C.3 Conditional Probability

- Find conditional probabilities.
- Find probabilities of independent events.

**Conditional Probability**

In some applications, it is important to know how to find the probability of an event given that another event has already occurred. For instance, suppose you are playing a card game and one of your opponents is dealt a card from a full deck of 52 cards. Because the deck contains four queens, the probability that the card is a queen (event $A$) is

$$P(A) = \frac{4}{52}.$$ 

Suppose that as the card is dealt, you catch a quick glance and notice that it is a black face card (event $B$). Given this fact, what is the probability that the card is a queen? This **conditional probability** is denoted by

$$P(A \mid B)$$

which is read as “the probability of $A$ given $B$” or, in this case, “the probability of drawing a queen, given that a black face card has been drawn.” To determine this probability, reduce the sample space (from 52 cards) to six black face cards, of which two are queens. So, the probability of $A$ given $B$ is

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{6} = \frac{1}{3}.$$ 

Note from Figure C.8 that

$$n(A \cap B) = 2 \quad \text{and} \quad n(B) = 6$$

which implies that

$$P(A \cap B) = \frac{2}{52} \quad \text{and} \quad P(B) = \frac{6}{52}.$$ 

Using these probabilities, the probability of $A$ given $B$ can be calculated as follows.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{2/52}{6/52} = \frac{2}{6} = \frac{1}{3}.$$ 

**Conditional Probability**

The **conditional probability** of an event $A$, given that event $B$ has occurred, is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$
Example 1  Finding Conditional Probabilities

Find the probability of \( A \) given \( B \) for the following.

a. \( P(B) = 0.52, P(A \cap B) = 0.13 \)

b. \( P(A) = 0.37, P(B) = 0.45, P(A \cup B) = 0.63 \)

SOLUTION

a. Because \( P(B) = 0.52 \) and \( P(A \cap B) = 0.13 \), the probability of \( A \) given \( B \) is

\[
P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{0.13}{0.52} = 0.25.
\]

b. Because \( P(A) = 0.37, P(B) = 0.45, \) and \( P(A \cup B) = 0.63 \), you can determine that the probability of \( A \cap B \) is

\[
P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.37 + 0.45 - 0.63 = 0.19.
\]

So, the probability of \( A \) given \( B \) is

\[
P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{0.19}{0.45} \approx 0.42.
\]

Example 2  Finding Conditional Probabilities

Figure C.9 shows the results of a study in which researchers examined a child’s IQ and the presence of a specific gene in the child. The events represented in the figure are as follows.

\[
\begin{align*}
A & = \text{child has the gene} \\
B & = \text{child has a high IQ} \\
A \cap B & = \text{child has the gene and a high IQ}
\end{align*}
\]

From the study, researchers determined that \( P(A) = 0.71, P(B) = 0.51, \) and \( P(A \cap B) = 0.32. \) (Source: Psychological Science)

a. What is the probability that a child has a high IQ, given that the child has the gene?

b. What is the probability that a child has the gene, given that the child has a high IQ?

SOLUTION

a. The probability that a child has a high IQ, given that the child has the gene is

\[
P(B | A) = \frac{P(A \cap B)}{P(A)} = \frac{0.32}{0.71} \approx 0.45.
\]

b. The probability that a child has the gene, given that the child has a high IQ is

\[
P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.51} \approx 0.63.
\]
Example 3  Effect of Having the Gene

Using the information given in Example 2, can you conclude that having the gene increases the probability that a child will have a high IQ?

SOLUTION  From Example 2, you know that the probability that a child has a high IQ (event $B$), given that the child has the gene (event $A$) is

$$P(B \mid A) = 0.45.$$  

To determine the effect that the gene has on IQ, calculate the probability that a child will have a high IQ, given that the child does not have the gene. This probability is

$$P(B \mid A') = \frac{P(A' \cap B)}{P(A')} = \frac{P(B) - P(A \cap B)}{1 - P(A)} = \frac{0.51 - 0.32}{1 - 0.71} = \frac{0.19}{0.29} = 0.66.$$  

A child who does not have the gene is approximately one-and-a-half times more likely to have a high IQ than one who has the gene. So, you can not conclude that having the gene increases the probability that a child will have a high IQ.

✓ CHECKPOINT 3

Using the result of Checkpoint 2, can you conclude that not having the gene increases the probability that a child will have a normal (i.e. not high) IQ?

Example 4  Finding Conditional Probability

A director of nursing is selected from a group of 21 nurses, of whom 16 are women and 5 are men. Twelve of the women and two of the men have BSN (Bachelor of Science in Nursing) degrees, as shown in Figure C.10. What is the probability that the person selected is a woman, given that the person has a BSN degree?

SOLUTION  Let $A$ represent the event that the person selected is a woman and let $B$ represent the event that the person has a BSN degree. Because $P(B) = 14/21$ and $P(A \cap B) = 12/21$, the probability that the person is a woman, given that the person has a BSN degree is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{12/21}{14/21} = \frac{12}{14} = \frac{6}{7}.$$  

✓ CHECKPOINT 4

In Example 4, find the probability that the person selected is a man, given that the person does not have a BSN degree.

STUDY TIP

Conditional probabilities can be illustrated by diagrams called tree diagrams, which are discussed in Appendix C.4.
Independent Events

Two events are **independent** if the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice.

**Independent Events**

Two events $A$ and $B$ are **independent** if

\[
P(A \mid B) = P(A) \quad \text{or if} \quad P(B \mid A) = P(B).
\]

Two events that are not independent are **dependent**.

To find the probability that two independent events will occur, *multiply* their probabilities.

**Probability of Independent Events**

If $A$ and $B$ are independent events, then the probability that both $A$ and $B$ will occur is

\[
P(A \cap B) = P(A) \cdot P(B).
\]

**Example 5** Finding the Probability of Independent Events

A computer generates two integers from 1 through 20 at random. What is the probability that both integers are less than 6?

**SOLUTION** The probability of generating a number from 1 through 5 is

\[
P(A) = \frac{5}{20} = \frac{1}{4}.
\]

So, the probability that both integers are less than 6 is

\[
P(A) \cdot P(A) = \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right) = \frac{1}{16}.
\]

**CHECKPOINT 5**

A coin is tossed and a twelve-sided die is rolled. Find the probability of getting a head and then rolling a 12.

Be sure you see the difference between mutually exclusive events and independent events. The next example compares these two concepts.
Example 6  Comparing Mutually Exclusive and Independent Events

a. One card is selected at random from a standard deck of 52 playing cards. What is the probability that it is a 10 or a face card?

b. Two cards are selected at random from a standard deck of 52 playing cards. The first card is replaced before the second card is drawn. What is the probability that the first card is a 10 and the second card is a face card?

SOLUTION  Let A represent the event “drawing a 10,” and let B represent the event “drawing a face card.”

a. In this case, only one card is being drawn. Moreover, because the two events A and B are mutually exclusive, the probability of drawing a 10 or a face card is

\[
P(A \cup B) = P(A) + P(B)
\]

\[
= \frac{4}{52} + \frac{12}{52} = \frac{16}{52} = \frac{4}{13} \approx 0.308.
\]

b. In this case, two cards are being drawn. Moreover, because the two events A and B are independent, and because the first card is replaced before the second is drawn, the probability that the first card is a 10 and the second is a face card is

\[
P(A \cap B) = P(A) \cdot P(B)
\]

\[
= \frac{4}{52} \cdot \frac{12}{52} = \frac{48}{2704} = 0.018.
\]

✓ CHECKPOINT 6

a. One card is selected at random from a standard deck of 52 playing cards. What is the probability that it is a club or a red jack?

b. Two cards are selected at random from a standard deck of 52 playing cards. The first card is replaced before the second card is drawn. What is the probability that the first card is a club and the second card is a red jack?

To find the probability that three or more independent events will occur, you can multiply the probabilities of the individual events. For instance, if A, B, and C are independent events, then the probability that all three events will occur is given by

\[
P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C).
\]
Example 7  Probability of Three or More Independent Events

a. If a coin is tossed five times, the probability that it turns up heads all five times is

\[ \text{Probability} = \left(\frac{1}{2}\right)^5 = \frac{1}{32}. \]

b. The probability of obtaining a total of 7 on two six-sided dice is \( \frac{6}{36} = \frac{1}{6} \). So, if two dice are tossed 10 times, the probability that the total is 7 every time is

\[ \text{Probability} = \left(\frac{1}{6}\right)^{10} = \frac{1}{60,466,176} \approx 0.000000017. \]

✓ CHECKPOINT 7

a. The probability that a particular elbow surgery is successful is 0.89. Find the probability that four elbow surgeries are successful.

b. Find the probability that none of the four elbow surgeries is successful.

(Hint: The probability of failure for one surgery is \( 1 - 0.89 = 0.11 \).) ■

Example 8  Finding the Probability of the Complement of an Event

An eyeglass lens manufacturer has determined that 1 out of every 1000 lenses it produces is faulty. What is the probability that an order of 200 lenses will have at least one faulty lens?

SOLUTION  To solve this problem as stated, you would need to find the probability of having exactly one faulty lens, exactly two faulty lenses, exactly three faulty lenses, and so on. However, using complements, you can simply find the probability that all lenses are perfect (event \( A \)) and then subtract this value from 1. Because the probability that any given lens is perfect is \( \frac{999}{1000} \), and choosing one perfect lens is independent from choosing another perfect lens, you can apply the formula for the probability of independent events to conclude that the probability that all 200 lenses are perfect is

\[ P(A) = \left(\frac{999}{1000}\right)^{200} \approx 0.8186. \]

So, the probability that at least one lens is faulty is

\[ P(A') = 1 - P(A) \approx 0.1814. \]

✓ CHECKPOINT 8

Use the result of Checkpoint 7(b) to find the probability that at least one of the four elbow surgeries is successful. ■

CONCEPT CHECK

1. Describe how to calculate conditional probability.

2. If the occurrence of one event has no effect on the occurrence of a second event, then the events are ______.

3. How do dependent events differ from independent events?

4. If \( A \) and \( B \) are independent events, then the probability that both \( A \) and \( B \) will occur is \( P(A \cap B) = _____ \).
The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 0.3, C.1, and C.2.

### Exercises C.3

**In Exercises 1–6, simplify the expression.**

1. \( \frac{3}{8} \cdot \frac{9}{18} \)
2. \( \frac{0.39}{0.87} \)
3. \( \frac{0.70 - 0.56}{1 - 0.23} \)
4. \( \frac{2}{30} \cdot \frac{6}{30} \)
5. \( (\frac{2}{5})^2 \)
6. \( 1 - (0.13)^2 \)

**In Exercises 7–10, find the probability for the experiment of selecting one letter of the alphabet at random.**

7. The letter is a vowel (a, e, i, o, or u).
8. The letter is used in the spellings of *Lhasa* or *Apsa*.
9. The letter is used in the spelling of *bananas*.
10. The letter is not a vowel.

### Skills Review C.3

In Exercises 1–4, find \( P(A \mid B) \).

1. \( P(B) = \frac{\frac{3}{4}}{\frac{1}{2}}, P(A \cap B) = \frac{1}{6} \)
2. \( P(B) = 0.02, P(A \cap B) = 0.31 \)
3. \( P(A) = \frac{\frac{1}{3}}, P(B) = \frac{\frac{1}{6}}, P(A \cup B) = \frac{\frac{1}{2}}{} \)
4. \( P(A) = \frac{\frac{1}{5}}, P(B) = \frac{\frac{1}{5}}, P(A \cup B) = \frac{\frac{1}{2}}{} \)

**Playing Cards** In Exercises 5–8, 1 card is selected at random from a standard deck of 52 playing cards.

5. What is the probability that the card is a 6, given that the card is not an ace or a face card?
6. What is the probability that the card is the ace of hearts, given that the card is red and not a face card?
7. What is the probability that the card is an 8, 9, or 10, given that it is not a face card?
8. What is the probability that the card is the 4 of clubs, given that it is black and not an ace?

**Effect of Advertising** In Exercises 9 and 10, a pharmaceutical company has been running a television advertisement for a new pain relief medication. A marketing research firm determined that the probability that an individual has heard the advertisement is 0.5, the probability that an individual bought the medication is 0.2, and the probability that an individual has heard the advertisement and bought the medication is 0.1.

9. What is the probability that an individual will purchase the medication, given that the person has heard the advertisement?
10. What is the probability that an individual has heard the advertisement, given that the person purchased the medication?

11. **Random Selection** An office building houses 20 medical offices. Of these, 5 are pediatricians' offices, 4 are general practitioners' offices, and 3 are both. If one of the offices is selected at random, what is the probability that it is a general practitioners' office, given that it is a pediatricians' office?

12. **Random Selection** In Exercise 11, what is the probability that the office selected is a pediatricians' office, given that it is a general practitioners' office?

13. **Random Selection** At a state teachers' convention, 30 teachers have a morning meeting and 20 have an afternoon meeting. If 10 teachers have both a morning and afternoon meeting, what is the probability that a teacher selected at random has a morning meeting, given that the person has an afternoon meeting?

14. **Random Selection** In Exercise 13, what is the probability that a teacher selected at random has an afternoon meeting, given that the person has a morning meeting?

15. **Consumer Preference** A research group surveyed 450 people and found that 350 listen to Station A, whereas 200 listen to Station B. One hundred of the people sampled listen to both stations. What is the probability that a person chosen at random listens to Station A, given that the person does not listen to Station B?
16. Random Selection  A food distribution center has 53 cases of food containing Product A or Product B. Forty-four cases contain Product A, 12 contain Product B, and 3 cases contain both products. If a case is selected at random, what is the probability that it contains Product A, given that it does not contain Product B?

17. Random Selection  A company distributes coupons for a new product to 200 customers. During the following week, 425 customers bought the new product, and of these, 115 use a coupon. What is the probability that a customer selected at random bought the new product, given that the customer received a coupon?

18. Random Selection  In Exercise 17, assume that the probability that a customer buys the product is 0.5. Can you conclude that receiving a coupon increases the probability that a customer will buy the product?

19. Breast Cancer  Research shows that one woman in eight will develop breast cancer. One woman in 600 carries a mutation of a gene. Eight out of 10 women with the mutation will develop breast cancer. Find the probability that a woman selected at random will develop breast cancer given that she has a mutation of the gene.  (Source: Susan G. Komen Breast Cancer Foundation)

20. Breast Cancer  Using the information given in Exercise 19, can you conclude that having a mutation of the gene increases the probability that a woman will develop breast cancer?

In Exercises 21–24, determine whether the two events are independent or dependent.

21. Selecting the first digit in a three-digit lottery at random. Selecting the last digit in a three-digit lottery at random.

22. Selecting a face card at random from a standard deck of 52 playing cards. Selecting an ace at random from the remaining cards.

23. Events $A$ and $B$, where $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{5}$, and $P(A \cap B) = \frac{1}{5}$.

24. Events $A$ and $B$, where $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{5}$, and $P(A \cap B) = \frac{1}{5}$.

25. Slot Machine  There are three wheels on a slot machine. Each wheel has 30 symbols, of which 6 are cherries. The lowest payoff is achieved when cherries appear on each wheel. With one pull, what is the probability of winning the lowest payoff?

26. Card Selection  Two cards are selected at random from an ordinary deck of 52 playing cards. Find the probability that two aces are selected, given that the two cards are drawn consecutively, without replacement.

27. Availability of Rescue Vehicles  A fire company keeps two rescue vehicles to serve the community. Because of the demand on the company’s time and the chance of mechanical failure, the probability that a specific vehicle is available when needed is 90%. If the availability of one vehicle is independent of the other, find the probability that (a) both vehicles are available at a given time, (b) neither vehicle is available at a given time, and (c) at least one is available at a given time.

28. Back-up System  A space shuttle has an independent back-up system for one of its communication networks. The probability that either system will function satisfactorily for the duration of a flight is 0.985. What is the probability that during a given flight (a) both systems function satisfactorily (b) both systems fail, and (c) at least one system functions satisfactorily.

29. Birth Prediction  Assume that the probability of the birth of a child of a particular gender is 50%. In a family with six children, what is the probability that (a) all the children are girls, (b) all the children are the same gender, and (c) there is at least one girl?

30. Quality Control  The probability that a computer chip on a space shuttle will fail is 0.02. How many chips should be carried on the shuttle as backups in order to ensure that there is a 0.9999 probability that at least one of the chips works?

31. Random Numbers  A computer generates two integers from 1 through 30 at random. What is the probability that (a) the numbers are both even, (b) one number is even and one is odd, (c) both numbers are less than 10, and (d) the same number is chosen twice?

32. Random Selection  The names of 30 people in an office are placed in a container for a “50-50” fund raising raffle. Each player pays $10 to play, and half of the money will be split among four winning tickets (that is, each winning ticket wins $37.50). After each winning ticket is drawn, it is replaced in the container. If you are one of the players in this raffle, what is the probability that your name will be drawn exactly three times?

33. Random Numbers  Use a graphing utility to generate three integers between 1 and 20 at random. What is the probability that all 3 numbers are less than 6? Try repeating this experiment until you obtain all 3 numbers less than 6. How many times did you have to repeat the experiment?

34. Random Numbers  Use a graphing utility to generate 10 numbers, each from the set \{1, 2, 3, 4, 5\}, at random. What is the probability of selecting the number 3 exactly twice? How many times did the computer actually select the number 3?

35. Prove that if $A$ and $B$ are independent events, then $A'$ and $B'$ are independent events.