10.2 ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

As a numerical technique, Gaussian elimination is rather unusual because it is *direct*. That is, a solution is obtained after a single application of Gaussian elimination. Once a "solution" has been obtained, Gaussian elimination offers no method of refinement. The lack of refinements can be a problem because, as the previous section shows, Gaussian elimination is sensitive to rounding error.

Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton's iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

The Jacobi Method

The first iterative technique is called the **Jacobi method,** after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

has a unique solution and (2) that the coefficient matrix A has no zeros on its main diagonal. If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

To begin the Jacobi method, solve the first equation for x_1, the second equation for x_2, and so on, as follows.

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\ &\vdots \\ x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}) \end{aligned}$$

Then make an *initial approximation* of the solution,

$(x_1, x_2, x_3, \ldots, x_n),$ **Initial approximation**

and substitute these values of x_i into the right-hand side of the rewritten equations to obtain the *first approximation*. After this procedure has been completed, one **iteration** has been performed. In the same way, the second approximation is formed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often **converges** to the actual solution. This procedure is illustrated in Example 1.

EXAMPLE 1 ***Applying the Jacobi Method***

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, write the system in the form

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2. \end{aligned}$$

Because you do not know the actual solution, choose

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0 \qquad \text{Initial approximation}$$

as a convenient initial approximation. So, the first approximation is

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429. \end{aligned}$$

Continuing this procedure, you obtain the sequence of approximations shown in Table 10.1.

TABLE 10.1

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

Because the last two columns in Table 10.1 are identical, you can conclude that to three significant digits the solution is

$$x_1 = 0.186, \qquad x_2 = 0.331, \qquad x_3 = -0.423.$$

For the system of linear equations given in Example 1, the Jacobi method is said to **converge.** That is, repeated iterations succeed in producing an approximation that is correct to three significant digits. As is generally true for iterative methods, greater accuracy would require more iterations.

The Gauss-Seidel Method

You will now look at a modification of the Jacobi method called the Gauss-Seidel method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification is no more difficult to use than the Jacobi method, and it often requires fewer iterations to produce the same degree of accuracy.

With the Jacobi method, the values of x_i obtained in the nth approximation remain unchanged until the entire $(n + 1)$th approximation has been calculated. With the Gauss-Seidel method, on the other hand, you use the new values of each x_i as soon as they are known. That is, once you have determined x_1 from the first equation, its value is then used in the second equation to obtain the new x_2. Similarly, the new x_1 and x_2 are used in the third equation to obtain the new x_3, and so on. This procedure is demonstrated in Example 2.

EXAMPLE 2 ***Applying the Gauss-Seidel Method***

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations given in Example 1.

Solution The first computation is identical to that given in Example 1. That is, using $(x_1, x_2, x_3) = (0, 0, 0)$ as the initial approximation, you obtain the following new value for x_1.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$

Now that you have a new value for x_1, however, use it to compute a new value for x_2. That is,

$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(0) \approx 0.156.$$

Similarly, use $x_1 = -0.200$ and $x_2 = 0.156$ to compute a new value for x_3. That is,

$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.156) \approx -0.508.$$

So the first approximation is $x_1 = -0.200$, $x_2 = 0.156$, and $x_3 = -0.508$. Continued iterations produce the sequence of approximations shown in Table 10.2.

TABLE 10.2

n	0	1	2	3	4	5
x_1	0.000	−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

Note that after only five iterations of the Gauss-Seidel method, you achieved the same accuracy as was obtained with seven iterations of the Jacobi method in Example 1.

Neither of the iterative methods presented in this section always converges. That is, it is possible to apply the Jacobi method or the Gauss-Seidel method to a system of linear equations and obtain a divergent sequence of approximations. In such cases, it is said that the method **diverges.**

EXAMPLE 3 ***An Example of Divergence***

Apply the Jacobi method to the system

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6, \end{aligned}$$

using the initial approximation $(x_1, x_2) = (0, 0)$, and show that the method diverges.

Solution As usual, begin by rewriting the given system in the form

$$\begin{aligned} x_1 &= -4 + 5x_2 \\ x_2 &= -6 + 7x_1. \end{aligned}$$

Then the initial approximation (0, 0) produces

$$\begin{aligned} x_1 &= -4 + 5(0) = -4 \\ x_2 &= -6 + 7(0) = -6 \end{aligned}$$

as the first approximation. Repeated iterations produce the sequence of approximations shown in Table 10.3.

TABLE 10.3

n	0	1	2	3	4	5	6	7
x_1	0	−4	−34	−174	−1244	−6124	−42,874	−214,374
x_2	0	−6	−34	−244	−1244	−8574	−42,874	−300,124

For this particular system of linear equations you can determine that the actual solution is $x_1 = 1$ and $x_2 = 1$. So you can see from Table 10.3 that the approximations given by the Jacobi method become progressively *worse* instead of better, and you can conclude that the method diverges.

The problem of divergence in Example 3 is not resolved by using the Gauss-Seidel method rather than the Jacobi method. In fact, for this particular system the Gauss-Seidel method diverges more rapidly, as shown in Table 10.4.

TABLE 10.4

n	0	1	2	3	4	5
x_1	0	−4	−174	−6124	−214,374	−7,503,124
x_2	0	−34	−1224	−42,874	−1,500,624	−52,521,874

With an initial approximation of $(x_1, x_2) = (0, 0)$, neither the Jacobi method nor the Gauss-Seidel method converges to the solution of the system of linear equations given in Example 3. You will now look at a special type of coefficient matrix A, called a **strictly diagonally dominant matrix,** for which it is guaranteed that both methods will converge.

Definition of Strictly Diagonally Dominant Matrix

An $n \times n$ matrix A is **strictly diagonally dominant** if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row. That is,

$$\begin{aligned}
|a_{11}| &> |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\
|a_{22}| &> |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\
&\vdots \\
|a_{nn}| &> |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|.
\end{aligned}$$

EXAMPLE 4 ***Strictly Diagonally Dominant Matrices***

Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

(a)
$$\begin{aligned}
3x_1 - x_2 &= -4 \\
2x_1 + 5x_2 &= 2
\end{aligned}$$

(b)
$$\begin{aligned}
4x_1 + 2x_2 - x_3 &= -1 \\
x_1 \qquad + 2x_3 &= -4 \\
3x_1 - 5x_2 + x_3 &= 3
\end{aligned}$$

Solution (a) The coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

is strictly diagonally dominant because $|3| > |-1|$ and $|5| > |2|$.

(b) The coefficient matrix

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 2 \\ 3 & -5 & 1 \end{bmatrix}$$

is not strictly diagonally dominant because the entries in the second and third rows do not conform to the definition. For instance, in the second row $a_{21} = 1$, $a_{22} = 0$, $a_{23} = 2$, and it is not true that $|a_{22}| > |a_{21}| + |a_{23}|$. Interchanging the second and third rows in the original system of linear equations, however, produces the coefficient matrix

$$A' = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix},$$

and this matrix is strictly diagonally dominant.

The following theorem, which is listed without proof, states that strict diagonal dominance is sufficient for the convergence of either the Jacobi method or the Gauss-Seidel method.

Theorem 10.1

Convergence of the Jacobi and Gauss-Seidel Methods

If A is strictly diagonally dominant, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has a unique solution to which the Jacobi method and the Gauss-Seidel method will converge for any initial approximation.

In Example 3 you looked at a system of linear equations for which the Jacobi and Gauss-Seidel methods diverged. In the following example you can see that by interchanging the rows of the system given in Example 3, you can obtain a coefficient matrix that is strictly diagonally dominant. After this interchange, convergence is assured.

EXAMPLE 5 ***Interchanging Rows to Obtain Convergence***

Interchange the rows of the system

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{aligned}$$

to obtain one with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to four significant digits.

Solution Begin by interchanging the two rows of the given system to obtain

$$\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4. \end{aligned}$$

Note that the coefficient matrix of this system is strictly diagonally dominant. Then solve for x_1 and x_2 as follows.

$$\begin{aligned} x_1 &= \tfrac{6}{7} + \tfrac{1}{7}x_2 \\ x_2 &= \tfrac{4}{5} + \tfrac{1}{5}x_1 \end{aligned}$$

Using the initial approximation $(x_1, x_2) = (0, 0)$, you can obtain the sequence of approximations shown in Table 10.5.

TABLE 10.5

n	0	1	2	3	4	5
x_1	0.0000	0.8571	0.9959	0.9999	1.000	1.000
x_2	0.0000	0.9714	0.9992	1.000	1.000	1.000

So you can conclude that the solution is $x_1 = 1$ and $x_2 = 1$.

Do not conclude from Theorem 10.1 that strict diagonal dominance is a necessary condition for convergence of the Jacobi or Gauss-Seidel methods. For instance, the coefficient matrix of the system

$$\begin{aligned} -4x_1 + 5x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

is not a strictly diagonally dominant matrix, and yet both methods converge to the solution $x_1 = 1$ and $x_2 = 1$ when you use an initial approximation of $(x_1, x_2) = (0, 0)$. (See Exercises 21–22.)

SECTION 10.2 ❑ EXERCISES

In Exercises 1–4, apply the Jacobi method to the given system of linear equations, using the initial approximation $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$. Continue performing iterations until two successive approximations are identical when rounded to three significant digits.

1. $\begin{aligned} 3x_1 - x_2 &= 2 \\ x_1 + 4x_2 &= 5 \end{aligned}$

2. $\begin{aligned} -4x_1 + 2x_2 &= -6 \\ 3x_1 - 5x_2 &= 1 \end{aligned}$

3. $\begin{aligned} 2x_1 - x_2 \phantom{{}+ x_3} &= 2 \\ x_1 - 3x_2 + x_3 &= -2 \\ -x_1 + x_2 - 3x_3 &= -6 \end{aligned}$

4. $\begin{aligned} 4x_1 + x_2 + x_3 &= 7 \\ x_1 - 7x_2 + 2x_3 &= -2 \\ 3x_1 \phantom{{}- 7x_2} + 4x_3 &= 11 \end{aligned}$

5. Apply the Gauss-Seidel method to Exercise 1.

6. Apply the Gauss-Seidel method to Exercise 2.

7. Apply the Gauss-Seidel method to Exercise 3.

8. Apply the Gauss-Seidel method to Exercise 4.

In Exercises 9–12, show that the Gauss-Seidel method diverges for the given system using the initial approximation $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

9. $\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 + x_2 &= 3 \end{aligned}$

10. $\begin{aligned} -x_1 + 4x_2 &= 1 \\ 3x_1 - 2x_2 &= 2 \end{aligned}$

11. $\begin{aligned} 2x_1 - 3x_2 \phantom{{}- 10x_3} &= -7 \\ x_1 + 3x_2 - 10x_3 &= 9 \\ 3x_1 \phantom{{}+ 3x_2} + x_3 &= 13 \end{aligned}$

12. $\begin{aligned} x_1 + 3x_2 - x_3 &= 5 \\ 3x_1 - x_2 \phantom{{}- x_3} &= 5 \\ x_2 + 2x_3 &= 1 \end{aligned}$

In Exercises 13–16, determine whether the matrix is strictly diagonally dominant.

13. $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

14. $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 12 & 6 & 0 \\ 2 & -3 & 2 \\ 0 & 6 & 13 \end{bmatrix}$

16. $\begin{bmatrix} 7 & 5 & -1 \\ 1 & -4 & 1 \\ 0 & 2 & -3 \end{bmatrix}$

17. Interchange the rows of the system of linear equations in Exercise 9 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

18. Interchange the rows of the system of linear equations in Exercise 10 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

19. Interchange the rows of the system of linear equations in Exercise 11 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

20. Interchange the rows of the system of linear equations in Exercise 12 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

In Exercises 21 and 22, the coefficient matrix of the system of linear equations is not strictly diagonally dominant. Show that the Jacobi and Gauss-Seidel methods converge using an initial approximation of $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

21. $\begin{aligned} -4x_1 + 5x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$

22. $\begin{aligned} 4x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 - 3x_2 - x_3 &= 7 \\ 3x_1 - x_2 + 4x_3 &= 5 \end{aligned}$

In Exercises 23 and 24, write a computer program that applies the Gauss-Siedel method to solve the system of linear equations.

23. $\begin{aligned} 4x_1 + x_2 - x_3 &= 3 \\ x_1 + 6x_2 - 2x_3 + x_4 - x_5 &= -6 \\ x_2 + 5x_3 - x_5 + x_6 &= -5 \\ 2x_2 + 5x_4 - x_5 - x_7 - x_8 &= 0 \\ -x_3 - x_4 + 6x_5 - x_6 - x_8 &= 12 \\ -x_3 - x_5 + 5x_6 &= -12 \\ -x_4 + 4x_7 - x_8 &= -2 \\ -x_4 - x_5 - x_7 + 5x_8 &= 2 \end{aligned}$

24. $\begin{aligned} 4x_1 - x_2 - x_3 &= 18 \\ -x_1 + 4x_2 - x_3 - x_4 &= 18 \\ -x_2 + 4x_3 - x_4 - x_5 &= 4 \\ -x_3 + 4x_4 - x_5 - x_6 &= 4 \\ -x_4 + 4x_5 - x_6 - x_7 &= 26 \\ -x_5 + 4x_6 - x_7 - x_8 &= 16 \\ -x_6 + 4x_7 - x_8 &= 10 \\ -x_7 + 4x_8 &= 32 \end{aligned}$