Section 1.3 Applications of Systems of Linear Equations

1. (a) Because there are three points, choose a second-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2$. Then substitute $x = 2, 3,$ and 4 into $p(x)$ and equate the results to $y = 5, 2,$ and 5, respectively.

$$\begin{aligned} a_0 + a_1(2) + a_2(2)^2 &= a_0 + 2a_1 + 4a_2 = 5 \\ a_0 + a_1(3) + a_2(3)^2 &= a_0 + 3a_1 + 9a_2 = 2 \\ a_0 + a_1(4) + a_2(4)^2 &= a_0 + 4a_1 + 16a_2 = 5 \end{aligned}$$

Form the augmented matrix

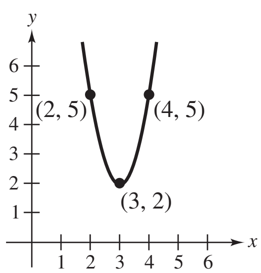
$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 1 & 3 & 9 & 2 \\ 1 & 4 & 16 & 5 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & -18 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Thus, $p(x) = 29 - 18x + 3x^2$.

(b)



3. (a) Because there are three points, choose a second-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2$. Then substitute $x = 2, 3,$ and 5 into $p(x)$ and equate the results to $y = 4, 6,$ and 10, respectively.

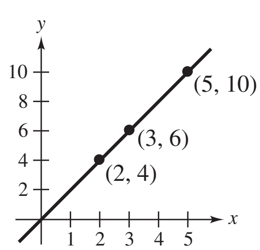
$$\begin{aligned} a_0 + a_1(2) + a_2(2)^2 &= a_0 + 2a_1 + 4a_2 = 4 \\ a_0 + a_1(3) + a_2(3)^2 &= a_0 + 3a_1 + 9a_2 = 6 \\ a_0 + a_1(5) + a_2(5)^2 &= a_0 + 5a_1 + 25a_2 = 10 \end{aligned}$$

Use Gauss-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 6 \\ 1 & 5 & 25 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, $p(x) = 2x$.

(b)



5. (a) Because there are three points, choose a second-degree polynomial, $p(z) = a_0 + a_1z + a_2z^2$. Then substitute $z = -1, 0, 1$ into $p(z)$ and equate the results to $y = 5, 7$, and 12, respectively.

$$\begin{aligned} a_0 + a_1(-1) + a_2(-1)^2 &= a_0 - a_1 + a_2 = 5 \\ a_0 + a_1(0) + a_2(0)^2 &= a_0 = 7 \\ a_0 + a_1(1) + a_2(1)^2 &= a_0 + a_1 + a_2 = 12 \end{aligned}$$

Then form the augmented matrix

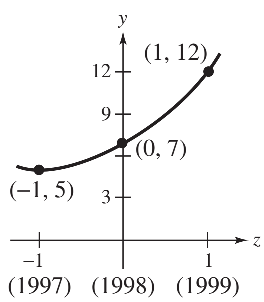
$$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 1 & 0 & 0 & 7 \\ 1 & 1 & 1 & 12 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

Thus, $p(z) = 7 + \frac{7}{2}z + \frac{3}{2}z^2$, $p(x) = 7 + \frac{7}{2}(x - 1998) + \frac{3}{2}(x - 1998)^2$.

(b)



7. Choose a fourth-degree polynomial and substitute $x = 1, 2, 3$, and 4 into $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. However, when you substitute $x = 3$ into $p(x)$ and equate it to $y = 2$ and $y = 3$ you get the contradictory equations

$$\begin{aligned} a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 &= 2 \\ a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 &= 3 \end{aligned}$$

and must conclude that the system containing these two equations will have no solution. By similar reasoning, you cannot choose $p(y) = b_0 + b_1y + b_2y^2 + b_3y^3 + b_4y^4$ because $y = 1$ corresponds to both $x = 1$ and $x = 2$.

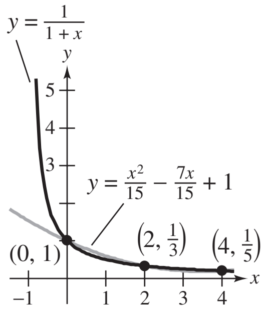
9. Letting $p(x) = a_0 + a_1x + a_2x^2$, substitute $x = 0$, 2, and 4 into $p(x)$ and equate the results to $y = 1$, 3, and 5, respectively.

$$\begin{aligned} a_0 + a_1(0) + a_2(0)^2 &= a_0 &&= 1 \\ a_0 + a_1(2) + a_2(2)^2 &= a_0 + 2a_1 + 4a_2 &&= 3 \\ a_0 + a_1(4) + a_2(4)^2 &= a_0 + 4a_1 + 16a_2 &&= 5 \end{aligned}$$

Use Gauss-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 16 & 5 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, $p(x) = 1 + x$. The graphs of $y = 1/p(x) = 1/(1 + x)$ and that of the function $y = 1 - \frac{7}{15}x + \frac{1}{15}x^2$ are shown below.



11. To begin, substitute $x = -1$ and $x = 1$ into $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and equate the results to $y = 2$ and $y = -2$, respectively.

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= 2 \\ a_0 + a_1 + a_2 + a_3 &= -2 \end{aligned}$$

Then, differentiate p, yielding $p'(x) = a_1 + 2a_2x + 3a_3x^2$. Substitute $x = -1$ and $x = 1$ into $p'(x)$ and equate the results to 0.

$$\begin{aligned} a_1 - 2a_2 + 3a_3 &= 0 \\ a_1 + 2a_2 + 3a_3 &= 0 \end{aligned}$$

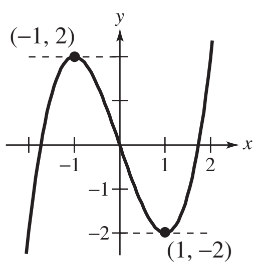
Combining these four equations into one system and forming the augmented matrix, you obtain

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 & -2 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

Use Gauss-Jordan elimination to find the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Thus, $p(x) = -3x + x^3$. The graph of $y = p(x)$ is shown below.



13. (a) Because you are given three points, choose a second-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2$. Because the x-values are large, use the translation $z = x - 1980$ to obtain $(-10, 203.3)$, $(0, 226.5)$, $(10, 248.7)$. Substituting the given points into $p(x)$ produces the following system of linear equations.

$$\begin{aligned}
a_0 + (-10)a_1 + (-10)^2a_2 &= a_0 - 10a_1 + 100a_2 = 203.3\\
a_0 + (0)a_1 + (0)^2a_2 &= a_0 = 226.5\\
a_0 + (10)a_1 + (10)^2a_2 &= a_0 + 10a_1 + 100a_2 = 248.7
\end{aligned}$$

Form the augmented matrix

$$\begin{bmatrix} 1 & -10 & 100 & 203.3\\ 1 & 0 & 0 & 226.5\\ 1 & 10 & 100 & 248.7 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 226.5\\ 0 & 1 & 0 & 2.27\\ 0 & 0 & 1 & -0.005 \end{bmatrix}$$

Thus $p(z) = 226.5 + 2.27z - 0.005z^2$ so that
$p(x) = 226.5 + 2.27(x - 1980) - 0.005(x - 1980)^2$.
To predict the population in 2000 and 2005, substitute these values into $p(x)$.

$$p(2000) = 226.5 + 2.27(20) - 0.005(20)^2 = 269.9$$

$$p(2005) = 226.5 + 2.27(25) - 0.005(25)^2 = 280.125$$

15. Choosing a second-degree polynomial approximation, $p(x) = a_0 + a_1x + a_2x^2$, substitute $x = 0, \frac{\pi}{2}$, and π into $p(x)$ and equate the results to $y = 0$, 1, and 0, respectively.

$$\begin{aligned} a_0 \qquad\qquad\qquad &= 0 \\ a_0 + \frac{\pi}{2}a_1 + \frac{\pi^2}{4}a_2 &= 1 \\ a_0 + \pi a_1 + \pi^2 a_2 &= 0 \end{aligned}$$

Then form the augmented matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{\pi}{2} & \frac{\pi^2}{4} & 1 \\ 1 & \pi & \pi^2 & 0 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{\pi} \\ 0 & 0 & 1 & -\frac{4}{\pi^2} \end{bmatrix}.$$

Thus,

$$p(x) = \frac{4}{\pi}x - \frac{4}{\pi^2}x^2 = \frac{4}{\pi^2}(\pi x - x^2).$$

Furthermore,

$$\begin{aligned} \sin\frac{\pi}{3} \approx p\left(\frac{\pi}{3}\right) &= \frac{4}{\pi^2}\left[\pi\left(\frac{\pi}{3}\right) - \left(\frac{\pi}{3}\right)^2\right] \\ &= \frac{4}{\pi^2}\left[\frac{2\pi^2}{9}\right] \\ &= \frac{8}{9} \approx 0.889. \end{aligned}$$

Note that $\sin \pi/3 = 0.866$ to three significant digits.

17. (i)
$$\begin{aligned} p(-1) &= a_0 + (-1)a_1 + (-1)^2a_2 = a_0 - a_1 + a_2 \\ p(0) &= a_0 + (0)a_1 + (0)^2a_2 = a_0 \\ p(1) &= a_0 + (1)a_1 + (1)^2a_2 = a_0 + a_1 + a_2 \end{aligned}$$

(ii)
$$\begin{aligned} a_0 - a_1 + a_2 &= 0 \\ a_0 \qquad\quad &= 0 \\ a_0 + a_1 + a_2 &= 0 \end{aligned}$$

(iii) Form the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and use Gauss-Jordan elimination to obtain the equivalent reduced row-echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus $a_0 = 0$, $a_1 = 0$, and $a_2 = 0$.

19. (a) Each of the network's six junctions gives rise to a linear equation as follows.

$$\begin{aligned}
\textbf{input} &= \textbf{output}\\
600 &= x_1 + x_3\\
x_1 &= x_2 + x_4\\
x_2 + x_5 &= 500\\
x_3 + x_6 &= 600\\
x_4 + x_7 &= x_6\\
500 &= x_5 + x_7
\end{aligned}$$

Rearrange these equations, form the augmented matrix, and use Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $x_7 = t$ and $x_6 = s$ be the free variables, write down the solution as follows.

$$\begin{aligned}
x_1 &= s\\
x_2 &= t\\
x_3 &= 600 - s\\
x_4 &= s - t\\
x_5 &= 500 - t\\
x_6 &= s\\
x_7 &= t
\end{aligned}$$

(b) If $x_6 = x_7 = 0$, then the solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 600$, $x_4 = 0$, $x_5 = 500$, $x_6 = 0$, $x_7 = 0$.

(c) If $x_5 = 1000$ and $x_6 = 0$, then the solution is $x_1 = 0$, $x_2 = -500$, $x_3 = 600$, $x_4 = 500$, $x_5 = 1000$, $x_6 = 0$, and $x_7 = -500$.

21. (a) Each of the network's four junctions gives rise to a linear equation, as follows.

$$\begin{aligned} \textbf{input} &= \textbf{output} \\ 200 + x_2 &= x_1 \\ x_4 &= x_2 + 100 \\ x_3 &= x_4 + 200 \\ x_1 + 100 &= x_3 \end{aligned}$$

Rearranging these equations and forming the augmented matrix, you obtain

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 1 & 0 & -1 & 0 & -100 \end{bmatrix}.$$

Gauss-Jordan elimination produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Letting $x_4 = t$, you have $x_1 = 100 + t$, $x_2 = -100 + t$, $x_3 = 200 + t$, and $x_4 = t$, where t is a real number.

(b) When $x_4 = t = 0$, then $x_1 = 100$, $x_2 = -100$, $x_3 = 200$.

(c) When $x_4 = t = 100$, then $x_1 = 200$, $x_2 = 0$, $x_3 = 300$.

23. Applying Kirchoff's first law to either junction produces

$$I_1 + I_3 = I_2$$

and applying the second law to the two paths produces

$$\begin{aligned} R_1I_1 + R_2I_2 &= 4I_1 + 3I_2 = 3 \\ R_2I_2 + R_3I_3 &= 3I_2 + I_3 = 4. \end{aligned}$$

Rearranging these equations, forming their augmented matrix, and using Gauss-Jordan elimination yields the following.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus, $I_1 = 0$, $I_2 = 1$, and $I_3 = 1$.

25. (a) To find the general solution, let A have a volts and B have b volts. Applying Kirchoff's first law to either junction produces

$$I_1 + I_3 = I_2$$

and applying the second law to the two paths produces

$$\begin{aligned} R_1I_1 + R_2I_2 &= I_1 + 2I_2 = a \\ R_2I_2 + R_3I_3 &= 2I_2 + 4I_3 = b. \end{aligned}$$

Rearranging these three equations and forming the augmented matrix yields

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & a \\ 0 & 2 & 4 & b \end{bmatrix}.$$

Gauss-Jordan elimination produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & (3a - b)/7 \\ 0 & 1 & 0 & (4a + b)/14 \\ 0 & 0 & 1 & (3b - 2a)/14 \end{bmatrix}.$$

When $a = 5$ and $b = 8$, then $I_1 = 1$, $I_2 = 2$, $I_3 = 1$.

(b) When $a = 2$ and $b = 6$, then $I_1 = 0$, $I_2 = 1$, $I_3 = 1$.