First-Order Linear Differential Equations

Definition of a First-Order Linear Differential Equation

A first-order linear differential equation is an equation of the form

\[ y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are functions of \( x \). An equation that is written in this form is said to be in standard form.

To solve a linear differential equation, write it in standard form to identify the functions \( P \) and \( Q \). Then integrate \( P(x) \) and form the expression

\[ u(x) = e^{\int P(x) \, dx} \]

which is called an integrating factor. The general solution of the equation is

\[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx. \]

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of

\[ y' + y = e^x. \]

Solution For this equation, \( P(x) = 1 \) and \( Q(x) = e^x \). So, the integrating factor is

\[ u(x) = e^{\int P(x) \, dx} = e^x. \]

This implies that the general solution is

\[ y = \frac{1}{e^x} \int e^x(e^x) \, dx \]
\[ = e^{-x} \left( \frac{1}{2}e^{2x} + C \right) \]
\[ = \frac{1}{2}e^x + Ce^{-x}. \]

In Example 1, the differential equation was given in standard form. For equations that are not written in standard form, you should first convert to standard form so that you can identify the functions \( P(x) \) and \( Q(x) \).
EXAMPLE 2 Solving a Linear Differential Equation

Find the general solution of
\[ xy' - 2y = x^2. \]

Assume \( x > 0. \)

**Solution** Begin by writing the equation in standard form.

\[ y' - \left( \frac{2}{x} \right)y = x \]

In this form, you can see that \( P(x) = -2/x \) and \( Q(x) = x. \) So,

\[
\int P(x) \, dx = - \int \frac{2}{x} \, dx = -2 \ln x = -\ln x^2
\]

which implies that the integrating factor is

\[ u(x) = e^{\int P(x) \, dx} = e^{-\ln x^2} = \frac{1}{x^2}. \]

This implies that the general solution is

\[
y = \frac{1}{u(x)} \int Q(x)u(x) \, dx
\]

\[
= \frac{1}{1/x^2} \int x \left( \frac{1}{x^2} \right) \, dx = -\ln x + C.
\]

**Guidelines for Solving a Linear Differential Equation**

1. Write the equation in standard form
   \[ y' + P(x)y = Q(x). \]
2. Find the integrating factor
   \[ u(x) = e^{\int P(x) \, dx}. \]
3. Evaluate the integral below to find the general solution.
   \[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx \]
Application

EXAMPLE 3 Finding a Balance

You are setting up a “continuous annuity” trust fund. For 20 years, money is continuously transferred from your checking account to the trust fund at the rate of $1000 per year (about $2.74 per day). The account earns 8% interest, compounded continuously. What is the balance in the account after 20 years?

Solution Let \( A \) represent the balance after \( t \) years. The balance increases in two ways: with interest \( \text{and} \) with additional deposits. The rate at which the balance is changing can be modeled by

\[
\frac{dA}{dt} = 0.08A + 1000.
\]

In standard form, this linear differential equation is

\[
\frac{dA}{dt} - 0.08A = 1000.
\]

which implies that \( P(t) = -0.08 \) and \( Q(t) = 1000 \). The general solution is

\[
A = -12,500 + C e^{0.08t}.
\]

Because \( A = 0 \) when \( t = 0 \), you can determine that \( C = 12,500 \). So, the revenue after 20 years is

\[
A = -12,500 + 12,500 e^{0.08(20)}
\]

\[
= -12,500 + 61,912.91
\]

\[
= $49,412.91.
\]

TAKE ANOTHER LOOK

Why an Integrating Factor Works

When both sides of the first-order linear differential equation

\[
y' + P(x)y = Q(x)
\]

are multiplied by the integrating factor \( e^{\int P(x)dx} \), you obtain

\[
y e^{\int P(x)dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}.
\]

Show that the left side is the derivative of \( ye^{\int P(x)dx} \), which implies that the general solution is given by

\[
y e^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx} dx.
\]
### WARM-UP C.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, simplify the expression.

1. $e^{-x}(e^{2x} + e^{x})$
2. $\frac{1}{e^x}(e^{-x} + e^{2x})$
3. $e^{-ln x}$
4. $e^{3 \ln x + 1}$

In Exercises 5–10, find the indefinite integral.

5. $\int (2 + e^{-2x}) \, dx$
6. $\int e^{2x}(xe^x + 1) \, dx$
7. $\int \frac{1}{2x + 5} \, dx$
8. $\int \frac{x + 1}{x^2 + 2x + 3} \, dx$
9. $\int (4x - 3)^2 \, dx$
10. $\int (1 - x^2)^2 \, dx$

### EXERCISES C.3

In Exercises 1–6, write the linear differential equation in standard form.

1. $x^3 - 2x^2y' + 3y = 0$
2. $y' - 5(2x - y) = 0$
3. $xy' + y = xe^x$
4. $xy' + y = x^3y$
5. $(x - 1)y' = y + 1$
6. $x = x^2(y' + y)$

In Exercises 7–18, solve the differential equation.

7. $\frac{dy}{dx} + 3y = 6$
8. $\frac{dy}{dx} + 5y = 15$
9. $\frac{dy}{dx} + y = e^{-x}$
10. $\frac{dy}{dx} + 3y = e^{-3x}$
11. $\frac{dy}{dx} + \frac{y}{x} = 3x + 4$
12. $\frac{dy}{dx} + \frac{2y}{x} = 3x + 1$
13. $y' + 5xy = x$
14. $y' + 5y = e^{5x}$
15. $(x - 1)y' + y = x^2 - 1$
16. $xy' + y = x^2 + 1$
17. $x^3y' + 2y = e^{1/x^2}$
18. $xy' + y = x^2 \ln x$

In Exercises 19–22, solve for $y$ in two ways.

19. $y' + y = 4$
20. $y' + 10y = 5$
21. $y' - 2xy = 2x$
22. $y' + 4xy = x$

In Exercises 23–26, match the differential equation with its solution.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' - 2x = 0$</td>
<td>(a) $y = Ce^x$</td>
</tr>
<tr>
<td>$y' - 2y = 0$</td>
<td>(b) $y = -\frac{1}{2} + Ce^x$</td>
</tr>
<tr>
<td>$y' - 2xy = 0$</td>
<td>(c) $y = x^2 + C$</td>
</tr>
<tr>
<td>$y' - 2xy = x$</td>
<td>(d) $y = Ce^{2x}$</td>
</tr>
</tbody>
</table>

In Exercises 27–34, find the particular solution that satisfies the initial condition.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' + y = 6e^x$</td>
<td>$y = 3$ when $x = 0$</td>
</tr>
<tr>
<td>$y' + 2y = e^{-2x}$</td>
<td>$y = 4$ when $x = 1$</td>
</tr>
<tr>
<td>$xy' + y = 0$</td>
<td>$y = 2$ when $x = 2$</td>
</tr>
<tr>
<td>$y' + y = x$</td>
<td>$y = 4$ when $x = 0$</td>
</tr>
<tr>
<td>$y' + 3x^2y = 3x^2$</td>
<td>$y = 6$ when $x = 0$</td>
</tr>
<tr>
<td>$y' + (2x - 1)y = 0$</td>
<td>$y = 2$ when $x = 1$</td>
</tr>
<tr>
<td>$xy' - 2y = -x^2$</td>
<td>$y = 5$ when $x = 1$</td>
</tr>
<tr>
<td>$x^2y' - 4xy = 10$</td>
<td>$y = 10$ when $x = 1$</td>
</tr>
</tbody>
</table>
35. Sales  The rate of change (in thousands of units) in sales $S$ is modeled by
\[
\frac{dS}{dt} = 0.2(100 - S) + 0.2r
\]
where $t$ is the time in years. Solve this differential equation and use the result to complete the table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

36. Sales  The rate of change in sales $S$ is modeled by
\[
\frac{dS}{dt} = k_1(L - S) + k_2t
\]
where $t$ is the time in years and $S = 0$ when $t = 0$. Solve this differential equation for $S$ as a function of $t$.

Elasticity of Demand  In Exercises 37 and 38, find the demand function $p = f(x)$. Recall from Section 3.5 that the price elasticity of demand was defined as $\eta = (p/x)/(dp/dx)$.

37. $\eta = 1 - \frac{400}{3x}$, $p = 340$ when $x = 20$

38. $\eta = 1 - \frac{500}{3x}$, $p = 2$ when $x = 100$

Supply and Demand  In Exercises 39 and 40, use the demand and supply functions to find the price $p$ as a function of time $t$. Begin by setting $D(t)$ equal to $S(t)$ and solving the resulting differential equation. Find the general solution, and then use the initial condition to find the particular solution.

39. $D(t) = 480 + 5p(t) - 2p'(t)$  Demand function
 $S(t) = 300 + 8p(t) + p'(t)$  Supply function
 $p(0) = 75.00$  Initial condition

40. $D(t) = 4000 + 5p(t) - 4p'(t)$  Demand function
 $S(t) = 2800 + 7p(t) + 2p'(t)$  Supply function
 $p(0) = 10000.00$  Initial condition

41. Investment  A brokerage firm opens a new real estate investment plan for which the earnings are equivalent to continuous compounding at the rate of $r$. The firm estimates that deposits from investors will create a net cash flow of $Pt$ dollars, where $t$ is the time in years. The rate of change in the total investment $A$ is modeled by
\[
\frac{dA}{dt} = rA + Pt.
\]
(a) Solve the differential equation and find the total investment $A$ as a function of $t$. Assume that $A = 0$ when $t = 0$.
(b) Find the total investment $A$ after 10 years given that $P = \$500,000$ and $r = 9\%$.

42. Investment  Let $A(t)$ be the amount in a fund earning interest at the annual rate of $r$, compounded continuously. If a continuous cash flow of $P$ dollars per year is withdrawn from the fund, then the rate of decrease of $A$ is given by the differential equation
\[
\frac{dA}{dt} = rA - P
\]
where $A = A_0$ when $t = 0$.
(a) Solve this equation for $A$ as a function of $t$.
(b) Use the result of part (a) to find $A$ when $A_0 = \$2,000,000$, $r = 7\%$, $P = \$250,000$, and $t = 5$ years.
(c) Find $A_0$ if a retired person wants a continuous cash flow of $\$40,000$ per year for 20 years. Assume that the person’s investment will earn $8\%$, compounded continuously.

43. Velocity  A booster rocket carrying an observation satellite is launched into space. The rocket and satellite have mass $m$ and are subject to air resistance proportional to the velocity $v$ at any time $t$. A differential equation that models the velocity of the rocket and satellite is
\[
m \frac{dv}{dt} = -mg - kv
\]
where $g$ is the acceleration due to gravity. Solve the differential equation for $v$ as a function of $t$.

44. Health  An infectious disease spreads through a large population according to the model
\[
\frac{dy}{dt} = \frac{1 - y}{4}
\]
where $y$ is the percent of the population exposed to the disease, and $t$ is the time in years.
(a) Solve this differential equation, assuming $y(0) = 0$.
(b) Find the number of years it takes for half of the population to have been exposed to the disease.
(c) Find the percentage of the population that has been exposed to the disease after 4 years.

45. Research Project  Use your school’s library, the Internet, or some other reference source to find an article in a scientific or business journal that uses a differential equation to model a real-life situation. Write a short paper describing the situation. If possible, describe the solution of the differential equation.