Binomial Distributions

Binomial Experiments

A binomial experiment is characterized by the following properties.

1. The experiment consists of a sequence of \( n \) identical experiments called trials.

2. Each trail has exactly two possible outcomes. These outcomes are called success and failure, and their respective probabilities are denoted by \( p \) and \( q \).

3. The probability of success (and of failure) does not change from one trial to the next.

4. The trials are independent. That is, the outcome of one trail does not affect the outcome of any other trial.

From the second property, note that the sum of \( p \) and \( q \) must equal 1. So, if you are given the probability of either success or failure in a binomial experiment, you can calculate the probability of the other using \( p = 1 - q \) or \( q = 1 - p \).

Example 1

Binomial Experiments

a. A coin is tossed five times. You call “heads” each time the coin is tossed. Each trial can result in a success (heads up) or a failure (tails up). For this binomial experiment,

\[ n = 5, \quad p = \frac{1}{2}, \quad \text{and} \quad q = \frac{1}{2}. \]

b. A six-sided die is tossed ten times. You note whether a 6 turns up on the die. Each trial can result in a success (a 6 turns up) or a failure (a 6 does not turn up). For this binomial experiment,

\[ n = 10, \quad p = \frac{1}{6}, \quad \text{and} \quad q = \frac{5}{6}. \]

c. A company submits income tax returns for 15 years. The probability that a return will be audited in any given year is 0.001. Each trial can result in a success (the return is not audited) or a failure (the return is audited). For this binomial experiment,

\[ n = 15, \quad p = 0.999, \quad \text{and} \quad q = 0.001. \]

Checkpoint 1

You pick a card from a standard deck of 52 playing cards. You note whether the card is a diamond. You repeat the experiment 12 times. Determine \( n, p, \) and \( q \).
The random variable that is usually associated with a binomial experiment is the number of successes that occur in \( n \) trials. This random variable is discrete because the only values it can have are

\[ \{0, 1, 2, 3, \ldots, n\} \quad \text{Range of random variable} \]

The probability that the random variable \( x \) has a value of \( m \), where \( 0 \leq m \leq n \), is denoted by \( P(m) \), and is given by the following formula.

### Example 2 Finding Probabilities for Binomial Experiments

A clothing store has determined that 30% of the people who enter the store will make a purchase. Eight people enter the store during a one-hour period. Find the probability that (a) exactly four people will make a purchase, and (b) at least one person will make a purchase.

**SOLUTION**

**a.** Using \( n = 8 \), \( p = 0.3 \), and \( m = 4 \), the probability that exactly four people will make a purchase is

\[
P(4) = \binom{8}{4}(0.3)^4(0.7)^4 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} (0.3)^4(0.7)^4
\]

\[
= 70(0.3)^4(0.7)^4
\]

\[
= 0.136.
\]

**b.** Using \( n = 8 \), \( p = 0.3 \), and \( m = 0 \), the probability that no one will make a purchase is

\[
P(0) = \binom{8}{0}(0.3)^0(0.7)^8 = (1)(1)(0.7)^8 = 0.058.
\]

So, the probability that at least one person will make a purchase is

\[
P(1 \leq x \leq 8) = 1 - P(0) = 1 - 0.058 = 0.942.
\]

**CHECKPOINT 2**

In Example 2, find the probability that all eight people will make a purchase.

*Hint: use \( n = 8 \), \( p = 0.3 \), and \( m = 8 \).*
In Example 2, note the solution of part (b) uses the fact that the sum of the probabilities for a discrete random variable must be 1. That is,

\[ P(0) + P(1) + P(2) + \cdots + P(n) = 1. \]

**Binomial Distributions**

The probability distribution associated with a binomial experiment is called a binomial distribution. The next example illustrates this concept.

**Example 3  Sketching the Graph of a Binomial Distribution**

Consider a binomial experiment that has eight trials. Find the binomial distribution for this experiment if (a) \( p = 0.5 \), and (b) \( p = 0.3 \).

**SOLUTION** The binomial distributions of these probabilities of success, and their corresponding graphical representations are shown in Figures C.37 and C.38.

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**Figure C.37** Binomial Distribution: \( n = 8, p = 0.5 \)

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<td>0.0467</td>
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**Figure C.38** Binomial Distribution: \( n = 8, p = 0.3 \)

**CHECKPOINT 3**

Use a graphing utility or a spreadsheet to verify the values of \( P(x) \) given in Figures C.37 and C.38. Does \( P(0) + P(1) + P(2) + \cdots + P(8) = 1? \) Explain why or why not.
In the binomial distribution shown in Figure C.37, note that the distribution is symmetrical about \( x = 4 \). It seems reasonable that the expected value (or mean) of this probability distribution is \( \mu = 4 \). You can verify this observation using the formula for the expected value of a discrete random variable.

\[
\mu = \sum_{i=0}^{8} x_i P(x_i) = 0(0.0039) + 1(0.0312) + \cdots + 8(0.0039) = 4
\]

You can use the formulas in Section 11.1 for expected value, variance, and standard deviation of a discrete probability distribution, but for a binomial distribution you can use simpler formulas. For example, it can be shown that for every binomial distribution, the expected value is simply \( \mu = np \). So for the binomial distribution shown in Figure C.37, the expected value is \( \mu = np = 8(0.5) = 4 \), confirming the above result. The formulas for expected value, variance, and standard deviation of a binomial distribution are summarized below.

### Expected Value, Variance, and Standard Deviation of a Binomial Distribution

In a binomial distribution that has \( n \) trials and a probability of success \( p \), the expected value of the random variable \( x \) is

\[
\mu = np.
\]

Mohoreover, the variance and standard deviation of the random variable are

\[
V(x) = np(1 - p)
\]

and

\[
\sigma = \sqrt{V(x)} = \sqrt{np(1 - p)}.
\]

### Example 4 Finding the Standard Deviations of Binomial Distributions

a. The standard deviation of a binomial distribution with 50 trials, for which the probability of success is 0.1, is

\[
\sigma = \sqrt{50(0.1)(1 - 0.1)} = \sqrt{4.5} \approx 2.12.
\]

b. The standard deviation of a binomial distribution with 500 trials, for which the probability of success is 0.1, is

\[
\sigma = \sqrt{500(0.1)(1 - 0.1)} = \sqrt{45} \approx 6.71.
\]

### ✓ Checkpoint 4

a. Find the standard deviation of a binomial distribution with 8 trials, for which the probability of success is 0.5.

b. Find the standard deviation of a binomial distribution with 8 trials, for which the probability of success is 0.3.
Approximating Binomial Distributions

So far in this section you have only calculated probabilities for binomial experiments in which the number of trials is relatively small. For large values of \( n \), the formula

\[
P(m) = \frac{n!}{(n - m)! \cdot m!} p^m (1 - p)^{n-m}
\]

can be very difficult to compute by hand (even with a calculator or computer). In such cases you can use a normal distribution to approximate the required probabilities.

**Example 5** Comparing Binomial and Normal Distributions

Sketch the graph of the binomial distribution with \( n = 16 \) and \( p = 0.5 \). Then compare this to the graph of the normal distribution with \( \mu = 16(0.5) = 8 \) and \( \sigma = \sqrt{16(0.5)(0.5)} = 2 \).

**SOLUTION** The binomial distribution with \( n = 16 \) and \( p = 0.5 \) and its graph are shown in Figure C.39. The graph of the continuous normal distribution with \( \mu = 8 \) and \( \sigma = 2 \) is superimposed over the discrete binomial distribution.

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<td>0.00024</td>
<td>0.00183</td>
<td>0.00855</td>
<td>0.02777</td>
<td>0.06665</td>
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<td>0.19638</td>
<td>0.17456</td>
<td>0.12219</td>
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<th>15</th>
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<tbody>
<tr>
<td>( P(x) )</td>
<td>0.02777</td>
<td>0.00855</td>
<td>0.00183</td>
<td>0.00024</td>
<td>0.00002</td>
</tr>
</tbody>
</table>

**FIGURE C.39**

From this comparison, you can see that a normal distribution can be used to approximate a binomial distribution. For example, suppose you wanted to find the probability that exactly eight successes will occur. From the table in Figure C.39, you know that this probability is \( P(8) = 0.19638 \). To approximate this value using the normal distribution, you can use an interval whose center is 8 and whose width is 1. Using the table for the standard normal distribution given in Appendix C.6.

\[
P(8) \approx P(7.5 \leq x \leq 8.5) = P(-0.25 \leq z \leq 0.25) = 2 \cdot P(0 \leq z \leq 0.25) = 0.1974.
\]

**✓ CHECKPOINT 5**

Use the graph in Figure C.37 to compare the graph of the binomial distribution with \( n = 8 \) and \( p = 0.5 \) to the graph of the normal distribution with \( \mu = 8(0.5) = 4 \) and \( \sigma = \sqrt{8(0.5)(0.5)} \approx 1.41 \).
Example 6  Approximating a Binomial Distribution

A medical supply manufacturer has found that 4% of a type of computer chip are defective. The manufacturer orders 2000 of these chips. Find the probability that (a) 70 chips or fewer are defective, and (b) 80 chips are defective.

**SOLUTION**

a. Use a model involving a binomial experiment in which and

To find the probability that 70 chips or fewer are defective, you could use the formula for binomial probabilities. This would be time consuming, however, because you would have to evaluate

Using a different tactic, you know that the expected value for the experiment is

and the standard deviation is

as shown in Figure C.40. So, you can approximate the probability

in the normal distribution with and For this distribution, the z-scores for 70.5 is

Using the table for the standard normal distribution given in Appendix C.6,

b. The probability that exactly 80 of the parts are defective can be approximated as follows

Using the table for the standard normal distribution given in Appendix C.6,

\[ P(0 \leq x \leq 70.5) = P(z \leq -1.08) = P(1.08 \leq z) \approx 0.1401. \]

b. The probability that exactly 80 of the parts are defective can be approximated as follows

\[ P(80) \approx P(79.5 \leq x \leq 80.5) = P(-0.06 \leq z \leq 0.06) \approx 0.0478 \]

**CHECKPOINT 6**

In Example 6, find the probability that (a) 80 chips or fewer are defective, and (b) exactly 70 chips are defective.

**CONCEPT CHECK**

1. The two possible outcomes of each trial of a binomial experiment are called ______ and ______.
2. Describe the meaning of each of the variables in the formula for the probability of \( m \) successes in a binomial experiment.
3. The expected value of a binomial distribution can be found using the formula ______.
4. Describe how to approximate a binomial distribution that has a large number of trials.
In Exercises 1–10, evaluate the expression.

1. \((0.5)^5(0.5)^4\)
2. \((0.4)^9(0.6)^8\)
3. \((0.3)^6(0.7)^7\)
4. \(\binom{16}{6}\)
5. \(\binom{12}{5}\)
6. \(\binom{5}{3}(0.5)^2\)
7. \(\binom{10}{0}(0.5)^3\)
8. \(\binom{5}{3}(0.89)^3(1 - 0.89)^3\)
9. \(\sqrt{1000}(0.06)(0.94)\)
10. \(\sqrt{5280}(0.003)(1 - 0.003)\)

In Exercises 1–6, determine the probability of the given result when tossing a coin 10 times.

1. 6 heads
2. 3 tails
3. 8 tails
4. 4 heads
5. 3 tails
6. 5 tails

7. **Blood Type** Seven out of every 100 people in the United States have Type O negative blood. If 6 people are chosen at random, what is the probability that exactly 2 of them have Type O negative blood? (Source: American Red Cross)

8. **Blood Type** Nine out of every 100 people in the United States have Type B positive blood. If 8 people are chosen at random, what is the probability that exactly 3 of them have Type B positive blood? (Source: American Red Cross)

9. **Eyeglasses** Five percent of the eyeglasses sold at an optical retailer have tinted lenses. If 25 pairs of glasses are sold on a particular day, what is the probability that at least one pair has tinted lenses?

10. **Contact Lenses** Twenty-five percent of the contact lenses sold at an optical retailer are tinted. If 25 customers buy contact lenses on a particular day, what is the probability that at least one customer buys tinted contact lenses?

11. **True-False Test** A true-false test has 20 questions. If you answer each of the questions randomly, what is the probability of answering exactly 12 questions correctly?

12. **Multiple Choice Test** A multiple-choice test has 20 questions. Each question has 3 possible answers, one of which is correct. If you answer each of the questions randomly, what is the probability of answering exactly 12 questions correctly?

13. **Radon Detection Test Kits** One out of every 100 radon detection test kits sold at a home improvement outlet gives inaccurate results. In a sample of 50 kits obtained from the outlet, what is the probability that exactly 48 give accurate results?

14. **Medical Sensors** One out of every 5000 medical sensors produced at a manufacturing facility is defective. If 10,000 sensors are produced in a given year, what is the probability that exactly 9998 are not defective?

In Exercises 15–18, use a graphing utility to graph the probability distribution for the given binomial experiment.

15. 4 trials with \(p = 1/3\)
16. 10 trials with \(p = 1/2\)
17. 6 trials with \(p = 0.4\)
18. 5 trials with \(p = 0.7\)

19. **Die Toss** A six-sided die is tossed three times. Success is achieved when a 6 or a 5 turns up on the die. Construct a probability distribution for this binomial experiment.

20. **Die Toss** Repeat Exercise 19 if success is also achieved when a 4 turns up on the die.

In Exercises 21–24, find the expected value of the random variable for 100 trials of a binomial experiment with the given probability.

21. \(p = 0.2\)
22. \(p = 0.7\)
23. \(p = \frac{9}{20}\)
24. \(p = \frac{7}{25}\)

25. **Customer Preference** One out of every 5 people buys premium gasoline. If a service station has 500 customers on a given day, what is the expected number of people who buy premium gasoline?

26. **Customer Preference** Repeat Exercise 25, assuming that the service station has only 200 customers on a given day.

In Exercises 27–30, find the variance and standard deviation for the indicated binomial experiment.

27. 60 trials, \(p = 1/3\)
28. 10 trials, \(p = 1/6\)
29. 50 trials, \(p = 0.44\)
30. 40 trials, \(p = 0.16\)
31. **Team Preference**  Thirty percent of the attendees at a baseball game are fans of the visiting team. The other attendees are fans of the home team. If 60 attendees are chosen at random, how many of them would you expect to be fans of the home team? What is the standard deviation?

32. **Patient Age**  In a doctor’s office, 80% of the patients are adults. If 15 patients are scheduled for an appointment on a given day, what is the expected number of adults? What is the standard deviation?

33. **Seat Belt Use**  Only 35% of the drivers in a particular city wear seat belts. Twenty drivers are stopped at random on a given day.
- (a) What is the expected number of drivers wearing seat belts?
- (b) Find the standard deviation and explain its meaning in the context of the problem.

34. **Seat Belt Use**  Repeat Exercise 33 assuming that 65% of the drivers wear seat belts.

35. **Shoe Rental**  At a bowling alley, 75% of the bowlers rent shoes. On a given day, the bowling alley had 500 bowlers. What is the probability that 400 or more rented shoes?

36. **Sale Incentive**  An ice cream stand chooses one day at random each week to sell ice cream cones at half-price. A customer goes to the stand once per week, 20 weeks per year. What is the probability that this customer will show up on a half-price day at least three times?

37. **Employee Benefits**  A study of small businesses in a particular city found that 83% of those that offer health insurance also provide dental insurance to their employees. If 60 small businesses from this city are chosen at random, what is the probability that at least 50 provide dental insurance?

38. **Car Maintenance**  Twenty-five percent of the customers of a car dealer return for the dealer’s routine maintenance program. If the dealer has 100 customers in a given week, what is the probability that at least 35 will return for the routine maintenance program?

39. **Coin Toss**  If 1000 coins are tossed, what is the probability of getting (a) exactly 500 heads, (b) exactly 490 heads, and (c) less than 490 heads?

40. **Die Toss**  A single six-sided die is tossed 600 times. Find the probability of getting (a) exactly 100 6’s, (b) exactly 95 2’s, and (c) more than 95 3’s.

41. **Sales Success**  The probability of success for a sales call is 0.4. What is the probability that a representative will have at least 10 successful sales out of 30 sales calls?

42. **Basketball**  During his basketball career, Michael Jordan made each free throw with a probability of 0.835. What is the probability that he would have made more than 90 free throws out of 100 attempted? *(Source: National Basketball Association)*

43. **Paycheck Errors**  The payroll department of a hospital has found that in one year, 0.9% of its paychecks are calculated incorrectly. The hospital has 500 employees.
- (a) What is the probability that in one month’s records no paycheck errors are made?
- (b) What is the probability that in one month’s records at least one paycheck error is made?

44. **Overtime Compensation**  A municipal accounting office has found that in one year, 0.4% of the police department paychecks have overtime compensation errors. The city employs 400 police officers.
- (a) What is the probability that in one month’s records no overtime compensation errors are made?
- (b) What is the probability that in one month’s records at least one overtime compensation error is made?

45. **Sports Selection**  Forty percent of the athletes in an high school play football. If 6 athletes are chosen from the high school at random, what is the probability that 3 of the athletes play football?

46. **Coffee Drinkers**  One in 6 coffee drinkers at a restaurant prefers coffee with cream, sugar, or both. If 4 coffee drinkers are chosen at random, what is the probability that at least 1 coffee drinker prefers coffee with cream, sugar, or both?

47. **Business Variety**  One in 5 businesses in the downtown district of a city are restaurants. If 20 businesses are chosen at random, what is the probability that at least 8 of them are restaurants?

48. **Music Classification**  Eighty percent of the songs played at a radio station are rock songs. If 10 songs are chosen at random from the station’s play list, what is the probability that at least 9 of the songs will be rock songs?

49. **Random Numbers**  Use a spreadsheet to conduct the following experiment. Choose 5 integers between 0 and 100 at random and calculate their mean. Conduct this procedure 500 times and construct a probability distribution using the following groups: 0–5, 5.2–15, 15.2–25, 25.2–35, 35.2–45, 45.2–55, 55.2–65, 65.2–75, 75.2–85, 85.2–95, and 95.2–100. Compare your results to those predicted by a binomial distribution with \( n = 10 \) and \( p = 0.5 \).

50. **Random Numbers**  Use a spreadsheet to conduct the following experiment. Choose 10 integers between 0 and 100 at random and calculate their mean. Conduct this procedure 1000 times and construct a probability distribution using the following groups: 0–5, 5.1–15, 15.1–25, 25.1–35, 35.1–45, 45.1–55, 55.1–65, 65.1–75, 75.1–85, 85.1–95, and 95.1–100. Compare your results to those predicted by a binomial distribution with \( n = 10 \) and \( p = 0.5 \).