CHAPTER 1
Systems of Linear Equations

Section 1.1 Introduction to Systems of Linear Equations

1. Because the equation is in the form $a_1x + a_2y = b$, it *is* linear in the variables x and y.

3. Because the equation cannot be written in the form $a_1x + a_2y = b$, it is *not* linear in the variables x and y.

5. Because the equation cannot be written in the form $a_1x + a_2y = b$, it is *not* linear in the variables x and y.

7. Choosing y as the free variable, let $y = t$ and obtain

$$\begin{aligned} 2x - 4t &= 0 \\ 2x &= 4t \\ x &= 2t \end{aligned}$$

Thus, you can describe the solution set as $x = 2t$ and $y = t$, where t is any real number.

9. Choosing y and z as the free variables, let $y = s$ and $z = t$, and obtain $x + s + t = 1$ or $x = 1 - s - t$. Thus, you can describe the solution set as $x = 1 - s - t$, $y = s$ and $z = t$, where s and t are any real numbers.

11. From Equation 2 we have $x_2 = 3$. Substituting this value into Equation 1 produces $x_1 - 3 = 2$ or $x_1 = 5$. Thus, the system has exactly one solution: $x_1 = 5$ and $x_2 = 3$.

13. From Equation 3 you can conclude that $z = 0$. Substituting this value into Equation 2 produces

$$\begin{aligned} 2y + 0 &= 3 \\ y &= \tfrac{3}{2}. \end{aligned}$$

Finally, by substituting $y = \frac{3}{2}$ and $z = 0$ into Equation 1, you obtain

$$\begin{aligned} -x + \tfrac{3}{2} - 0 &= 0 \\ x &= \tfrac{3}{2}. \end{aligned}$$

Thus, the system has exactly one solution: $x = \frac{3}{2}$, $y = \frac{3}{2}$, and $z = 0$.

15. Begin by rewriting the system in row-echelon form as follows.
The equations are interchanged.

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ 5x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

The first equation is multiplied by $\frac{1}{2}$.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 &= 0 \\ 5x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

Adding -5 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 &= 0 \\ -\tfrac{1}{2}x_2 - x_3 &= 0 \end{aligned}$$

The second equation is multiplied by -2.

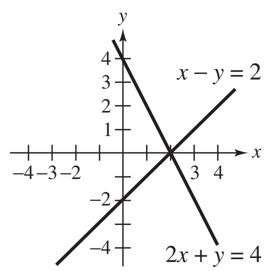
$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t. Because $x_2 = 2x_3$ and $x_1 = -\frac{1}{2}x_2$, you can describe the solution set as

$$x_1 = -t,\ x_2 = 2t,\ x_3 = t,\ t \text{ is any real number.}$$

17. $2x + y = 4$
$x - y = 2$

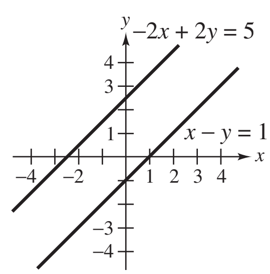
Adding the first equation to the second equation produces a new second equation, $3x = 6$, or $x = 2$. Hence, $y = 0$, and the solution is $x = 2$, $y = 0$. This is the point where the two lines intersect.



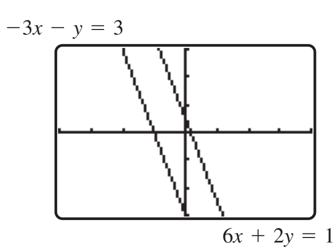
19. Adding 2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - y &= 1 \\ 0 &= 7 \end{aligned}$$

Because the second equation is an absurdity, you can conclude that the original system of equations has no solution. Geometrically, the two lines are parallel.

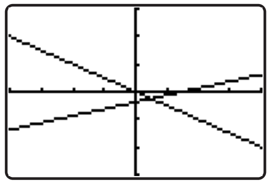


21. (a)



(b) The system is inconsistent.

23. (a)



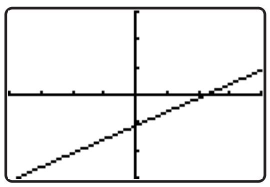
(b) The system is consistent.

(c) The solution is approximately $x = \frac{1}{2}$, $y = -\frac{1}{4}$.

(d) Adding $-\frac{1}{4}$ times the first equation to the second equation produces the new second equation $3y = -\frac{3}{4}$, or $y = -\frac{1}{4}$. Hence, $x = \frac{1}{2}$, and the solution is $x = \frac{1}{2}$, $y = -\frac{1}{4}$.

(e) The solutions in (c) and (d) are the same.

25. (a)



(b) The system is consistent.

(c) There are infinite solutions.

(d) The second equation is the result of multiplying both sides of the first equation by -0.2. A parametric representation of the solution set is given by

$x = \frac{9}{4} + 2t$, $y = t$, t is any real number.

(e) The solutions in (c) and (d) are consistent.

27. Adding -3 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 &= -1 \end{aligned}$$

Now, using back-substitution you can conclude that the system has exactly one solution: $x_1 = -1$ and $x_2 = -1$.

29. Interchanging the two equations produces the system

$$\begin{aligned} u + 2v &= 120 \\ 2u + v &= 120. \end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} u + 2v &= 120 \\ -3v &= -120 \end{aligned}$$

Solving the second equation you can have $v = 40$. Substituting this value into the first equation gives $u + 80 = 120$ or $u = 40$. Thus, the system has exactly one solution: $u = 40$ and $v = 40$.

31. Dividing the first equation by 9 produces a new first equation.

$$x - \tfrac{1}{3}y = -\tfrac{1}{9}$$
$$\tfrac{1}{5}x + \tfrac{2}{5}y = -\tfrac{1}{3}$$

Adding $-\frac{1}{5}$ times the first equation to the second equation produces a new second equation.

$$x - \tfrac{1}{3}y = -\tfrac{1}{9}$$
$$\tfrac{7}{15}y = -\tfrac{14}{45}$$

Multiplying the second equation by $\frac{15}{7}$ produces a new second equation.

$$x - \tfrac{1}{3}y = -\tfrac{1}{9}$$
$$y = -\tfrac{2}{3}$$

Now, using back-substitution, you can substitute $y = -\frac{2}{3}$ into the first equation to obtain $x + \frac{2}{9} = -\frac{1}{9}$ or $x = -\frac{1}{3}$. Thus you can conclude that the system has exactly one solution: $x = -\frac{1}{3}$ and $y = -\frac{2}{3}$.

33. To begin, change the form of the first equation.

$$\tfrac{1}{2}x + \tfrac{1}{3}y = \tfrac{23}{6}$$
$$x - 2y = 5$$

Multiplying the first equation by 2 yields a new first equation.

$$x + \tfrac{2}{3}y = \tfrac{23}{3}$$
$$x - 2y = 5$$

Subtracting the first equation from the second equation yields a new second equation.

$$x + \tfrac{2}{3}y = \tfrac{23}{3}$$
$$-\tfrac{8}{3}y = -\tfrac{8}{3}$$

Dividing the second equation by $-\frac{8}{3}$ yields a new second equation.

$$x + \tfrac{2}{3}y = \tfrac{23}{3}$$
$$y = 1$$

Now, using back-substitution you can conclude that the system has exactly one solution: $x = 7$ and $y = 1$.

35. Multiplying the first equation by 50 and the second equation by 100 produces a new system.

$$\begin{aligned} x_1 - 2.5x_2 &= -9.5 \\ 3x_1 + 4x_2 &= 52 \end{aligned}$$

Adding -3 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 2.5x_2 &= -9.5 \\ 11.5x_2 &= 80.5 \end{aligned}$$

Now, using back-substitution, you can conclude that the system has exactly one solution: $x_1 = 8$ and $x_2 = 7$.

37. Adding -2 times the first equation to the second equation yields a new second equation.

$$\begin{aligned} x + y + z &= 6 \\ -3y - z &= -9 \\ 3x - z &= 0 \end{aligned}$$

Adding -3 times the first equation to the third equation yields a new third equation.

$$\begin{aligned} x + y + z &= 6 \\ -3y - z &= -9 \\ -3y - 4z &= -18 \end{aligned}$$

Dividing the second equation by -3 yields a new second equation.

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ -3y - 4z &= -18 \end{aligned}$$

Adding 3 times the second equation to the third equation yields a new third equation.

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ -3z &= -9 \end{aligned}$$

Dividing the third equation by -3 yields a new third equation.

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ z &= 3 \end{aligned}$$

Now, using back-substitution you can conclude that the system has exactly one solution: $x = 1$, $y = 2$, and $z = 3$.

39. Dividing the first equation by 3 yields a new first equation.

$$\begin{aligned} x_1 - \tfrac{2}{3}x_2 + \tfrac{4}{3}x_3 &= \tfrac{1}{3} \\ x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - 3x_2 + 6x_3 &= 8 \end{aligned}$$

Subtracting the first equation from the second equation yields a new second equation.

$$\begin{aligned} x_1 - \tfrac{2}{3}x_2 + \tfrac{4}{3}x_3 &= \tfrac{1}{3} \\ \tfrac{5}{3}x_2 - \tfrac{10}{3}x_3 &= \tfrac{8}{3} \\ 2x_1 - 3x_2 + 6x_3 &= 8 \end{aligned}$$

Adding -2 times the first equation to the third equation yields a new third equation.

$$\begin{aligned} x_1 - \tfrac{2}{3}x_2 + \tfrac{4}{3}x_3 &= \tfrac{1}{3} \\ \tfrac{5}{3}x_2 - \tfrac{10}{3}x_3 &= \tfrac{8}{3} \\ -\tfrac{5}{3}x_2 + \tfrac{10}{3}x_3 &= \tfrac{22}{3} \end{aligned}$$

At this point you should recognize that Equations 2 and 3 cannot both be satisfied. Thus, the original system of equations has no solution.

41. Dividing the first equation by 2 yields a new first equation.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ 4x_1 \qquad + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Adding -4 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ -2x_2 + 8x_3 &= 2 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Adding 2 times the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ -2x_2 + 8x_3 &= 2 \\ 4x_2 - 16x_3 &= -4 \end{aligned}$$

Dividing the second equation by -2 yields a new second equation.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ x_2 - 4x_3 &= -1 \\ 4x_2 - 16x_3 &= -4 \end{aligned}$$

Adding -4 times the second equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ x_2 - 4x_3 &= -1 \\ 0 &= 0 \end{aligned}$$

Adding $-\frac{1}{2}$ times the second equation to the first equation produces a new first equation.

$$\begin{aligned} x_1 \qquad + \tfrac{1}{2}x_3 &= \tfrac{5}{2} \\ x_2 - 4x_3 &= -1 \end{aligned}$$

Choosing $x_3 = t$ as the free variable, the solution is $x_1 = \frac{5}{2} - \frac{1}{2}t$, $x_2 = -1 + 4t$, and $x_3 = t$, where t is any real number.

43. Adding -5 times the first equation to the second equation yields a new second equation.

$$\begin{aligned} x - 3y + 2z &= 18 \\ 0 &= -72 \end{aligned}$$

Because the second equation is an absurdity, you can conclude that the original system of equations has no solution.

45. Adding -2 times the first equation to the second, 3 times the first equation to the third, and -1 times the first equation to the fourth, produces

$$\begin{aligned} x + y + z + w &= 6 \\ y - 2z - 3w &= -12 \\ 7y + 4z + 5w &= 22 \\ y - 2z &= -6. \end{aligned}$$

Adding -7 times the second equation to the third, and -1 times the second equation to the fourth, produces

$$\begin{aligned} x + y + z + w &= 6 \\ y - 2z - 3w &= -12 \\ 18z + 26w &= 106 \\ 3w &= 6. \end{aligned}$$

Using back-substitution, you find that the original system has exactly one solution: $x = 1, y = 0, z = 3$, and $w = 2$.

47. Using a computer or graphing calculator, you obtain
$x_1 = 11.2415, x_2 = -60.9029, x_3 = 40.7674, x_4 = 27.4267$ (answers might vary slightly).

49. Using a computer or graphing calculator you obtain
$x_1 = 8.1124, x_2 = -4.5588, x_3 = -9.0448$ (answers might vary slightly).

51. $x = y = z = 0$ is clearly a solution.

Dividing the first equation by 4 yields a new first equation.

$$\begin{aligned} x + \tfrac{3}{4}y + \tfrac{17}{4}z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

Adding -5 times the first equation to the second equation yields a new second equation.

$$\begin{aligned} x + \tfrac{3}{4}y + \tfrac{17}{4}z &= 0 \\ \tfrac{1}{4}y + \tfrac{3}{4}z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

Adding -4 times the first equation to the third equation yields a new third equation.

$$\begin{aligned} x + \tfrac{3}{4}y + \tfrac{17}{4}z &= 0 \\ \tfrac{1}{4}y + \tfrac{3}{4}z &= 0 \\ -y + 2z &= 0 \end{aligned}$$

Multiplying the second equation by 4 yields a new second equation.

$$\begin{aligned} x + \tfrac{3}{4}y + \tfrac{17}{4}z &= 0 \\ y + 3z &= 0 \\ -y + 2z &= 0 \end{aligned}$$

Adding the second equation to the third equation yields a new third equation.

$$\begin{aligned} x + \tfrac{3}{4}y + \tfrac{17}{4}z &= 0 \\ y + 3z &= 0 \\ 5z &= 0 \end{aligned}$$

Dividing the third equation by 5 yields a new third equation.

$$\begin{aligned} x + \tfrac{3}{4}y + \tfrac{17}{4}z &= 0 \\ y + 3z &= 0 \\ z &= 0 \end{aligned}$$

Now, using back-substitution, you can conclude that the system has exactly one solution: $x = 0$, $y = 0$, and $z = 0$.

53. $x = y = z = 0$ is clearly a solution.

Dividing the first equation by 5 yields a new first equation.

$$\begin{aligned} x + y - \tfrac{1}{5}z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

Adding -10 times the first equation to the second equation yields a new second equation.

$$\begin{aligned} x + y - \tfrac{1}{5}z &= 0 \\ -5y + 4z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

Adding -5 times the first equation to the third equation yields a new third equation.

$$\begin{aligned} x + y - \tfrac{1}{5}z &= 0 \\ -5y + 4z &= 0 \\ 10y - 8z &= 0 \end{aligned}$$

Dividing the second equation by -5 yields a new second equation.

$$\begin{aligned} x + y - \tfrac{1}{5}z &= 0 \\ y - \tfrac{4}{5}z &= 0 \\ 10y - 8z &= 0 \end{aligned}$$

Adding -10 times the second equation to the third equation yields a new third equation.

$$\begin{aligned} x + y - \tfrac{1}{5}z &= 0 \\ y - \tfrac{4}{5}z &= 0 \\ 0 &= 0 \end{aligned}$$

Adding -1 times the second equation to the first equation yields a new first equation.

$$\begin{aligned} x + \tfrac{3}{5}z &= 0 \\ y - \tfrac{4}{5}z &= 0 \end{aligned}$$

Choosing $z = t$ as the free variable you find the solution to be $x = -\frac{3}{5}t$, $y = \frac{4}{5}t$, and $z = t$, where t is any real number.

55. (a) True, you can describe the entire solution set using parametric representation.

$ax + by = c$

Choosing $y = t$ as the free variable, the solution is $x = \frac{c}{a} - \frac{b}{a}t$, $y = t$, where t is any real number.

(b) False, for example consider the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

which is an inconsistent system.

(c) False, a consistent system may have only one solution.

57. Since $x_1 = t$ and $x_2 = 3t - 4 = 3x_1 - 4$, one answer is the system

$$\begin{aligned} 3x_1 - x_2 &= 4 \\ -3x_1 + x_2 &= -4 \end{aligned}$$

Letting $x_2 = t$, you get $x_1 = \frac{4 + t}{3} = \frac{4}{3} + \frac{t}{3}$.

59. Substituting $X = 1/x$ and $Y = 1/y$ into the original system yields

$$\begin{aligned} 12X - 12Y &= 7 \\ 3X + 4Y &= 0. \end{aligned}$$

Reduce this system to row-echelon form. Dividing the first equation by 12 yields a new first equation.

$$\begin{aligned} X - Y &= \tfrac{7}{12} \\ 3X + 4Y &= 0 \end{aligned}$$

Adding -3 times the first equation to the second equation yields a new second equation.

$$\begin{aligned} X - Y &= \tfrac{7}{12} \\ 7Y &= -\tfrac{7}{4} \end{aligned}$$

Dividing the second equation by 7 yields a new second equation.

$$\begin{aligned} X - Y &= \tfrac{7}{12} \\ Y &= -\tfrac{1}{4} \end{aligned}$$

Thus, $Y = -1/4$ and $X = 1/3$. Because $X = 1/x$ and $Y = 1/y$, the solution of the original system of equations is $x = 3$ and $y = -4$.

61. Reduce the system to row-echelon form. Dividing the first equation by $\cos\theta$ yields a new first equation.

$$\begin{aligned} x + \left(\frac{\sin\theta}{\cos\theta}\right)y &= \frac{1}{\cos\theta} \\ (-\sin\theta)x + (\cos\theta)y &= 0 \end{aligned}$$

Multiplying the first equation by $\sin\theta$ and adding to the second equation yields a new second equation.

$$\begin{aligned} x + \left(\frac{\sin\theta}{\cos\theta}\right)y &= \frac{1}{\cos\theta} \\ \left(\frac{1}{\cos\theta}\right)y &= \frac{\sin\theta}{\cos\theta} \quad \left(\text{Since } \frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta}\right) \end{aligned}$$

Multiplying the second equation by $\cos\theta$ yields a new second equation.

$$\begin{aligned} x + \left(\frac{\sin\theta}{\cos\theta}\right)y &= \frac{1}{\cos\theta} \\ y &= \sin\theta \end{aligned}$$

Substituting $y = \sin\theta$ into the first equation yields

$$x + \left(\frac{\sin\theta}{\cos\theta}\right)\sin\theta = \frac{1}{\cos\theta}$$

$$x = \frac{1 - \sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta.$$

Thus, the solution of the original system of equations is $x = \cos\theta$ and $y = \sin\theta$.

63. Reducing the system to row-echelon form, you have

$$x + \frac{k}{4}y = \frac{3}{2}$$
$$y = \frac{6}{k+2}.$$

Now, if $k = -2$, there is no solution. Otherwise, you can see by back-substitution that there would be exactly one solution. Hence, the answer is all k except -2.

65. Reduce the system to row-echelon form.

$$\begin{aligned} x + \quad ky &= 0 \\ (1 - k^2)y &= 0 \end{aligned}$$
$$\begin{aligned} x + \quad ky &= 0 \\ y &= 0,\ 1 - k^2 \neq 0 \end{aligned}$$
$$\begin{aligned} x \quad &= 0 \\ y &= 0,\ 1 - k^2 \neq 0 \end{aligned}$$

Thus, if $1 - k^2 \neq 0$, that is if $k \neq \pm 1$, the system will have exactly one solution.

67. To begin, reduce the system to row-echelon form.

$$x + 2y + kz = 6$$
$$(8 - 3k)z = -14$$

This system will have a solution unless $8 - 3k = 0$, that is, $k = \frac{8}{3}$.

69. Reducing the system to row-echelon form, you have

$$\begin{aligned} x + \quad y + \quad kz &= 3 \\ (k-1)y + \quad (1-k)z &= -1 \\ (1-k)y + \quad (1-k^2)z &= 1 - 3k \end{aligned}$$

$$\begin{aligned} x + \quad y + \quad kz &= 3 \\ (k-1)y + \quad (1-k)z &= -1 \\ (-k^2 - k + 2)z &= -3k. \end{aligned}$$

If $-k^2 - k + 2 = 0$, then there is no solution. Hence, if $k = 1$ or $k = -2$, there is not a unique solution.

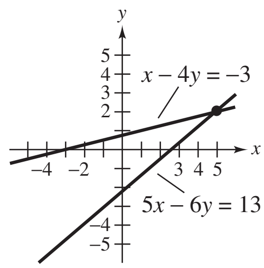
71. (a) All three of the lines will intersect in exactly one point (corresponding to the solution point).

(b) All three of the lines will coincide (every point on these lines is a solution point).

(c) The three lines have no common point.

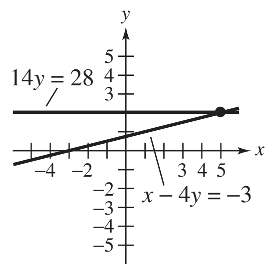
73. Answers vary. Hint: Choose three different values for x and solve the resulting system of linear equations in the variables a, b, and c.

75. $$\begin{aligned} x - 4y &= -3 \\ 5x - 6y &= 13 \end{aligned}$$



$$\begin{aligned} x - 4y &= -3 \\ 14y &= 28 \end{aligned}$$

$$\begin{aligned} x - 4y &= -3 \\ y &= 2 \end{aligned}$$



$$\begin{aligned} x &= 5 \\ y &= 2 \end{aligned}$$