Chapter 1 Review Exercises

1. Choosing y and z as the free variables and letting $y = s$ and $z = t$, you have

$$\begin{aligned} -4x + 2s - 6t &= 1 \\ -4x &= 1 - 2s + 6t \\ x &= -\tfrac{1}{4} + \tfrac{1}{2}s - \tfrac{3}{2}t. \end{aligned}$$

Thus, the solution set may be described as $x = -\frac{1}{4} + \frac{1}{2}s - \frac{3}{2}t$, $y = s$, $z = t$ where s and t are real numbers.

3. This matrix has the characteristic stair-step pattern of leading 1's so that it is in row-echelon form. However, the leading 1 in row three of column four has 1's above it, so the matrix is *not* in reduced row-echelon form.

5. Because the first row begins with -1, this matrix is not in row-echelon form.

7. This matrix corresponds to the system

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$

Choosing $x_2 = t$ as the free variable you find that the solution set can be described as $x_1 = -2t$, $x_2 = t$, and $x_3 = 0$, where t is a real number.

9. Row reduce the augmented matrix for this system.

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$$

Converting back to a linear system, the solution is $x = \frac{1}{2}$ and $y = \frac{3}{2}$.

11. Begin by rearranging the given equations as

$$\begin{aligned} x - y &= -4 \\ 2x - 3y &= 0 \end{aligned}$$

Row reduce the augmented matrix for this system.

$$\begin{bmatrix} 1 & -1 & -4 \\ 2 & -3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -4 \\ 0 & -1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -4 \\ 0 & 1 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -12 \\ 0 & 1 & -8 \end{bmatrix}$$

Converting back to a linear system, the solution is $x = -12$ and $y = -8$.

13. Row reduce the augmented matrix for this system.

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Converting back to a linear system, the solution is $x = 0$ and $y = 0$.

15. The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 9 \\ -1 & 1 & 1 \end{bmatrix}$$

which is equivalent to the reduced row-echelon matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because the second row corresponds to $0 = 1$, which is an absurdity, you can conclude that the system has no solution.

17. Multiplying both equations by 100 and forming the augmented matrix produces

$$\begin{bmatrix} 20 & 30 & 14 \\ 40 & 50 & 20 \end{bmatrix}.$$

Gauss-Jordan elimination yields the following.

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{10} \\ 40 & 50 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{10} \\ 0 & -10 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{7}{10} \\ 0 & 1 & \frac{4}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{4}{5} \end{bmatrix}$$

Hence, the solution is $x_1 = -\frac{1}{2}$ and $x_2 = \frac{4}{5}$.

19. Expanding the second equation, $3x + 2y = 0$, the augmented matrix for this system is

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

which is equivalent to the reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus, the solution set is $x = 0$ and $y = 0$.

21. The augmented matrix for this system is

$$\begin{bmatrix} -1 & 1 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ 5 & 4 & 2 & 4 \end{bmatrix}.$$

which is equivalent to the reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Thus, the solution is $x = 2$, $y = -3$, and $z = 3$.

23. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 2 & 3 & 3 & 3 \\ 6 & 6 & 12 & 13 \\ 12 & 9 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus, $x = \frac{1}{2}$, $y = -\frac{1}{3}$, and $z = 1$.

25. The augmented matrix for this system is

$$\begin{bmatrix} 1 & -2 & 1 & -6 \\ 2 & -3 & 0 & -7 \\ -1 & 3 & -3 & 11 \end{bmatrix}$$

which is equivalent to the reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing $z = t$ as the free variable you find that the solution set can be described by $x = 4 + 3t$, $y = 5 + 2t$, and $z = t$, where t is a real number.

27. Use the Gauss-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the solution is $x = \frac{3}{2} - 2t$, $y = 1 + 2t$, $z = t$, where t is any real number.

29. The augmented matrix for this system is

$$\begin{bmatrix} 2 & 1 & 1 & 2 & -1 \\ 5 & -2 & 1 & -3 & 0 \\ -1 & 3 & 2 & 2 & 1 \\ 3 & 2 & 3 & -5 & 12 \end{bmatrix}$$

which is equivalent to the reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Thus, the solution is $x_1 = 1$, $x_2 = 4$, $x_3 = -3$, and $x_4 = -2$.

31. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & -2 & -8 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 1 & 7 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence, the solution is $x_1 = x_2 = x_3 = 0$.

33. The augmented matrix for this system is

$$\begin{bmatrix} 2 & -8 & 4 & 0 \\ 3 & -10 & 7 & 0 \\ 0 & 10 & 5 & 0 \end{bmatrix}$$

which is equivalent to the reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing $x_3 = t$ as the free variable, you find that the solution set can be described by $x_1 = -4t$, $x_2 = -\frac{1}{2}t$, and $x_3 = t$, where t is a real number.

35. Forming the augmented matrix

$$\begin{bmatrix} k & 1 & 0 \\ 1 & k & 1 \end{bmatrix}$$

and using Gauss-Jordan elimination, you obtain

$$\begin{bmatrix} 1 & k & 1 \\ k & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & k & 1 \\ 0 & 1-k^2 & -k \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & k & 1 \\ 0 & 1 & \frac{k}{k^2-1} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{-1}{k^2-1} \\ 0 & 1 & \frac{k}{k^2-1} \end{bmatrix}, \quad k^2 - 1 \neq 0.$$

Thus, the system is inconsistent if $k = \pm 1$.

37. Row reduce the augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ a & b & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & (b-2a) & (-9-3a) \end{bmatrix}$$

(a) There will be no solution if $b - 2a = 0$ <u>and</u> $-9 - 3a \neq 0$.That is, if $b = 2a$ and $a \neq -3$.

(b) There will be exactly one solution if $b \neq 2a$.

(c) There will be an infinite number of solutions if $b = 2a$ <u>and</u> $a = -3$. That is, if $a = -3$ and $b = -6$.

39. You can show that two matrices of the same size are row equivalent if they both row reduce to the same matrix. The two given matrices are row equivalent since each is row equivalent to the identity matrix.

41. Adding a multiple of row one to each row yields the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & -n & -2n & \cdots & -(n-1)n \\ 0 & -2n & -4n & \cdots & -2(n-1)n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -(n-1)n & -2(n-1)n & \cdots & -(n-1)(n-1)n \end{bmatrix}$$

Every row below row two is a multiple of row two. Therefore, reduce these rows to zeros.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & -n & -2n & \cdots & -(n-1)n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Dividing row two by $-n$ yields a new second row.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Adding -2 times row two to row one yields a new first row.

$$\begin{bmatrix} 1 & 0 & -1 & \cdots & 2-n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

This matrix is in reduced row-echelon from.

43. (a) False. See page 3, following Example 2.

(b) True. See page 5, Example 4(b).

45. (a) Because there are three points, choose a second degree polynomial, $p(x) = a_0 + a_1x + a_2x^2$. By substituting the values at each point into this equation you obtain the system

$$\begin{aligned} a_0 + 2a_1 + 4a_2 &= 5 \\ a_0 + 3a_1 + 9a_2 &= 0 \\ a_0 + 4a_1 + 16a_2 &= 20. \end{aligned}$$

Forming the augmented matrix

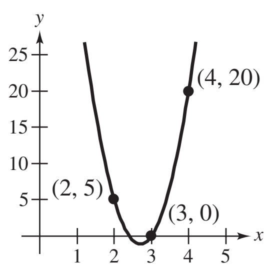
$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 1 & 3 & 9 & 0 \\ 1 & 4 & 16 & 20 \end{bmatrix}$$

and using Gauss-Jordan elimination you obtain

$$\begin{bmatrix} 1 & 0 & 0 & 90 \\ 0 & 1 & 0 & -\frac{135}{2} \\ 0 & 0 & 1 & \frac{25}{2} \end{bmatrix}$$

Thus, $p(x) = 90 - \frac{135}{2}x + \frac{25}{2}x^2$.

(b)



47. Establish the first year as $x = 0$ and substitute the values at each point into $p(x) = a_0 + a_1x + a_2x^2$ to obtain the system

$$\begin{aligned} a_0 &= 50 \\ a_0 + a_1 + a_2 &= 60 \\ a_0 + 2a_1 + 4a_2 &= 75. \end{aligned}$$

Forming the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 50 \\ 1 & 1 & 1 & 60 \\ 1 & 2 & 4 & 75 \end{bmatrix}$$

and using Gauss-Jordan elimination, you obtain

$$\begin{bmatrix} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & \frac{15}{2} \\ 0 & 0 & 1 & \frac{5}{2} \end{bmatrix}.$$

Thus, $p(x) = 50 + \frac{15}{2}x + \frac{5}{2}x^2$. To predict the sales in the fourth year, evaluate $p(x)$ when $x = 3$.

$p(3) = 50 + \frac{15}{2}(3) + \frac{5}{2}(3)^2 = \$95$.

49. (a) There are three points: (0, 80), (4, 68), and (80, 30). Because you are given three points, choose a second-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2$. Substituting the given points into $p(x)$ produces the following system of linear equations.

$$\begin{aligned} a_0 + (0)a_1 + (0)^2a_2 &= a_0 &&= 80 \\ a_0 + (4)a_1 + (4)^2a_2 &= a_0 + 4a_1 + 16a_2 &&= 68 \\ a_0 + (80)a_1 + (80)^2a_2 &= a_0 + 80a_1 + 6400a_2 &&= 30 \end{aligned}$$

(b) Form the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 80 \\ 1 & 4 & 16 & 68 \\ 1 & 80 & 6400 & 30 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 80 \\ 0 & 1 & 0 & -\frac{25}{8} \\ 0 & 0 & 1 & \frac{1}{32} \end{bmatrix}$$

Thus $p(x) = 80 - \frac{25}{8}x + \frac{1}{32}x^2$.

(c) The graphing utility gives $a_0 = 80$, $a_1 = -3.125$, and $a_2 = 0.03125$. In other words $p(x) = 80 - 3.125x + 0.03125x^2$.

(d) The results to (b) and (c) are the same.

(e) There is precisely one polynomial function of degree $n - 1$ (or less) that fits n distinct points.

51. (a) First find the equations corresponding to each node in the network.

input **output**

$$\begin{aligned} x_1 + 200 &= x_2 + x_4 \\ x_6 + 100 &= x_1 + x_3 \\ x_2 + x_3 &= x_5 + 300 \\ x_4 + x_5 &= x_6 \end{aligned}$$

Rearranging this system and forming the augmented matrix, you have

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & -200 \\ 1 & 0 & 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

The equivalent reduced row-echelon matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing $x_3 = r$, $x_5 = s$, and $x_6 = t$ as the free variables, you obtain

$$\begin{aligned} x_1 &= 100 - r + t \\ x_2 &= 300 - r + s \\ x_4 &= -s + t. \end{aligned}$$

(b) When $x_3 = 100 = r$, $x_5 = 50 = s$, and $x_6 = 50 = t$, you have

$$\begin{aligned} x_1 &= 100 - 100 + 50 = 50 \\ x_2 &= 300 - 100 + 50 = 250 \\ x_4 &= -50 + 50 = 0. \end{aligned}$$