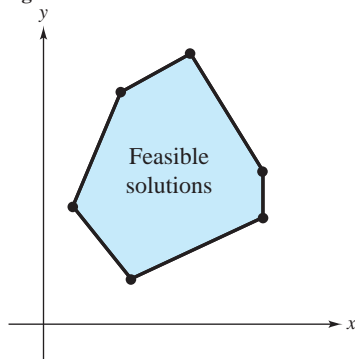


9.2 LINEAR PROGRAMMING INVOLVING TWO VARIABLES

Figure 9.10



The objective function has its optimal value at one of the vertices of the region determined by the constraints.

Many applications in business and economics involve a process called **optimization**, in which we are required to find the minimum cost, the maximum profit, or the minimum use of resources. In this section we discuss one type of optimization problem called **linear programming**.

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints**. The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions**.

For example, consider a linear programming problem in which we are asked to maximize the value of

$$z = ax + by \quad \text{Objective function}$$

subject to a set of constraints that determine the region indicated in Figure 9.10. Because every point in the region satisfies each constraint, it is not clear how we should go about finding the point that yields a maximum value of z . Fortunately, it can be shown that if there is an optimal solution, it must occur at one of the vertices of the region. In other words, *we can find the maximum value by testing z at each of the vertices*, as illustrated in Example 1.

Theorem 9.1

Optimal Solution of a Linear Programming Problem

If a linear programming problem has a solution, it must occur at a vertex of the set of feasible solutions. If the problem has more than one solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

EXAMPLE 1 Solving a Linear Programming Problem

Find the maximum value of

$$z = 3x + 2y \quad \text{Objective function}$$

subject to the following constraints.

$$\left. \begin{array}{l} x \geq 0 \\ y \geq 0 \\ x + 2y \leq 4 \\ x - y \leq 1 \end{array} \right\} \quad \text{Constraints}$$

Solution The constraints form the region shown in Figure 9.11. At the four vertices of this region, the objective function has the following values.

$$\text{At } (0, 0): z = 3(0) + 2(0) = 0$$

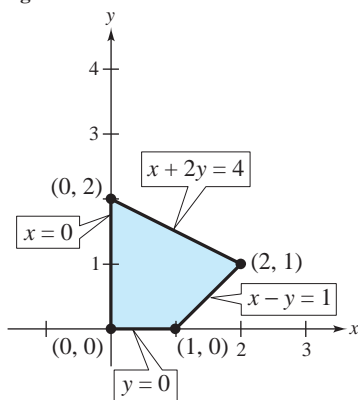
$$\text{At } (1, 0): z = 3(1) + 2(0) = 3$$

$$\text{At } (2, 1): z = 3(2) + 2(1) = 8 \quad (\text{Maximum value of } z)$$

$$\text{At } (0, 2): z = 3(0) + 2(2) = 4$$

Thus, the maximum value of z is 8, and this occurs when $x = 2$ and $y = 1$.

Figure 9.11



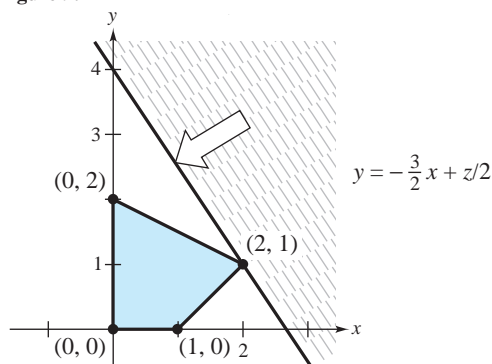
REMARK: In Example 1, try testing some of the interior points in the region. You will see that the corresponding values of z are less than 8.

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in the form

$$y = -\frac{3}{2}x + \frac{z}{2}.$$

This equation represents a family of lines, each of slope $-3/2$. Of these infinitely many lines, we want the one that has the largest z -value, while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-3/2$, we want the one that has the largest y -intercept *and* intersects the given region, as shown in Figure 9.12. It should be clear that such a line will pass through one (or more) of the vertices of the region.

Figure 9.12



We outline the graphical method for solving a linear programming problem as follows.

Graphical Method of Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the following steps.

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are called *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, *if* an optimal solution exists, then it will occur at a vertex.)

These guidelines will work regardless of whether the objective function is to be maximized *or* minimized. For instance, in Example 1 the same test used to find the maximum value of z can be used to conclude that the minimum value of z is 0, and this occurs at the vertex $(0, 0)$.

EXAMPLE 2 Solving a Linear Programming Problem

Find the maximum value of the objective function

$$z = 4x + 6y \quad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left. \begin{array}{l} -x + y \leq 11 \\ x + y \leq 27 \\ 2x + 5y \leq 90. \end{array} \right\} \text{Constraints}$$

Solution The region bounded by the constraints is shown in Figure 9.13. By testing the objective function at each vertex, we obtain the following.

$$\begin{array}{l} \text{At } (0, 0): z = 4(0) + 6(0) = 0 \\ \text{At } (0, 11): z = 4(0) + 6(11) = 66 \\ \text{At } (5, 16): z = 4(5) + 6(16) = 116 \\ \text{At } (15, 12): z = 4(15) + 6(12) = 132 \quad \text{(Maximum value of } z) \\ \text{At } (27, 0): z = 4(27) + 6(0) = 108 \end{array}$$

Thus, the maximum value of z is 132, and this occurs when $x = 15$ and $y = 12$.

In the next example, we show that the same basic procedure can be used to solve a linear programming problem in which the objective function is to be *minimized*.

Figure 9.13

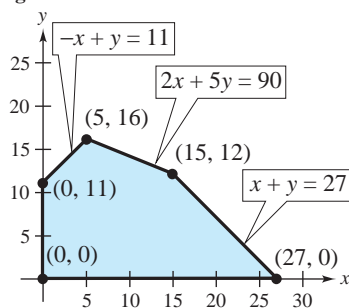
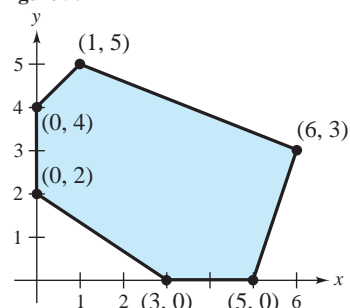


Figure 9.14

**EXAMPLE 3** *Minimizing an Objective Function*

Find the minimum value of the objective function

$$z = 5x + 7y \quad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left. \begin{aligned} 2x + 3y &\geq 6 \\ 3x - y &\leq 15 \\ -x + y &\leq 4 \\ 2x + 5y &\leq 27. \end{aligned} \right\} \text{Constraints}$$

Solution The region bounded by the constraints is shown in Figure 9.14. By testing the objective function at each vertex, we obtain the following.

$$\text{At } (0, 2): z = 5(0) + 7(2) = 14 \quad \text{(Minimum value of } z\text{)}$$

$$\text{At } (0, 4): z = 5(0) + 7(4) = 28$$

$$\text{At } (1, 5): z = 5(1) + 7(5) = 40$$

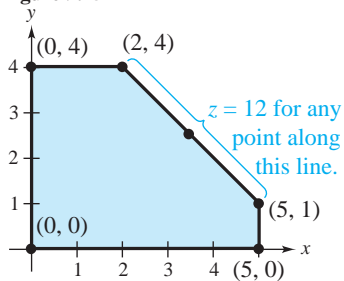
$$\text{At } (6, 3): z = 5(6) + 7(3) = 51$$

$$\text{At } (5, 0): z = 5(5) + 7(0) = 25$$

$$\text{At } (3, 0): z = 5(3) + 7(0) = 15$$

Thus, the minimum value of z is 14, and this occurs when $x = 0$ and $y = 2$.

Figure 9.15



REMARK: In Example 3, note that the steps used to find the minimum value are precisely the same ones we would use to find the maximum value. In other words, once we have evaluated the objective function at the vertices of the feasible region, we simply choose the largest value as the maximum and the smallest value as the minimum.

When solving a linear programming problem, it is possible that the maximum (or minimum) value occurs at *two* different vertices. For instance, at the vertices of the region shown in Figure 9.15, the objective function

$$z = 2x + 2y \quad \text{Objective function}$$

has the following values.

$$\text{At } (0, 0): z = 2(0) + 2(0) = 0$$

$$\text{At } (0, 4): z = 2(0) + 2(4) = 8$$

$$\text{At } (2, 4): z = 2(2) + 2(4) = 12 \quad \text{(Maximum value of } z\text{)}$$

$$\text{At } (5, 1): z = 2(5) + 2(1) = 12 \quad \text{(Maximum value of } z\text{)}$$

$$\text{At } (5, 0): z = 2(5) + 2(0) = 10$$

In this case, we can conclude that the objective function has a maximum value not only at the vertices $(2, 4)$ and $(5, 1)$, it also has a maximum value (of 12) at *any point on the line segment connecting these two vertices*.

Some linear programming problems have no optimal solution. This can occur if the region determined by the constraints is *unbounded*. Example 4 illustrates such a problem.

EXAMPLE 4 An Unbounded Region

Find the maximum value of

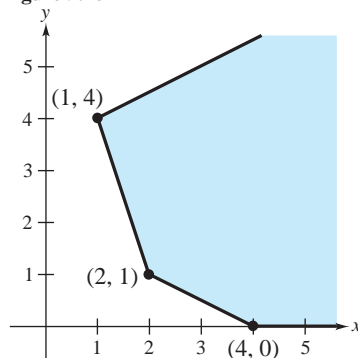
$$z = 4x + 2y \quad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left. \begin{array}{l} x + 2y \geq 4 \\ 3x + y \geq 7 \\ -x + 2y \leq 7. \end{array} \right\} \text{Constraints}$$

Solution The region determined by the constraints is shown in Figure 9.16. For this unbounded region, there is no maximum value of z . To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. By choosing x to be large, we can obtain values of $z = 4(x) + 2(0) = 4x$ that are as large as we want. Thus, there is no maximum value of z .

Figure 9.16



Applications

EXAMPLE 5 *An Application: Minimum Cost*

In Example 4 in Section 9.1, we set up a system of linear equations for the following problem. The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Now, suppose that dietary drink X costs \$0.12 per cup and drink Y costs \$0.15 per cup. How many cups of each drink should be consumed each day to minimize the cost and still meet the stated daily requirements?

Solution We begin by letting x be the number of cups of dietary drink X and y be the number of cups of dietary drink Y. Moreover, to meet the minimum daily requirements, the following inequalities must be satisfied.

$$\left. \begin{array}{l} \text{For calories: } 60x + 60y \geq 300 \\ \text{For vitamin A: } 12x + 6y \geq 36 \\ \text{For vitamin C: } 10x + 30y \geq 90 \\ x \geq 0 \\ y \geq 0 \end{array} \right\} \text{Constraints}$$

The cost C is given by

$$C = 0.12x + 0.15y. \quad \text{Objective function}$$

The graph of the region corresponding to the constraints is shown in Figure 9.17. To determine the minimum cost, we test C at each vertex of the region as follows.

$$\text{At } (0, 6): C = 0.12(0) + 0.15(6) = 0.90$$

$$\text{At } (1, 4): C = 0.12(1) + 0.15(4) = 0.72$$

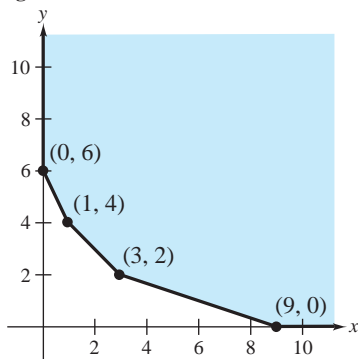
$$\text{At } (3, 2): C = 0.12(3) + 0.15(2) = 0.66$$

$$\text{At } (9, 0): C = 0.12(9) + 0.15(0) = 1.08$$

(Minimum value of C)

Thus, the minimum cost is \$0.66 per day, and this occurs when three cups of drink X and two cups of drink Y are consumed each day.

Figure 9.17



SECTION 9.2 EXERCISES

In Exercises 1–16, find the minimum and maximum values of the given objective function, subject to the indicated constraints.

1. Objective function:

$$z = 3x + 2y$$

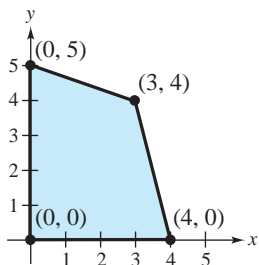
Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$x + 3y \leq 15$$

$$4x + y \leq 16$$



3. Objective function:

$$z = 5x + 0.5y$$

Constraints:

(See Exercise 1.)

5. Objective function:

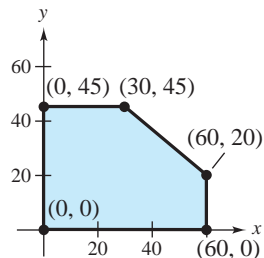
$$z = 10x + 7y$$

Constraints:

$$0 \leq x \leq 60$$

$$0 \leq y \leq 45$$

$$5x + 6y \leq 420$$



2. Objective function:

$$z = 4x + 3y$$

Constraints:

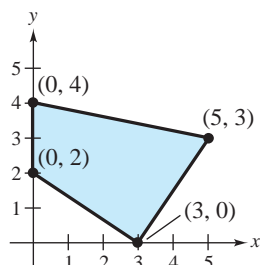
$$x \geq 0$$

$$y \geq 0$$

$$2x + 3y \geq 6$$

$$3x - 2y \leq 9$$

$$x + 5y \leq 20$$



4. Objective function:

$$z = x + 6y$$

Constraints:

(See Exercise 2.)

6. Objective function:

$$z = 50x + 35y$$

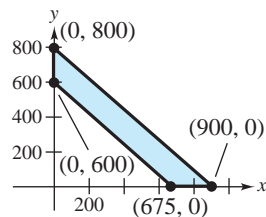
Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$8x + 9y \leq 7200$$

$$8x + 9y \geq 5400$$



7. Objective function:

$$z = 25x + 30y$$

Constraints:

(See Exercise 5.)

9. Objective function:

$$z = 4x + 5y$$

Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$4x + 3y \geq 27$$

$$x + y \geq 8$$

$$3x + 5y \geq 30$$

11. Objective function:

$$z = 2x + 7y$$

Constraints:

(See Exercise 9.)

13. Objective function:

$$z = 4x + y$$

Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$x + 2y \leq 40$$

$$x + y \geq 30$$

$$2x + 3y \geq 72$$

15. Objective function:

$$z = x + 4y$$

Constraints:

(See Exercise 13.)

8. Objective function:

$$z = 16x + 18y$$

Constraints:

(See Exercise 6.)

10. Objective function:

$$z = 4x + 5y$$

Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$2x + 2y \leq 10$$

$$x + 2y \leq 6$$

12. Objective function:

$$z = 2x - y$$

Constraints:

(See Exercise 10.)

14. Objective function:

$$z = x$$

Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$2x + 3y \leq 60$$

$$2x + y \leq 28$$

$$4x + y \leq 48$$

16. Objective function:

$$z = y$$

Constraints:

(See Exercise 14.)

In Exercises 17–20, maximize the given objective function subject to the constraints $3x_1 + x_2 \leq 15$ and $4x_1 + 3x_2 \leq 30$ where $x_1, x_2 \geq 0$.

17. $z = 2x_1 + x_2$

19. $z = x_1 + x_2$

18. $z = 5x_1 + x_2$

20. $z = 3x_1 + x_2$

- 21.** A merchant plans to sell two models of home computers at costs of \$250 and \$400, respectively. The \$250 model yields a profit of \$45 and the \$400 model yields a profit of \$50. The merchant estimates that the total monthly demand will not exceed 250 units. Find the number of units of each model that should be stocked in order to maximize profit. Assume that the merchant does not want to invest more than \$70,000 in computer inventory.
- 22.** A fruit grower has 150 acres of land available to raise two crops, A and B. It takes one day to trim an acre of crop A and two days to trim an acre of crop B, and there are 240 days per year available for trimming. It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking. Find the number of acres of each fruit that should be planted to maximize profit, assuming that the profit is \$140 per acre for crop A and \$235 per acre for crop B.
- 23.** A farming cooperative mixes two brands of cattle feed. Brand X costs \$25 per bag and contains 2 units of nutritional element A, 2 units of element B, and 2 units of element C. Brand Y costs \$20 per bag and contains 1 unit of nutritional element A, 9 units of element B, and 3 units of element C. Find the number of bags of each brand that should be mixed to produce a mixture having a minimum cost per bag. The minimum requirements of nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively.
- 24.** Two gasolines, type A and type B, have octane ratings of 80 and 92, respectively. Type A costs \$0.83 per liter and type B costs \$0.98 per liter. Determine the blend of minimum cost with an octane rating of at least 90. [Hint: Let x be the fraction of each liter that is type A and y be the fraction that is type B.]

In Exercises 25–30, the given linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. (In each problem, the objective function is to be maximized.)

- | | |
|--|--|
| <p>25. Objective function:
 $z = 2.5x + y$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $3x + 5y \leq 15$
 $5x + 2y \leq 10$</p> | <p>26. Objective function:
 $z = x + y$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $-x + y \leq 1$
 $-x + 2y \leq 4$</p> |
|--|--|

- 27.** Objective function:
 $z = -x + 2y$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $x \leq 10$
 $x + y \leq 7$

- 29.** Objective function:
 $z = 3x + 4y$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $x + y \leq 1$
 $2x + y \geq 4$

- 28.** Objective function:
 $z = x + y$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $-x + y \leq 0$
 $-3x + y \geq 3$

- 30.** Objective function:
 $z = x + 2y$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $x + 2y \leq 4$
 $2x + y \leq 4$

In Exercises 31 and 32, determine t -values such that the given objective function has a maximum value at the indicated vertex.

- 31.** Objective function:
 $z = x + ty$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $x \leq 1$
 $y \leq 1$
 (a) (0, 0) (b) (1, 0)
 (c) (1, 1) (d) (0, 1)

- 32.** Objective function:
 $z = 3x + ty$
 Constraints:
 $x \geq 0$
 $y \geq 0$
 $x + 2y \leq 4$
 $x - y \leq 1$
 (a) (0, 0) (b) (1, 0)
 (c) (2, 1) (d) (0, 2)
 [Note: For $t = 2$, this problem is equivalent to that given in Example 1.]