8.4 COMPLEX VECTOR SPACES AND INNER PRODUCTS

All the vector spaces we have studied thus far in the text are *real vector spaces* since the scalars are real numbers. A **complex vector space** is one in which the scalars are complex numbers. Thus, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are vectors in a complex vector space, then a linear combination is of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

where the scalars $c_1, c_2, \ldots, c_m$ are complex numbers. The complex version of R^n is the complex vector space C^n consisting of ordered n-tuples of complex numbers. Thus, a vector in C^n has the form

$$\mathbf{v} = (a_1 + b_1 i, a_2 + b_2 i, \ldots, a_n + b_n i).$$

It is also convenient to represent vectors in C^n by column matrices of the form

$$\mathbf{v} = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix}.$$

As with R^n, the operations of addition and scalar multiplication in C^n are performed component by component.

EXAMPLE 1 ***Vector Operations in C^n***

Let

$$\mathbf{v} = (1 + 2i, 3 - i) \quad \text{and} \quad \mathbf{u} = (-2 + i, 4)$$

be vectors in the complex vector space C^2, and determine the following vectors.

(a) $\mathbf{v} + \mathbf{u}$ (b) $(2 + i)\mathbf{v}$ (c) $3\mathbf{v} - (5 - i)\mathbf{u}$

Solution (a) In column matrix form, the sum $\mathbf{v} + \mathbf{u}$ is

$$\mathbf{v} + \mathbf{u} = \begin{bmatrix} 1 + 2i \\ 3 - i \end{bmatrix} + \begin{bmatrix} -2 + i \\ 4 \end{bmatrix} = \begin{bmatrix} -1 + 3i \\ 7 - i \end{bmatrix}.$$

(b) Since $(2 + i)(1 + 2i) = 5i$ and $(2 + i)(3 - i) = 7 + i$, we have

$$(2 + i)\mathbf{v} = (2 + i)(1 + 2i, 3 - i) = (5i, 7 + i).$$

(c)
$$\begin{aligned} 3\mathbf{v} - (5 - i)\mathbf{u} &= 3(1 + 2i, 3 - i) - (5 - i)(-2 + i, 4) \\ &= (3 + 6i, 9 - 3i) - (-9 + 7i, 20 - 4i) \\ &= (12 - i, -11 + i) \end{aligned}$$

Many of the properties of R^n are shared by C^n. For instance, the scalar multiplicative identity is the scalar 1 and the additive identity in C^n is $\mathbf{0} = (0, 0, 0, \ldots, 0)$. The **standard basis** for C^n is simply

$$\begin{aligned}\mathbf{e}_1 &= (1, 0, 0, \ldots, 0)\\ \mathbf{e}_2 &= (0, 1, 0, \ldots, 0)\\ &\vdots\\ \mathbf{e}_n &= (0, 0, 0, \ldots, 1)\end{aligned}$$

which is the standard basis for R^n. Since this basis contains n vectors, it follows that the dimension of C^n is n. Other bases exist; in fact, any linearly independent set of n vectors in C^n can be used, as we demonstrate in Example 2.

EXAMPLE 2 *Verifying a Basis*

Show that

$$S = \{\overbrace{(i, 0, 0)}^{\mathbf{v}_1}, \overbrace{(i, i, 0)}^{\mathbf{v}_2}, \overbrace{(0, 0, i)}^{\mathbf{v}_3}\}$$

is a basis for C^3.

Solution Since C^3 has a dimension of 3, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis if it is linearly independent. To check for linear independence, we set a linear combination of the vectors in S equal to $\mathbf{0}$ as follows.

$$\begin{aligned}c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= (0, 0, 0)\\ (c_1 i, 0, 0) + (c_2 i, c_2 i, 0) + (0, 0, c_3 i) &= (0, 0, 0)\\ ((c_1 + c_2)i, c_2 i, c_3 i) &= (0, 0, 0)\end{aligned}$$

This implies that

$$\begin{aligned}(c_1 + c_2)i &= 0\\ c_2 i &= 0\\ c_3 i &= 0.\end{aligned}$$

Therefore, $c_1 = c_2 = c_3 = 0$, and we conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

EXAMPLE 3 *Representing a Vector in C^n by a Basis*

Use the basis S in Example 2 to represent the vector

$$\mathbf{v} = (2, i, 2 - i).$$

Solution By writing

$$\begin{aligned}\mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= ((c_1 + c_2)i, c_2 i, c_3 i) \\ &= (2, i, 2 - i)\end{aligned}$$

we obtain

$$\begin{aligned}(c_1 + c_2)i &= 2 \\ c_2 i &= i \\ c_3 i &= 2 - i\end{aligned}$$

which implies that $c_2 = 1$ and

$$c_1 = \frac{2 - i}{i} = -1 - 2i \quad \text{and} \quad c_3 = \frac{2 - i}{i} = -1 - 2i.$$

Therefore,

$$\mathbf{v} = (-1 - 2i)\mathbf{v}_1 + \mathbf{v}_2 + (-1 - 2i)\mathbf{v}_3.$$

Try verifying that this linear combination yields $(2, i, 2 - i)$.

Other than C^n, there are several additional examples of complex vector spaces. For instance, the set of $m \times n$ complex matrices with matrix addition and scalar multiplication forms a complex vector space. Example 4 describes a complex vector space in which the vectors are functions.

EXAMPLE 4 ***The Space of Complex-Valued Functions***

Consider the set S of *complex-valued* functions of the form

$$\mathbf{f}(x) = \mathbf{f}_1(x) + i\mathbf{f}_2(x)$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ are real-valued functions of a real variable. The set of complex numbers form the scalars for S and vector addition is defined by

$$\begin{aligned}\mathbf{f}(x) + \mathbf{g}(x) &= [\mathbf{f}_1(x) + i\mathbf{f}_2(x)] + [\mathbf{g}_1(x) + i\mathbf{g}_2(x)] \\ &= [\mathbf{f}_1(x) + \mathbf{g}_1(x)] + i[\mathbf{f}_2(x) + \mathbf{g}_2(x)].\end{aligned}$$

It can be shown that S, scalar multiplication, and vector addition form a complex vector space. For instance, to show that S is closed under scalar multiplication, we let $c = a + bi$ be a complex number. Then

$$\begin{aligned}c\mathbf{f}(x) &= (a + bi)[\mathbf{f}_1(x) + i\mathbf{f}_2(x)] \\ &= [a\mathbf{f}_1(x) - b\mathbf{f}_2(x)] + i[b\mathbf{f}_1(x) + a\mathbf{f}_2(x)]\end{aligned}$$

is in S.

The definition of the Euclidean inner product in C^n is similar to that of the standard dot product in R^n, except that here the second factor in each term is a complex conjugate.

Definition of Euclidean Inner Product in C^n

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in C^n. The **Euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}.$$

REMARK: Note that if $\mathbf{u}$ and $\mathbf{v}$ happen to be "real," then this definition agrees with the standard inner (or dot) product in R^n.

EXAMPLE 5 ***Finding the Euclidean Inner Product in C^3***

Determine the Euclidean inner product of the vectors

$$\mathbf{u} = (2 + i, 0, 4 - 5i) \quad \text{and} \quad \mathbf{v} = (1 + i, 2 + i, 0).$$

Solution

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3}\\ &= (2 + i)(1 - i) + 0(2 - i) + (4 - 5i)(0)\\ &= 3 - i\end{aligned}$$

Several properties of the Euclidean inner product C^n are stated in the following theorem.

Theorem 8.7 Properties of the Euclidean Inner Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in C^n and let k be a complex number. Then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
4. $\mathbf{u} \cdot (k\mathbf{v}) = \overline{k}(\mathbf{u} \cdot \mathbf{v})$
5. $\mathbf{u} \cdot \mathbf{u} \geq 0$
6. $\mathbf{u} \cdot \mathbf{u} = 0$if and only if $\mathbf{u} = \mathbf{0}$.

Proof We prove the first property and leave the proofs of the remaining properties to you. Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, \ldots, v_n).$$

Then

$$\begin{aligned}\overline{\mathbf{v} \cdot \mathbf{u}} &= \overline{v_1\overline{u_1} + v_2\overline{u_2} + \cdots + v_n\overline{u_n}}\\ &= \overline{v_1\overline{u_1}} + \overline{v_2\overline{u_2}} + \cdots + \overline{v_n\overline{u_n}}\\ &= \overline{v_1}u_1 + \overline{v_2}u_2 + \cdots + \overline{v_n}u_n\end{aligned}$$

$$= u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}$$
$$= \mathbf{u} \cdot \mathbf{v}.$$

We now use the Euclidean inner product in C^n to define the Euclidean norm (or length) of a vector in C^n and the Euclidean distance between two vectors in C^n.

Definition of Norm and Distance in C^n

The **Euclidean norm (**or **length)** of $\mathbf{u}$ in C^n is denoted by $\|\mathbf{u}\|$ and is

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2}.$$

The **Euclidean distance** between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| .$$

The Euclidean norm and distance may be expressed in terms of components as

$$\|\mathbf{u}\| = (|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2)^{1/2}$$
$$d(\mathbf{u}, \mathbf{v}) = (|u_1 - v_1|^2 + |u_2 - v_2|^2 + \cdots + |u_n - v_n|^2)^{1/2}.$$

EXAMPLE 6 ***Finding the Euclidean Norm and Distance in C^n***

Determine the norms of the vectors

$$\mathbf{u} = (2 + i, 0, 4 - 5i) \qquad \text{and} \qquad \mathbf{v} = (1 + i, 2 + i, 0)$$

and find the distance between $\mathbf{u}$ and $\mathbf{v}$.

Solution The norms of $\mathbf{u}$ and $\mathbf{v}$ are given as follows.

$$\begin{aligned}\|\mathbf{u}\| &= (|u_1|^2 + |u_2|^2 + |u_3|^2)^{1/2}\\ &= [(2^2 + 1^2) + (0^2 + 0^2) + (4^2 + 5^2)]^{1/2}\\ &= (5 + 0 + 41)^{1/2} = \sqrt{46}\end{aligned}$$

$$\begin{aligned}\|\mathbf{v}\| &= (|v_1|^2 + |v_2|^2 + |v_3|^2)^{1/2}\\ &= [(1^2 + 1^2) + (2^2 + 1^2) + (0^2 + 0^2)]^{1/2}\\ &= (2 + 5 + 0)^{1/2} = \sqrt{7}\end{aligned}$$

The distance between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\begin{aligned}d(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\|\\ &= \|(1, -2 - i, 4 - 5i)\|\\ &= [(1^2 + 0^2) + ((-2)^2 + (-1)^2) + (4^2 + 5^2)]^{1/2}\\ &= (1 + 5 + 41)^{1/2} = \sqrt{47}.\end{aligned}$$

Complex Inner Product Spaces

The Euclidean inner product is the most commonly used inner product in C^n. However, on occasion it is useful to consider other inner products. To generalize the notion of an inner product, we use the properties listed in Theorem 8.7.

Definition of a Complex Inner Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a complex vector space. A function that associates with $\mathbf{u}$ and $\mathbf{v}$ the complex number $\langle \mathbf{u}, \mathbf{v} \rangle$ is called a **complex inner product** if it satisfies the following properties.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

A complex vector space with a complex inner product is called a **complex inner product space** or **unitary space.**

EXAMPLE 7 ***A Complex Inner Product Space***

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in the complex space C^2. Show that the function defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v_1} + 2u_2\overline{v_2}$$

is a complex inner product.

Solution We verify the four properties of a complex inner product as follows.

1. $\overline{\langle \mathbf{v}, \mathbf{u} \rangle} = \overline{v_1\overline{u_1} + 2v_2\overline{u_2}} = u_1\overline{v_1} + 2u_2\overline{v_2} = \langle \mathbf{u}, \mathbf{v} \rangle$
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = (u_1 + v_1)\overline{w_1} + 2(u_2 + v_2)\overline{w_2}$
   $= (u_1\overline{w_1} + 2u_2\overline{w_2}) + (v_1\overline{w_1} + 2v_2\overline{w_2})$
   $= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = (ku_1)\overline{v_1} + 2(ku_2)\overline{v_2} = k(u_1\overline{v_1} + 2u_2\overline{v_2}) = k\langle \mathbf{u}, \mathbf{v} \rangle$
4. $\langle \mathbf{u}, \mathbf{u} \rangle = u_1\overline{u_1} + 2u_2\overline{u_2} = |u_1|^2 + 2|u_2|^2 \geq 0$

   Moreover, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $u_1 = u_2 = 0$.

Since all properties hold, $\langle \mathbf{u}, \mathbf{v} \rangle$ is a complex inner product.

SECTION 8.4 EXERCISES

In Exercises 1–8, perform the indicated operation using $\mathbf{u} = (i, 3 - i)$, $\mathbf{v} = (2 + i, 3 + i)$, and $\mathbf{w} = (4i, 6)$.

1. $3\mathbf{u}$
2. $4i\mathbf{w}$
3. $(1 + 2i)\mathbf{w}$
4. $i\mathbf{v} + 3\mathbf{w}$
5. $\mathbf{u} - (2 - i)\mathbf{v}$
6. $(6 + 3i)\mathbf{v} - (2 + 2i)\mathbf{w}$
7. $\mathbf{u} + i\mathbf{v} + 2i\mathbf{w}$
8. $2i\mathbf{v} - (3 - i)\mathbf{w} + \mathbf{u}$

In Exercises 9–12, determine whether S is a basis for C^n.

9. $S = \{(1, i), (i, -1)\}$
10. $S = \{(1, i), (i, 1)\}$
11. $S = \{(i, 0, 0), (0, i, i), (0, 0, 1)\}$
12. $S = \{(1 - i, 0, 1), (2, i, 1 + i), (1 - i, 1, 1)\}$

In Exercises 13–16, express $\mathbf{v}$ as a linear combination of the following basis vectors.

(a) $\{(i, 0, 0), (i, i, 0), (i, i, i)\}$

(b) $\{(1, 0, 0), (1, 1, 0), (0, 0, 1 + i)\}$

13. $\mathbf{v} = (1, 2, 0)$
14. $\mathbf{v} = (1 - i, 1 + i, -3)$
15. $\mathbf{v} = (-i, 2 + i, -1)$
16. $\mathbf{v} = (i, i, i)$

In Exercises 17–24, determine the Euclidean norm of $\mathbf{v}$.

17. $\mathbf{v} = (i, -i)$
18. $\mathbf{v} = (1, 0)$
19. $\mathbf{v} = 3(6 + i, 2 - i)$
20. $\mathbf{v} = (2 + 3i, 2 - 3i)$
21. $\mathbf{v} = (1, 2 + i, -i)$
22. $\mathbf{v} = (0, 0, 0)$
23. $\mathbf{v} = (1 - 2i, i, 3i, 1 + i)$
24. $\mathbf{v} = (2, -1 + i, 2 - i, 4i)$

In Exercises 25–30, determine the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$.

25. $\mathbf{u} = (1, 0)$, $\mathbf{v} = (i, i)$
26. $\mathbf{u} = (2 + i, 4, -i)$, $\mathbf{v} = (2 + i, 4, -i)$
27. $\mathbf{u} = (i, 2i, 3i)$, $\mathbf{v} = (0, 1, 0)$
28. $\mathbf{u} = (\sqrt{2}, 2i, -i)$, $\mathbf{v} = (i, i, i)$
29. $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$
30. $\mathbf{u} = (1, 2, 1, -2i)$, $\mathbf{v} = (i, 2i, i, 2)$

In Exercises 31–34, determine whether the set of vectors is linearly independent or linearly dependent.

31. $\{(1, i), (i, -1)\}$
32. $\{(1 + i, 1 - i, 1), (i, 0, 1), (-2, -1 + i, 0)\}$
33. $\{(1, i, 1 + i), (0, i, -i), (0, 0, 1)\}$
34. $\{(1 + i, 1 - i, 0), (1 - i, 0, 0), (0, 1, 1)\}$

In Exercises 35–38, determine whether the given function is a complex inner product, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

35. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 + u_2 v_2$
36. $\langle \mathbf{u}, \mathbf{v} \rangle = (u_1 + v_1) + 2(u_2 + v_2)$
37. $\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1\overline{v_1} + 6u_2\overline{v_2}$
38. $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2$
39. Let $\mathbf{v}_1 = (i, 0, 0)$ and $\mathbf{v}_2 = (i, i, 0)$. If $\mathbf{v}_3 = (z_1, z_2, z_3)$ and the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is *not* a basis for C^3, what does this imply about z_1, z_2, and z_3?
40. Let $\mathbf{v}_1 = (i, i, i)$ and $\mathbf{v}_2 = (1, 0, 1)$. Determine a vector $\mathbf{v}_3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for C^3.

In Exercises 41–45, prove the given property where **u, v,** and **w** are vectors in C^n and k is a complex number.

41. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
42. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
43. $\mathbf{u} \cdot (k\mathbf{v}) = \overline{k}(\mathbf{u} \cdot \mathbf{v})$
44. $\mathbf{u} \cdot \mathbf{u} \geq 0$
45. $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
46. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be a complex inner product and k a complex number. How are $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle \mathbf{u}, k\mathbf{v} \rangle$ related?

In Exercises 47 and 48, determine the linear transformation $T: C^m \rightarrow C^n$ that has the given characteristics.

47. $T(1, 0) = (2 + i, 1)$, $T(0, 1) = (0, -i)$
48. $T(i, 0) = (2 + i, 1)$, $T(0, i) = (0, -i)$

In Exercises 49–52, the linear transformation $T: C^m \rightarrow C^n$ is given by $T(\mathbf{v}) = A\mathbf{v}$. Find the image of $\mathbf{v}$ and the preimage of $\mathbf{w}$.

49. $A = \begin{bmatrix} 1 & 0 \\ i & i \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 + i \\ 1 - i \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

50. $A = \begin{bmatrix} 0 & i & 1 \\ i & 0 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} i \\ 0 \\ 1 + i \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

51. $A = \begin{bmatrix} 1 & 0 \\ i & 0 \\ i & i \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 - i \\ 3 + 2i \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 2i \\ 3i \end{bmatrix}$

52. $A = \begin{bmatrix} 0 & 1 & 1 \\ i & i & -1 \\ 0 & i & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 - i \\ 1 + i \\ i \end{bmatrix}$

53. Find the kernel of the linear transformation given in Exercise 49.

54. Find the kernel of the linear transformation given in Exercise 50.

In Exercises 55 and 56, find the image of $\mathbf{v} = (i, i)$ for the indicated composition, where T_1 and T_2 are given by the following matrices.

$$A_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}$$

55. $T_2 \circ T_1$

56. $T_1 \circ T_2$

57. Determine which of the following sets are subspaces of the vector space of 2×2 complex matrices.

(a) The set of 2×2 symmetric matrices.

(b) The set of 2×2 matrices A satisfying $(\bar{A})^T = A$.

(c) The set of 2×2 matrices in which all entries are real.

(d) The set of 2×2 diagonal matrices.

58. Determine which of the following sets are subspaces of the vector space of complex-valued functions (see Example 4).

(a) The set of all functions f satisfying $f(i) = 0$.

(b) The set of all functions f satisfying $f(0) = 1$.

(c) The set of all functions f satisfying $f(i) = f(-i)$.

8.5 UNITARY AND HERMITIAN MATRICES

Problems involving diagonalization of complex matrices, and the associated eigenvalue problems, require the concept of **unitary** and **Hermitian** matrices. These matrices roughly correspond to orthogonal and symmetric real matrices. In order to define unitary and Hermitian matrices, we first introduce the concept of the **conjugate transpose** of a complex matrix.

Definition of the Conjugate Transpose of a Complex Matrix

The **conjugate transpose** of a complex matrix A, denoted by A^*, is given by

$$A^* = \bar{A}^T$$

where the entries of $\bar{A}$ are the complex conjugates of the corresponding entries of A.

Note that if A is a matrix with real entries, then $A^* = A^T$. To find the conjugate transpose of a matrix, we first calculate the complex conjugate of each entry and then take the transpose of the matrix, as shown in the following example.

EXAMPLE 1 ***Finding the Conjugate Transpose of a Complex Matrix***

Determine A^* for the matrix

$$A = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix}.$$