

CHAPTER 8 □ REVIEW EXERCISES

In Exercises 1–6, perform the given operation.

1. Find $u + z$: $u = 2 - 4i$, $z = 4i$
2. Find $u - z$: $u = 4$, $z = 8i$
3. Find uz : $u = 4 - 2i$, $z = 4 + 2i$
4. Find uz : $u = 2i$, $z = 1 - 2i$
5. Find $\frac{u}{z}$: $u = 6 - 2i$, $z = 3 - 3i$
6. Find $\frac{u}{z}$: $u = 7 + i$, $z = i$

In Exercises 7–10, find all zeros of the polynomial.

7. $x^2 - 4x + 8$
8. $x^2 - 4x + 7$
9. $3x^2 + 3x + 3$
10. $x^3 + 2x^2 + 2x + 1$

In Exercises 11–14, perform the operation using

$$A = \begin{bmatrix} 4 - i & 2 \\ 3 & 3 + i \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 + i & i \\ 2i & 2 + i \end{bmatrix}.$$

11. $A + B$
12. $2iB$
13. $\det(A - B)$
14. $3BA$

In Exercises 15–20, perform the operation using $w = 2 - 2i$, $v = 3 + i$, and $z = -1 + 2i$.

15. \bar{z}
16. \bar{v}
17. $|w|$
18. $|v\bar{z}|$
19. \overline{wv}
20. $|\overline{zw}|$

In Exercises 21–24, perform the indicated operation.

21. $\frac{2 + i}{2 - i}$
22. $\frac{1 + i}{-1 + 2i}$

$$23. \frac{(1 - 2i)(1 + 2i)}{3 - 3i} \qquad 24. \frac{5 + 2i}{(-2 + 2i)(2 - 3i)}$$

In Exercises 25 and 26, find A^{-1} (if it exists).

$$25. A = \begin{bmatrix} 3 - i & -1 - 2i \\ -\frac{23}{5} + \frac{11}{5}i & 2 + 3i \end{bmatrix}$$

$$26. A = \begin{bmatrix} 5 & 1 - i \\ 0 & i \end{bmatrix}$$

In Exercises 27–30, determine the polar form of the complex number.

27. $4 + 4i$
28. $3 + 2i$
29. $7 - 4i$
30. $\sqrt{3} + i$

In Exercises 31–34, find the standard form of the complex number.

$$31. 5 \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right]$$

$$32. 4 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right)$$

$$33. 6 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \qquad 34. 7 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$$

In Exercises 35–38, perform the indicated operation. Leave the result in polar form.

$$35. \left[4 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \right] \left[3 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right]$$

$$36. \left[\frac{1}{2} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \right] \left[2 \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right) \right]$$

$$37. \frac{9[\cos(\pi/2) + i \sin(\pi/2)]}{6[\cos(2\pi/3) + i \sin(2\pi/3)]}$$

$$38. \frac{4[\cos(\pi/4) + i \sin(\pi/4)]}{7[\cos(\pi/3) + i \sin(\pi/3)]}$$

In Exercises 39–42, find the indicated power of the number and express the result in polar form.

$$39. (-1 - i)^4$$

$$40. (2i)^3$$

$$41. \left[\sqrt{2} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right]^7$$

$$42. \left[5 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right]^4$$

In Exercises 43–46, express the roots in standard form.

$$43. \text{Square roots: } 25 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$44. \text{Cube roots: } 27 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$45. \text{Cube roots: } i$$

$$46. \text{Fourth roots: } 16 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

In Exercises 47 and 48, determine the conjugate transpose of the matrix.

$$47. A = \begin{bmatrix} -1 + 4i & 3 + i \\ 3 - i & 2 + i \end{bmatrix}$$

$$48. A = \begin{bmatrix} 5 & 2 - i & 3 + 2i \\ 2 + 2i & 3 - 2i & i \\ 3i & 2 + i & -1 - 2i \end{bmatrix}$$

In Exercises 49–52, find the indicated vector using $\mathbf{u} = (4i, 2 + i)$, $\mathbf{v} = (3, -i)$, and $\mathbf{w} = (3 - i, 4 + i)$.

$$49. 7\mathbf{u} - \mathbf{v}$$

$$50. 3i\mathbf{w} + (4 - i)\mathbf{v}$$

$$51. i\mathbf{u} + i\mathbf{v} - i\mathbf{w}$$

$$52. (3 + 2i)\mathbf{u} - (-2i)\mathbf{w}$$

In Exercises 53 and 54, determine the Euclidean norm of the vector.

$$53. \mathbf{v} = (3 - 5i, 2i)$$

$$54. \mathbf{v} = (3i, -1 - 5i, 3 + 2i)$$

In Exercises 55 and 56, find the Euclidean distance between the vectors.

$$55. \mathbf{v} = (2 - i, i), \quad \mathbf{u} = (i, 2 - i)$$

$$56. \mathbf{v} = (2 + i, -1 + 2i, 3i), \quad \mathbf{u} = (4 - 2i, 3 + 2i, 4)$$

In Exercises 57–60, determine whether the matrix is unitary.

$$57. \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$58. \begin{bmatrix} \frac{2+i}{4} & \frac{1+i}{4} \\ \frac{i}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$$59. \begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix}$$

$$60. \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & i & 0 \\ \frac{1+i}{2} & 0 & \frac{-1-i}{2} \end{bmatrix}$$

In Exercises 61 and 62, determine whether the matrix is Hermitian.

$$61. \begin{bmatrix} 1 & -1 + i & 2 - i \\ 1 - i & 3 & i \\ 2 + i & -i & 4 \end{bmatrix}$$

$$62. \begin{bmatrix} 9 & 2 - i & 2 \\ 2 + i & 0 & -1 - i \\ 2 & -1 + i & 3 \end{bmatrix}$$

In Exercises 63 and 64, find the eigenvalues and corresponding eigenvectors of the matrix.

$$63. \begin{bmatrix} 4 & 2 - i \\ 2 + i & 0 \end{bmatrix}$$

$$64. \begin{bmatrix} 2 & 0 & -i \\ 0 & 3 & 0 \\ i & 0 & 2 \end{bmatrix}$$

65. Prove that if A is an invertible matrix, then A^* is also invertible.

66. Determine all complex numbers z such that $z = -\bar{z}$.

67. Prove that if the product of two complex numbers is zero, then one of the numbers must be zero.

68. (a) Find the determinant of the following Hermitian matrix.

$$\begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$$

(b) Prove that the determinant of any Hermitian matrix is real.

69. Let A and B be Hermitian matrices. Prove that $AB = BA$ if and only if AB is Hermitian.

70. Let \mathbf{u} be a unit vector in C^n . Define $H = I - 2\mathbf{u}\mathbf{u}^*$. Prove that H is an $n \times n$ Hermitian and unitary matrix.

71. Use mathematical induction to prove DeMoivre's Theorem.

72. Prove that if z is a zero of a polynomial equation with real coefficients, then the conjugate of z is also a zero.

73. Show that if $z_1 + z_2$ and $z_1\bar{z}_2$ are both nonzero real numbers, then z_1 and z_2 are both real numbers.
74. Prove that if z and w are complex numbers, then $|z + w| \leq |z| + |w|$.
75. Prove that for all vectors \mathbf{u} and \mathbf{v} in a complex inner product space,
- $$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}[\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 + i\|\mathbf{u} + i\mathbf{v}\|^2 - i\|\mathbf{u} - i\mathbf{v}\|^2].$$

CHAPTER 8 PROJECTS

1 Population Growth and Dynamical Systems - II

In the projects for Chapter 7, you were asked to model the population of two species using a system of differential equations of the form

$$y_1'(t) = ay_1(t) + by_2(t)$$

$$y_2'(t) = cy_1(t) + dy_2(t).$$

The constants a , b , c , and d depend on the particular species being studied. In Chapter 7, you looked at an example of a predator–prey relationship, in which $a = 0.5$, $b = 0.6$, $c = -0.4$, and $d = 3.0$. Now consider a slightly different model.

$$y_1'(t) = 0.6y_1(t) + 0.8y_2(t), \quad y_1(0) = 36$$

$$y_2'(t) = -0.8y_1(t) + 0.6y_2(t), \quad y_2(0) = 121$$

1. Use the diagonalization technique to find the general solutions $y_1(t)$ and $y_2(t)$ at any time $t > 0$. Although the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

are complex, the same principles apply, and you can obtain complex exponential solutions.

2. Convert your complex solutions to real solutions by observing that if $\lambda = a + bi$ is a (complex) eigenvalue of A with (complex) eigenvector \mathbf{v} , then the real and imaginary parts of $e^{\lambda t}\mathbf{v}$ form a linearly independent pair of (real) solutions. You will need to use the formula $e^{i\theta} = \cos \theta + i \sin \theta$.
3. Use the initial conditions to find the explicit form of the (real) solutions to the original equations.
4. If you have access to a computer or graphing calculator, graph the solutions obtained in part (3) over the domain $0 \leq t \leq 3$. At what moment are the two populations equal?
5. Interpret the solution in terms of the long-term population trend for the two species. Does one species ultimately disappear? Why or why not? Contrast this solution to that obtained for the model in Chapter 7.
6. If you have access to a computer or graphing calculator that can numerically solve differential equations, use it to graph the solutions to the original system of equations. Does this numerical approximation appear to be accurate?