide z_1 by z_2:
Divide moduli and add arguments.

EXAMPLE 3 ***Multiplying and Dividing in Polar Form***

Determine z_1z_2 and z_1/z_2 for the complex numbers

$$z_1 = 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = \frac{1}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

Solution Because you are given the polar forms of z_1 and z_2, you can apply Theorem 8.4 as follows.

$$z_1z_2 = \overbrace{(5)\left(\frac{1}{3}\right)}^{\text{multiply}}\left[\cos\underbrace{\left(\frac{\pi}{4} + \frac{\pi}{6}\right)}_{\text{add}} + i\sin\underbrace{\left(\frac{\pi}{4} + \frac{\pi}{6}\right)}_{\text{add}}\right] = \frac{5}{3}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

$$\frac{z_1}{z_2} = \overbrace{\frac{5}{1/3}}^{\text{divide}}\left[\cos\underbrace{\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}_{\text{subtract}} + i\sin\underbrace{\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}_{\text{subtract}}\right] = 15\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

REMARK: Try performing the multiplication and division in Example 3 using the standard forms

$$z_1 = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i \quad \text{and} \quad z_2 = \frac{\sqrt{3}}{6} + \frac{1}{6}i.$$

DeMoivre's Theorem

The final topic in this section involves procedures for finding powers and roots of complex numbers. Repeated use of multiplication in the polar form yields

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ z^2 &= r(\cos\theta + i\sin\theta)\, r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta) \\ z^3 &= r(\cos\theta + i\sin\theta)\, r^2(\cos 2\theta + i\sin 2\theta) = r^3(\cos 3\theta + i\sin 3\theta). \end{aligned}$$

Similarly,

$$\begin{aligned} z^4 &= r^4(\cos 4\theta + i\sin 4\theta) \\ z^5 &= r^5(\cos 5\theta + i\sin 5\theta). \end{aligned}$$

This pattern leads to the following important theorem, named after the French mathematician Abraham DeMoivre (1667–1754). You are asked to prove this theorem in Chapter Review Exercise 71.

Theorem 8.5

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is any positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

EXAMPLE 4 ***Raising a Complex Number to an Integer Power***

Find $\left(-1 + \sqrt{3}i\right)^{12}$ and write the result in standard form.

Solution First convert to polar form. For $-1 + \sqrt{3}i$,

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2 \quad \text{and} \quad \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

which implies that $\theta = 2\pi/3$. So,

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

By DeMoivre's Theorem,

$$\begin{aligned} \left(-1 + \sqrt{3}i\right)^{12} &= \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{12} \\ &= 2^{12}\left[\cos\frac{12(2\pi)}{3} + i\sin\frac{12(2\pi)}{3}\right] \end{aligned}$$

$$= 4096(\cos 8\pi + i \sin 8\pi)$$

$$= 4096[1 + i(0)] = 4096.$$

Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial of degree n has n zeros in the complex number system. So, a polynomial like $p(x) = x^6 - 1$ has six zeros, and in this case you can find the six zeros by factoring and using the quadratic formula.

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$$

Consequently, the zeros are

$$x = \pm 1, \qquad x = \frac{-1 \pm \sqrt{3}i}{2}, \qquad \text{and} \qquad x = \frac{1 \pm \sqrt{3}i}{2}.$$

Each of these numbers is called a sixth root of 1. In general, the nth root of a complex number is defined as follows.

Definition of nth Root of a Complex Number

The complex number $w = a + bi$ is an ***n*th root** of the complex number z if

$$z = w^n = (a + bi)^n.$$

DeMoivre's Theorem is useful in determining roots of complex numbers. To see how this is done, let w be an nth root of z, where

$$w = s(\cos \beta + i \sin \beta) \qquad \text{and} \qquad z = r(\cos \theta + i \cos \theta).$$

Then, by DeMoivre's Theorem you have $w^n = s^n(\cos n\beta + i \sin n\beta)$, and because $w^n = z$, it follows that

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Now, because the right and left sides of this equation represent equal complex numbers, you can equate moduli to obtain $s^n = r$ which implies that $s = \sqrt[n]{r}$ and equate principal arguments to conclude that θ and $n\beta$ must differ by a multiple of 2π. Note that r is a positive real number and so $s = \sqrt[n]{r}$ is also a positive real number. Consequently, for some integer k, $n\beta = \theta + 2\pi k$, which implies that

$$\beta = \frac{\theta + 2\pi k}{n}.$$

Finally, substituting this value for β into the polar form of w produces the result stated in the following theorem.

Theorem 8.6

*n*th Roots of a Complex Number

For any positive integer n, the complex number

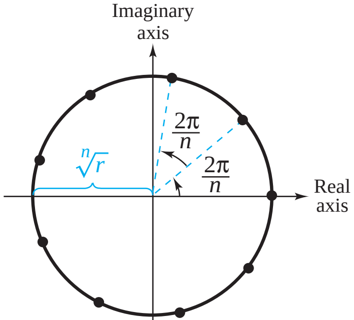
$$z = r(\cos\theta + i\sin\theta)$$

has exactly n distinct roots. These n roots are given by

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Figure 8.9



*n*th Roots of a Complex Number

REMARK: Note that when k exceeds $n - 1$, the roots begin to repeat. For instance, if $k = n$, the angle is

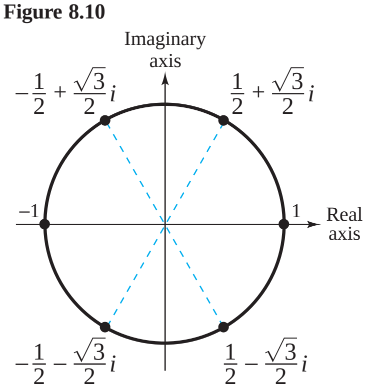
$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

which yields the same value for the sine and cosine as $k = 0$.

The formula for the nth roots of a complex number has a nice geometric interpretation, as shown in Figure 8.9. Note that because the nth roots all have the same modulus (length) $\sqrt[n]{r}$, they will lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, the n roots are equally spaced along the circle, because successive nth roots have arguments that differ by $2\pi/n$.

You have already found the sixth roots of 1 by factoring and the quadratic formula. Try solving the same problem using Theorem 8.6 to see if you get the roots shown in Figure 8.10. When Theorem 8.6 is applied to the real number 1, the nth roots are given a special name—the ***n*th roots of unity.**

Figure 8.10



6th Roots of Unity

EXAMPLE 5 ***Finding the nth Roots of a Complex Number***

Determine the fourth roots of i.

Solution In polar form, you can write i as

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

so that $r = 1$, $\theta = \pi/2$. Then, by applying Theorem 8.6, you have

$$i^{1/4} = \sqrt[4]{1}\left[\cos\left(\frac{\pi/2}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi/2}{4} + \frac{2k\pi}{4}\right)\right]$$

$$= \cos\left(\frac{\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{8} + \frac{k\pi}{2}\right).$$

Setting $k = 0, 1, 2,$ and 3 you obtain the four roots

$$z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$

$$z_2 = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}$$

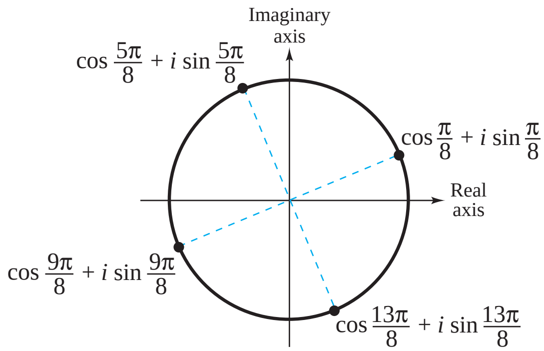
$$z_3 = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}$$

$$z_4 = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$$

as shown in Figure 8.11.

REMARK: In Figure 8.11 note that when each of the four angles, $\pi/8$, $5\pi/8$, $9\pi/8$, and $13\pi/8$ is multiplied by 4, the result is of the form $(\pi/2) + 2k\pi$.

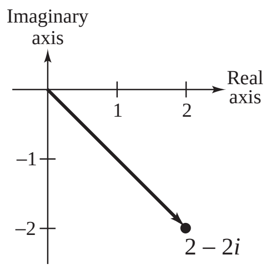
Figure 8.11



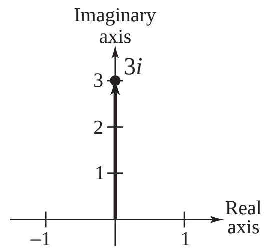
SECTION 8.3 ❑ EXERCISES

In Exercises 1–4, express the complex number in polar form.

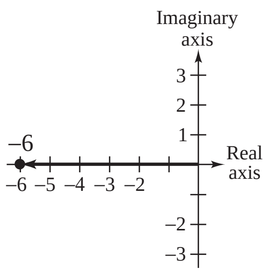
1.



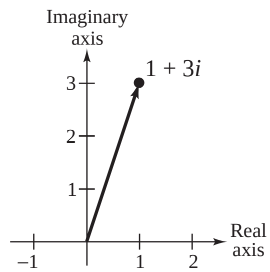
2.



3.



4.



In Exercises 5–16, represent the complex number graphically, and give the polar form of the number. (Use the principle argument.)

5. $-2 - 2i$

6. $\sqrt{3} + i$

7. $-2(1 + \sqrt{3}\,i)$

8. $\frac{5}{2}(\sqrt{3} - i)$

9. $6i$

10. 4

11. 7

12. $-2i$

13. $1 + 6i$

14. $2\sqrt{2} - i$

15. $-3 - i$

16. $-4 + 2i$

In Exercises 17–26, represent the complex number graphically, and give the standard form of the number.

17. $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

18. $5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

19. $\frac{3}{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

20. $\frac{3}{4}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

21. $3.75\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

22. $8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

23. $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

24. $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

25. $7(\cos 0 + i\sin 0)$

26. $6(\cos\pi + i\sin\pi)$

In Exercises 27–34, perform the indicated operation and leave the result in polar form.

27. $\left[3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$

28. $\left[\frac{3}{4}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]\left[6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]$

29. $[0.5(\cos\pi + i\sin\pi)][0.5(\cos[-\pi] + i\sin[-\pi])]$

30. $\left[3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]\left[\frac{1}{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]$

31. $\dfrac{2[\cos(2\pi/3) + i\sin(2\pi/3)]}{4[\cos(2\pi/9) + i\sin(2\pi/9)]}$

32. $\dfrac{\cos(5\pi/3) + i\sin(5\pi/3)}{\cos\pi + i\sin\pi}$

33. $\dfrac{12[\cos(\pi/3) + i\sin(\pi/3)]}{3[\cos(\pi/6) + i\sin(\pi/6)]}$

34. $\dfrac{9[\cos(3\pi/4) + i\sin(3\pi/4)]}{5[\cos(-\pi/4) + i\sin(-\pi/4)]}$

In Exercises 35–44, use DeMoivre's Theorem to find the indicated powers of the given complex number. Express the result in standard form.

35. $(1 + i)^4$

36. $(2 + 2i)^6$

37. $(-1 + i)^{10}$

38. $(\sqrt{3} + i)^7$

39. $(1 - \sqrt{3}i)^3$

40. $\left[5\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)\right]^3$

41. $\left[3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^4$

42. $\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)^{10}$

43. $\left[2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]^8$

44. $\left[5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right]^4$

In Exercises 45–56, (a) use DeMoivre's Theorem to find the indicated roots, (b) represent each of the roots graphically, and (c) express each of the roots in standard form.

45. Square roots: $16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

46. Square roots: $9\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

47. Fourth roots: $16\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

48. Fifth roots: $32\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

49. Square roots: $-25i$

50. Fourth roots: $625i$

51. Cube roots: $-\dfrac{125}{2}\left(1 + \sqrt{3}i\right)$

52. Cube roots: $-4\sqrt{2}(1 - i)$

53. Cube roots: 8

54. Fourth roots: i

55. Fourth roots: 1

56. Cube roots: 1000

In Exercises 57–62, find all the solutions to the equation and represent your solutions graphically.

57. $x^4 - i = 0$

58. $x^3 + 1 = 0$

59. $x^5 + 243 = 0$

60. $x^4 - 81 = 0$

61. $x^3 + 64i = 0$

62. $x^4 + i = 0$

63. Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ with $z_2 \neq 0$ prove that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

64. Show that the complex conjugate of $z = r(\cos\theta + i\sin\theta)$ is

$$\bar{z} = r[\cos(-\theta) + i\sin(-\theta)].$$

65. Use the polar form of z and $\bar{z}$ in Exercise 64 to find each of the following.

(a) $z\bar{z}$ (b) $z/\bar{z},\ \bar{z} \neq 0$

66. Show that the negative of $z = r(\cos\theta + i\sin\theta)$ is

$$-z = r[\cos(\theta + \pi) + i\sin(\theta + \pi)].$$

67. ***Writing***

(a) Let $z = r(\cos\theta + i\sin\theta) = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$. Sketch z, iz, and z/i in the complex plane.

(b) What is the geometric effect of multiplying a complex number z by i? What is the geometric effect of dividing z by i?

68. ***Calculus*** Recall that the Maclaurin series for e^x, $\sin x$, and $\cos x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

(a) Substitute $x = i\theta$ into the series for e^x and show that $e^{i\theta} = \cos\theta + i\sin\theta$.

(b) Show that any complex number $z = a + bi$ can be expressed in polar form as $z = re^{i\theta}$.

(c) Prove that if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.

(d) Prove the amazing formula $e^{i\pi} = -1$.