Section 1.2 Gaussian Elimination and Gauss-Jordan Elimination

1. Because the matrix has 3 rows and 3 columns, it has size 3×3.

3. Because the matrix has 1 row and 5 columns, it has size 1×5.

5. Because the matrix has 4 rows and 5 columns, it has size 4×5.

7. The matrix satisfies all three conditions in the definition of row-echelon form. Moreover, because each column that has a leading 1 (columns one and two) has zeros elsewhere, the matrix is in reduced row-echelon form.

9. Because the matrix has two non-zero rows without leading 1's, it is not in row-echelon form.

11. Because the matrix has a non-zero row without a leading 1, it is not in row-echelon form.

13. Because the matrix is in reduced row-echelon form, convert back to a system of linear equations

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 2 \end{aligned}$$

Thus, the solution set is $x_1 = 0$ and $x_2 = 2$.

15. Because the matrix is in row-echelon form, convert back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 \qquad &= 3 \\ x_2 - 2x_3 &= 1 \\ x_3 &= -1 \end{aligned}$$

Solve this system by back-substitution.

$$\begin{aligned} x_2 - 2(-1) &= 1 \\ x_2 &= -1 \end{aligned}$$

Substituting $x_2 = -1$ into equation 1,

$$\begin{aligned} x_1 - (-1) &= 3 \\ x_1 &= 2. \end{aligned}$$

Thus, the solution set is $x_1 = 2$, $x_2 = -1$, and $x_3 = -1$.

17. Because the matrix is in row-echelon form, convert back to a system of linear equations.

$$\begin{aligned} x_1 + 2x_2 \qquad + x_4 &= 4 \\ x_2 + 2x_3 + x_4 &= 3 \\ x_3 + 2x_4 &= 1 \\ x_4 &= 4 \end{aligned}$$

Solve this system by back-substitution.

$$x_3 = 1 - 2x_4 = 1 - 2(4) = -7$$

$$x_2 = 3 - 2x_3 - x_4 = 3 - 2(-7) - 4 = 13$$

$$x_1 = 4 - 2x_2 - x_4 = 4 - 2(13) - 4 = -26$$

Thus, the solution is: $x_1 = -26$, $x_2 = 13$, $x_3 = -7$, and $x_4 = 4$.

19. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & 8 \end{bmatrix}.$$

Adding -2 times the first row to the second row yields a new second row.

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & -3 & -6 \end{bmatrix}$$

Dividing the second row by -3 yields a new second row.

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

Converting back to a system of linear equations produces

$$\begin{aligned} x + 2y &= 7 \\ y &= 2. \end{aligned}$$

Finally, using back-substitution you find that $x = 3$ and $y = 2$.

21. The augmented matrix for this system is

$$\begin{bmatrix} -1 & 2 & 1.5 \\ 2 & -4 & 3 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} -1 & 2 & 1.5 \\ 2 & -4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -\frac{3}{2} \\ 2 & -4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -\frac{3}{2} \\ 0 & 0 & 6 \end{bmatrix}$$

Because the second row of this matrix corresponds to the equation $0 = 6$, you can conclude that the original system has no solution.

23. The augmented matrix for this system is

$$\begin{bmatrix} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix}.$$

Dividing the first row by -3 yields a new first row.

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix}$$

Adding -3 times the first row to the second row yields a new second row.

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 4 & -8 & 32 \end{bmatrix}$$

Adding -4 times the first row to the third row yields a new third row.

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix}$$

Dividing the second row by 9 yields a new second row.

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix}$$

Adding $\frac{4}{3}$ times the second row to the third row yields a new third row.

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Converting back to a system of linear equations produces

$$\begin{aligned} x - \tfrac{5}{3}y &= \tfrac{22}{3} \\ y &= -2. \end{aligned}$$

Finally, using back-substitution you find that the solution set is $x = 4$ and $y = -2$.

25. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{bmatrix}$$

Back substitution now yields

$$\begin{aligned} x_3 &= 2 \\ x_2 &= 11 - 7x_3 = 11 - (7)2 = -3 \\ x_1 &= -2 + 3x_3 = -2 + 3(2) = 4. \end{aligned}$$

Hence, the solution is: $x_1 = 4$, $x_2 = -3$, and $x_3 = 2$.

27. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix}.$$

Subtracting the first row from the second row yields a new second row.

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

Adding -2 times the first row to the third row yields a new third row.

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & -3 & 9 & -6 \end{bmatrix}$$

Multiplying the second row by -1 yields a new second row.

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{bmatrix}$$

Adding 3 times the second row to the third row yields a new third row.

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding -1 times the second row to the first row yields a new first row.

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Converting back to a system of linear equations produces

$$\begin{aligned} x_1 \quad - 2x_3 &= 1 \\ x_2 - 3x_3 &= 2. \end{aligned}$$

Finally, choosing $x_3 = t$ as the free variable you find the solution set to be $x_1 = 1 + 2t$, $x_2 = 2 + 3t$, and $x_3 = t$, where t is any real number.

29. The augmented matrix for this system is

$$\begin{bmatrix} 4 & 12 & -7 & -20 & 22 \\ 3 & 9 & -5 & -28 & 30 \end{bmatrix}.$$

Dividing the first row by 4 yields a new first row.

$$\begin{bmatrix} 1 & 3 & -\frac{7}{4} & -5 & \frac{11}{2} \\ 3 & 9 & -5 & -28 & 30 \end{bmatrix}$$

Adding -3 times the first row to the second row yields a new second row.

$$\begin{bmatrix} 1 & 3 & -\frac{7}{4} & -5 & \frac{11}{2} \\ 0 & 0 & \frac{1}{4} & -13 & \frac{27}{2} \end{bmatrix}$$

Multiplying the second row by 4 yields a new second row.

$$\begin{bmatrix} 1 & 3 & -\frac{7}{4} & -5 & \frac{11}{2} \\ 0 & 0 & 1 & -52 & 54 \end{bmatrix}$$

Converting back to a system of linear equations produces

$$\begin{aligned} x + 3y - \tfrac{7}{4}z - 5w &= \tfrac{11}{2} \\ z - 52w &= 54. \end{aligned}$$

Choosing $y = s$ and $w = t$ as the free variables you find the solution set can be represented by $x = 100 - 3s + 96t$, $y = s$, $z = 54 + 52t$, and $w = t$, where s and t are any real numbers.

31. The augmented matrix for this system is

$$\begin{bmatrix} 3 & 3 & 12 & 6 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 3 & 3 & 12 & 6 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 2 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 12 & 6 \\ 0 & 3 & 12 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $z = t$ be the free variable, the solution is: $x = 0$, $y = 2 - 4t$, $z = t$, where t is any real number.

33. Using a computer or graphing calculator, the augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & -0.5278 & 23.5361 \\ 0 & 1 & 0 & -4.1111 & 18.5444 \\ 0 & 0 & 1 & -2.1389 & 7.4306 \end{bmatrix}.$$

Letting $x_4 = t$ be the free variable, you obtain

$$\begin{aligned} x_1 &= 23.5361 + 0.5278t \\ x_2 &= 18.5444 + 4.1111t \\ x_3 &= 7.4306 + 2.1389t \\ x_4 &= t \end{aligned}$$

35. Using a computer or graphing calculator, the augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Hence, the solution is: $x_1 = 2$, $x_2 = -2$, $x_3 = 3$, $x_4 = -5$, and $x_5 = 1$.

37. The corresponding system of equations is

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence, $x_3 = t$ can be any real number, $x_2 = -x_3 = -t$ and $x_1 = 0$.

39. The corresponding system of equations is

$$\begin{aligned} x_1 \qquad + x_4 &= 0 \\ x_3 \qquad &= 0 \\ 0 &= 0 \end{aligned}$$

Hence, $x_4 = t$ can be any real number, $x_3 = 0$, $x_2 = s$ can be any real number, and $x_1 = -x_4 = -t$.

41. (a) Because there are two rows and three columns, there are two equations and two unknowns.

(b) Gaussian elimination produces the following.

$$\begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & k & 2 \\ 0 & 4+3k & 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & k & 2 \\ 0 & 1 & \frac{7}{4+3k} \end{bmatrix}$$

The system is consistent if $4 + 3k \neq 0$, or if $k \neq -\frac{4}{3}$.

(c) Because there are two rows and three columns, there are two equations and three unknowns.

(d) Gaussian elimination produces the following.

$$\begin{bmatrix} 1 & k & 2 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & k & 2 & 0 \\ 0 & 4+3k & 7 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & k & 2 & 0 \\ 0 & 1 & \frac{7}{4+3k} & 0 \end{bmatrix}$$

Notice that $4 + 3k \neq 0$, or $k \neq -\frac{4}{3}$. But if $-\frac{4}{3}$ is substituted for k in the original matrix, Guassian elimination produces the following.

$$\begin{bmatrix} 1 & -\frac{4}{3} & 2 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{4}{3} & 2 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

The system is consistent. Thus, the original system is consistent for all real k.

43. Begin by forming the augmented matrix for the system

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ a & b & c & 0 \end{bmatrix}.$$

Then use Gauss-Jordan elimination as follows.

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ a & b & c & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & b-a & c & -2a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & b-a & c & -2a \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & a-b+c & -2b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & a-b+c & -2b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+b+c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+b+c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & a+b+c \end{bmatrix}$$

Converting back to a system of linear equations

$$\begin{aligned} x &= 1 \\ y &= 1 \\ z &= 1 \\ 0 &= a + b + c. \end{aligned}$$

The system

(a) will have a unique solution if $a + b + c = 0$,
(b) will have no solution if $a + b + c \neq 0$, and
(c) cannot have an infinite number of solutions.

45. Solve each pair of equations by Gaussian elimination as follows.

(a) Equations 1 and 2:

$$\begin{bmatrix} 4 & -2 & 5 & 16 \\ 1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{5}{6} & \frac{8}{3} \\ 0 & 1 & -\frac{5}{6} & -\frac{8}{3} \end{bmatrix} \Rightarrow x = \frac{8}{3} - \frac{5}{6}t,$$

$$y = -\frac{8}{3} + \frac{5}{6}t, z = t$$

(b) Equations 1 and 3:

$$\begin{bmatrix} 4 & -2 & 5 & 16 \\ -1 & -3 & 2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{11}{14} & \frac{36}{14} \\ 0 & 1 & -\frac{13}{14} & -\frac{40}{14} \end{bmatrix} \Rightarrow x = \frac{18}{7} - \frac{11}{14}t,$$

$$y = -\frac{20}{7} + \frac{13}{14}t, z = t$$

(c) Equations 2 and 3:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -3 & 2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -3 \end{bmatrix} \Rightarrow x = 3 - t$$

$$y = -3 + t, z = t$$

(d) Each of these systems has an infinite number of solutions.

47. Use Gauss-Jordan elimination as follows.

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

49. Begin by finding all possible first rows.

$$[0 \quad 0], \qquad [0 \quad 1], \qquad [1 \quad 0], \qquad [1 \quad k].$$

For each of these, examine the possible second rows.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$$

These represent all possible 2×2 reduced row-echelon matrices.

51. (a) True, in the notation $m \times n$, m is the number of rows of the matrix. Thus, a 6×3 matrix has 6 rows.

(b) True, on page 18, the first sentence reads, "It can be shown that every matrix is row-equivalent to a matrix in row-echelon form."

(c) False, consider the row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

which gives the solution $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 3$.

(d) True, Theorem 1.1 states that if a homogeneous system has fewer equations than variables, then it must have an infinite number of solutions.

53. First off, a and c cannot both be zero. So, assume $a \neq 0$, and use row reduction as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ 0 & \frac{-cb}{a} + d \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Hence, $ad - bc \neq 0$. Similarly, if $c \neq 0$, interchange rows and proceed as above. So the original matrix is row equivalent to the identity if and only if $ad - bc \neq 0$.

55. Form the augmented matrix for this system

$$\begin{bmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 0 \end{bmatrix}$$

and reduce the system using elementary row operations.

$$\begin{bmatrix} 1 & \lambda - 2 & 0 \\ \lambda - 2 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \lambda - 2 & 0 \\ 0 & \lambda^2 - 4\lambda + 3 & 0 \end{bmatrix}$$

To have a nontrivial solution you must have

$$\begin{aligned} \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 1)(\lambda - 3) &= 0. \end{aligned}$$

Thus, if $\lambda = 1$ or $\lambda = 3$, the system will have nontrivial solutions.

57. To show that it is possible you need give only one example, such as

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

which has fewer equations than variables and obviously has no solution.

59. $2 \cos \alpha - \sin\beta = 0$
$4 \cos \alpha + 2 \sin\beta = 4$

(-2) times equation 1 added to equation 2 produces

$$4 \sin\beta = 4 \quad \Rightarrow \quad \sin\beta = 1 \quad \Rightarrow \quad \beta = \frac{\pi}{2}.$$

Then $\cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}, \frac{5\pi}{3}$.

61. $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a - c & b - d \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a - c & b - d \\ a & b \end{bmatrix} \Rightarrow \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} \Rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

The rows have been interchanged. In general, the second and third elementary row operations can be used in this manner to interchange two rows of a matrix. Hence, the first elementary row operation is, in fact, redundant.