Angles and Degree Measure

An angle has three parts: an initial ray, a terminal ray, and a vertex (the point of intersection of the two rays), as shown in Figure D.24. An angle is in standard position if its initial ray coincides with the positive $x$-axis and its vertex is at the origin. It is assumed that you are familiar with the degree measure of an angle.* It is common practice to use $\theta$ (the Greek lowercase theta) to represent both an angle and its measure. Angles between $0^\circ$ and $90^\circ$ are acute, and angles between $90^\circ$ and $180^\circ$ are obtuse.

Positive angles are measured counterclockwise, and negative angles are measured clockwise. For instance, Figure D.25 shows an angle whose measure is $-45^\circ$. You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved. For example, Figure D.25 shows that the angle measuring $-45^\circ$ has the same terminal ray as the angle measuring $315^\circ$. Such angles are coterminal. In general, if $\theta$ is any angle, then

$$\theta + n(360^\circ), \quad n \text{ is a nonzero integer}$$

is coterminal with $\theta$.

An angle that is larger than $360^\circ$ is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure D.26. You can form an angle whose measure is less than $-360^\circ$ by revolving a terminal ray more than one full revolution clockwise.

NOTE It is common to use the symbol $\theta$ to refer to both an angle and its measure. For instance, in Figure D.26, you can write the measure of the smaller angle as $\theta = 45^\circ$.

*For a more complete review of trigonometry, see Precalculus, 6th edition, by Larson and Hostetler (Boston, Massachusetts: Houghton Mifflin, 2004).
Radian Measure

To assign a radian measure to an angle $\theta$, consider $\theta$ to be a central angle of a circle of radius 1, as shown in Figure D.27. The radian measure of $\theta$ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a unit circle (of radius 1) is $2\pi$. This implies that the radian measure of an angle measuring $360^\circ$ is $2\pi$. In other words, $360^\circ = 2\pi$ radians.

Using radian measure for $\theta$, the length $s$ of a circular arc of radius $r$ is $s = r\theta$, as shown in Figure D.28.

You should know the conversions of the common angles shown in Figure D.29. For other angles, use the fact that $180^\circ$ is equal to $\pi$ radians.

**EXAMPLE 1** Conversions Between Degrees and Radians

a. $40^\circ = (40 \text{ deg}) \left( \frac{\pi \text{ rad}}{180 \text{ deg}} \right) = \frac{2\pi}{9}$ radians

b. $540^\circ = (540 \text{ deg}) \left( \frac{\pi \text{ rad}}{180 \text{ deg}} \right) = 3\pi$ radians

c. $-270^\circ = (-270 \text{ deg}) \left( \frac{\pi \text{ rad}}{180 \text{ deg}} \right) = -\frac{3\pi}{2}$ radians

d. $-\frac{\pi}{2}$ radians $= \left( -\frac{\pi}{2} \text{ rad} \right) \left( \frac{180 \text{ deg}}{\pi \text{ rad}} \right) = -90^\circ$

e. $2$ radians $= (2 \text{ rad}) \left( \frac{180 \text{ deg}}{\pi \text{ rad}} \right) = \frac{360}{\pi} \approx 114.59^\circ$

f. $\frac{9\pi}{2}$ radians $= \left( \frac{9\pi}{2} \text{ rad} \right) \left( \frac{180 \text{ deg}}{\pi \text{ rad}} \right) = 810^\circ$
The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal side of an angle in standard position. The six trigonometric functions, **sine**, **cosine**, **tangent**, **cotangent**, **secant**, and **cosecant** (abbreviated as sin, cos, etc.), are defined below from both viewpoints.

### Definition of the Six Trigonometric Functions

**Right triangle definitions, where** \(0 < \theta < \frac{\pi}{2}\) \((\text{see Figure D.30})\).

- \(\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}\)
- \(\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}\)
- \(\tan \theta = \frac{\text{opposite}}{\text{adjacent}}\)
- \(\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}}\)
- \(\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}\)
- \(\cot \theta = \frac{\text{adjacent}}{\text{opposite}}\)

**Circular function definitions, where \(\theta\) is any angle \((\text{see Figure D.31})\).**

- \(\sin \theta = \frac{y}{r}\)
- \(\cos \theta = \frac{x}{r}\)
- \(\tan \theta = \frac{y}{x}, x \neq 0\)
- \(\csc \theta = \frac{r}{y}, y \neq 0\)
- \(\sec \theta = \frac{r}{x}, x \neq 0\)
- \(\cot \theta = \frac{x}{y}, y \neq 0\)

The following trigonometric identities are direct consequences of the definitions. \((\phi\) is the Greek letter phi.)

### Trigonometric Identities

**Pythagorean Identities:**

- \(\sin^2 \theta + \cos^2 \theta = 1\)
- \(1 + \tan^2 \theta = \sec^2 \theta\)
- \(1 + \cot^2 \theta = \csc^2 \theta\)

**Sum or Difference of Two Angles:**

- \(\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi\)
- \(\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi\)
- \(\tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}\)

**Law of Cosines:**

- \(a^2 = b^2 + c^2 - 2bc \cos A\)

**Reduction Formulas:**

- \(\sin(-\theta) = -\sin \theta\)
- \(\cos(-\theta) = \cos \theta\)
- \(\tan(-\theta) = -\tan \theta\)

**Half-Angle Formulas:**

- \(\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}\)
- \(\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}\)

**Double-Angle Formulas:**

- \(\cos 2\theta = 2 \cos^2 \theta - 1\)
- \(\sin 2\theta = 2 \sin \theta \cos \theta\)

**Reciprocal Identities:**

- \(\csc \theta = \frac{1}{\sin \theta}\)
- \(\sec \theta = \frac{1}{\cos \theta}\)
- \(\cot \theta = \frac{1}{\tan \theta}\)

**Quotient Identities:**

- \(\tan \theta = \frac{\sin \theta}{\cos \theta}\)
- \(\cot \theta = \frac{\cos \theta}{\sin \theta}\)
Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions: (1) decimal approximations with a calculator and (2) exact evaluations using trigonometric identities and formulas from geometry. When using a calculator to evaluate a trigonometric function, remember to set the calculator to the appropriate mode—degree mode or radian mode.

**EXAMPLE 2  Exact Evaluation of Trigonometric Functions**

Evaluate the sine, cosine, and tangent of \( \frac{\pi}{3} \).

**Solution** Begin by drawing the angle \( \theta = \frac{\pi}{3} \) in standard position, as shown in Figure D.32. Then, because \( 60^\circ = \frac{\pi}{3} \) radians, you can draw an equilateral triangle with sides of length 1 and \( \theta \) as one of its angles. Because the altitude of this triangle bisects its base, you know that \( x = \frac{1}{2} \). Using the Pythagorean Theorem, you obtain

\[
y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.
\]

Now, knowing the values of \( x, y, \) and \( r \), you can write the following.

\[
\sin \frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2},
\]

\[
\cos \frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2},
\]

\[
\tan \frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.
\]

**NOTE** All angles in this text are measured in radians unless stated otherwise. For example, when \( \sin 3 \) is written, the sine of 3 radians is meant, and when \( \sin 3^\circ \) is written, the sine of 3 degrees is meant.

The degree and radian measures of several common angles are shown in the table below, along with the corresponding values of the sine, cosine, and tangent (see Figure D.33).

**Common First Quadrant Angles**

<table>
<thead>
<tr>
<th>Degrees</th>
<th>0</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>0</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>Undefined</td>
</tr>
</tbody>
</table>
The quadrant signs for the sine, cosine, and tangent functions are shown in Figure D.34. To extend the use of the table on the preceding page to angles in quadrants other than the first quadrant, you can use the concept of a reference angle (see Figure D.35), with the appropriate quadrant sign. For instance, the reference angle for \( \frac{3\pi}{4} \) is \( \pi/4 \), and because the sine is positive in Quadrant II, you can write

\[
\sin \frac{3\pi}{4} = + \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.
\]

Similarly, because the reference angle for 330° is 30°, and the tangent is negative in Quadrant IV, you can write

\[
\tan 330^\circ = -\tan 30^\circ = -\frac{\sqrt{3}}{3}.
\]

**EXAMPLE 3** Trigonometric Identities and Calculators

Evaluate each trigonometric expression.

**a.** \( \sin \left( -\frac{\pi}{3} \right) \)

**b.** \( \sec 60^\circ \)

**c.** \( \cos(1.2) \)

**Solution**

**a.** Using the reduction formula \( \sin(-\theta) = -\sin \theta \), you can write

\[
\sin \left( -\frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.
\]

**b.** Using the reciprocal identity \( \sec \theta = 1/\cos \theta \), you can write

\[
\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{1/2} = 2.
\]

**c.** Using a calculator, you obtain

\( \cos(1.2) \approx 0.3624 \).

Remember that 1.2 is given in radian measure. Consequently, your calculator must be set in radian mode.
Solving Trigonometric Equations

How would you solve the equation \( \sin \theta = 0 \)? You know that \( \theta = 0 \) is one solution, but this is not the only solution. Any one of the following values of \( \theta \) is also a solution.

\[ \ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots \]

You can write this infinite solution set as \( \{n\pi: n \text{ is an integer}\} \).

**EXAMPLE 4  Solving a Trigonometric Equation**

Solve the equation

\[ \sin \theta = -\frac{\sqrt{3}}{2}. \]

**Solution**  To solve the equation, you should consider that the sine is negative in Quadrants III and IV and that

\[ \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}. \]

So, you are seeking values of \( \theta \) in the third and fourth quadrants that have a reference angle of \( \pi/3 \). In the interval \([0, 2\pi]\), the two angles fitting these criteria are

\[ \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}. \]

By adding integer multiples of \( 2\pi \) to each of these solutions, you obtain the following general solution.

\[ \theta = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2n\pi, \quad \text{where } n \text{ is an integer}. \]

See Figure D.36.

**EXAMPLE 5  Solving a Trigonometric Equation**

Solve \( \cos 2\theta = 2 - 3 \sin \theta \), where \( 0 \leq \theta \leq 2\pi \).

**Solution**  Using the double-angle identity \( \cos 2\theta = 1 - 2 \sin^2 \theta \), you can rewrite the equation as follows.

\[
\begin{align*}
\cos 2\theta &= 2 - 3 \sin \theta \\
1 - 2 \sin^2 \theta &= 2 - 3 \sin \theta \quad \text{Trigonometric identity} \\
0 &= 2 \sin^2 \theta - 3 \sin \theta + 1 \\
0 &= (2 \sin \theta - 1)(\sin \theta - 1) \quad \text{Factor.}
\end{align*}
\]

If \( 2 \sin \theta - 1 = 0 \), then \( \sin \theta = 1/2 \) and \( \theta = \pi/6 \) or \( \theta = 5\pi/6 \). If \( \sin \theta - 1 = 0 \), then \( \sin \theta = 1 \) and \( \theta = \pi/2 \). So, for \( 0 \leq \theta \leq 2\pi \), there are three solutions.

\[ \theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \text{or} \quad \frac{\pi}{2} \]
**Graphs of Trigonometric Functions**

A function $f$ is periodic if there exists a nonzero number $p$ such that $f(x + p) = f(x)$ for all $x$ in the domain of $f$. The smallest such positive value of $p$ (if it exists) is the period of $f$. The sine, cosine, secant, and cosecant functions each have a period of $2\pi$, and the other two trigonometric functions, tangent and cotangent, have a period of $\pi$, as shown in Figure D.37.

The graphs of the six trigonometric functions

**Figure D.37**

Note in Figure D.37 that the maximum value of $\sin x$ and $\cos x$ is 1 and the minimum value is $-1$. The graphs of the functions $y = a \sin bx$ and $y = a \cos bx$ oscillate between $-a$ and $a$, and so have an amplitude of $|a|$. Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a \sin bx$ and $y = a \cos bx$ each have a period of $2\pi/|b|$. The table below summarizes the amplitudes and periods for some types of trigonometric functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Period</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = a \sin bx$ or $y = a \cos bx$</td>
<td>$\frac{2\pi}{</td>
<td>b</td>
</tr>
<tr>
<td>$y = a \tan bx$ or $y = a \cot bx$</td>
<td>$\frac{\pi}{</td>
<td>b</td>
</tr>
<tr>
<td>$y = a \sec bx$ or $y = a \csc bx$</td>
<td>$\frac{2\pi}{</td>
<td>b</td>
</tr>
</tbody>
</table>
EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Sketch the graph of \( f(x) = 3 \cos 2x \).

**Solution** The graph of \( f(x) = 3 \cos 2x \) has an amplitude of 3 and a period of \( 2\pi/2 = \pi \). Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval \([0, \pi]\), using the following pattern.

- **Maximum:** \((0, 3)\)
- **Minimum:** \((\pi/2, -3)\)
- **Maximum:** \((\pi, 3)\)

By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure D.38.

Horizontal shifts, vertical shifts, and reflections can be applied to the graphs of trigonometric functions, as illustrated in Example 7.

EXAMPLE 7 Shifts of Graphs of Trigonometric Functions

Sketch the graph of each function.

a. \( f(x) = \sin \left(x + \frac{\pi}{2}\right) \)  
   b. \( f(x) = 2 + \sin x \)  
   c. \( f(x) = 2 + \sin \left(x - \frac{\pi}{4}\right) \)

**Solution**

a. To sketch the graph of \( f(x) = \sin \left(x + \frac{\pi}{2}\right) \), shift the graph of \( y = \sin x \) to the left \( \pi/2 \) units, as shown in Figure D.39(a).

b. To sketch the graph of \( f(x) = 2 + \sin x \), shift the graph of \( y = \sin x \) upward two units, as shown in Figure D.39(b).

c. To sketch the graph of \( f(x) = 2 + \sin \left(x - \frac{\pi}{4}\right) \), shift the graph of \( y = \sin x \) upward two units and to the right \( \pi/4 \) units, as shown in Figure D.39(c).

![Figure D.39](image-url)
In Exercises 1 and 2, determine two coterminal angles in degree measure (one positive and one negative) for each angle.

1. (a) \( \theta = 36^\circ \)  
(b) \( \theta = -120^\circ \)

2. (a) \( \theta = 300^\circ \)  
(b) \( \theta = -420^\circ \)

In Exercises 3 and 4, determine two coterminal angles in radian measure (one positive and one negative) for each angle.

3. (a) \( \theta = \frac{\pi}{6} \)  
(b) \( \theta = -\frac{4\pi}{3} \)

4. (a) \( \theta = -\frac{9\pi}{4} \)  
(b) \( \theta = \frac{8\pi}{9} \)

In Exercises 5 and 6, rewrite each angle in radian measure as a multiple of \( \pi \) and as a decimal accurate to three decimal places.

5. (a) \( 30^\circ \)  
(b) \( 150^\circ \)  
(c) \( 315^\circ \)  
(d) \( 120^\circ \)

6. (a) \( -20^\circ \)  
(b) \( -240^\circ \)  
(c) \( -270^\circ \)  
(d) \( 144^\circ \)

In Exercises 7 and 8, rewrite each angle in degree measure.

7. (a) \( \frac{3\pi}{2} \)  
(b) \( \frac{7\pi}{6} \)  
(c) \( -\frac{7\pi}{12} \)  
(d) \(-2.367 \)

8. (a) \( \frac{7\pi}{3} \)  
(b) \( -\frac{11\pi}{30} \)  
(c) \( \frac{11\pi}{6} \)  
(d) \(0.438 \)

9. Let \( r \) represent the radius of a circle, \( \theta \) the central angle (measured in radians), and \( s \) the length of the arc subtended by the angle. Use the relationship \( s = r\theta \) to complete the table.

<table>
<thead>
<tr>
<th>( r )</th>
<th>8 ft</th>
<th>15 in.</th>
<th>85 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>12 ft</td>
<td>96 in.</td>
<td>8642 mi</td>
</tr>
<tr>
<td>( \theta )</td>
<td>1.6</td>
<td>( \frac{3\pi}{4} )</td>
<td>4</td>
</tr>
</tbody>
</table>
In Exercises 19–22, evaluate the sine, cosine, and tangent of each angle without using a calculator.

19. (a) $60^\circ$
   (b) $120^\circ$
   (c) $\frac{\pi}{4}$
   (d) $\frac{5\pi}{4}$

20. (a) $-30^\circ$
   (b) $150^\circ$
   (c) $-\frac{\pi}{6}$
   (d) $\frac{\pi}{2}$

21. (a) $225^\circ$
   (b) $-225^\circ$
   (c) $\frac{5\pi}{3}$
   (d) $\frac{11\pi}{6}$

22. (a) $750^\circ$
   (b) $510^\circ$
   (c) $\frac{10\pi}{3}$
   (d) $\frac{17\pi}{3}$

In Exercises 23–26, use a calculator to evaluate each trigonometric function. Round your answers to four decimal places.

23. (a) $\sin 10^\circ$
   (b) $\csc 10^\circ$

24. (a) $\sec 225^\circ$
   (b) $\csc 135^\circ$

25. (a) $\tan \frac{\pi}{9}$
   (b) $\tan \frac{10\pi}{9}$

26. (a) $\cot(1.35)$
   (b) $\tan(1.35)$

In Exercises 27–30, find two solutions of each equation. Give your answers in radians ($0 \leq \theta < 2\pi$). Do not use a calculator.

27. (a) $\cos \theta = \frac{\sqrt{3}}{2}$
   (b) $\cos \theta = -\frac{\sqrt{2}}{2}$

28. (a) $\sec \theta = 2$
   (b) $\sec \theta = -2$

29. (a) $\tan \theta = 1$
   (b) $\cot \theta = -\sqrt{3}$

30. (a) $\sin \theta = \frac{\sqrt{3}}{2}$
   (b) $\sin \theta = -\frac{\sqrt{3}}{2}$

In Exercises 31–38, solve the equation for $0 \leq \theta < 2\pi$.

31. $2\sin^2 \theta = 1$
32. $\tan^2 \theta = 3$
33. $\tan^2 \theta - \tan \theta = 0$
34. $2\cos^2 \theta - \cos \theta = 1$
35. $\sec \theta \csc \theta = 2 \csc \theta$
36. $\sin \theta = \cos \theta$
37. $\cos^2 \theta + \sin \theta = 1$
38. $\cos \frac{\theta}{2} - \cos \theta = 1$

39. **Airplane Ascent** An airplane leaves the runway climbing at an angle of $18^\circ$ with a speed of 275 feet per second (see figure). Find the altitude $a$ of the plane after 1 minute.

40. **Height of a Mountain** In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is $3.5^\circ$. After you drive 13 miles closer to the mountain, the angle of elevation is $9^\circ$. Approximate the height of the mountain.

41. In Exercises 41–44, determine the period and amplitude of each function.

   41. (a) $y = 2 \sin 2x$
   (b) $y = \frac{1}{2} \sin \pi x$

   42. (a) $y = \frac{3}{2} \cos \frac{x}{2}$
   (b) $y = -2 \sin \frac{x}{3}$

   43. $y = 3 \sin 4\pi x$

   44. $y = \frac{2}{3} \cos \frac{\pi x}{10}$
In Exercises 45–48, find the period of the function.
45. \( y = 5 \tan 2x \)  
46. \( y = 7 \tan 2\pi x \)
47. \( y = \sec 5x \)  
48. \( y = \csc 4x \)

**Writing** In Exercises 49 and 50, use a graphing utility to graph each function \( f \) in the same viewing window for \( c = -2 \), \( c = -1 \), \( c = 1 \), and \( c = 2 \). Give a written description of the change in the graph caused by changing \( c \).

49. (a) \( f(x) = c \sin x \)  
   (b) \( f(x) = \cos(cx) \)  
   (c) \( f(x) = \cos(\pi x - c) \)
50. (a) \( f(x) = \sin x + c \)  
   (b) \( f(x) = -\sin(2\pi x - c) \)  
   (c) \( f(x) = c \cos x \)

In Exercises 51–62, sketch the graph of the function.
51. \( y = \sin \frac{x}{2} \)  
52. \( y = 2 \cos 2x \)
53. \( y = -\sin \frac{2\pi x}{3} \)  
54. \( y = 2 \tan x \)
55. \( y = \csc \frac{x}{2} \)  
56. \( y = \tan 2x \)
57. \( y = 2 \sec 2x \)  
58. \( y = \csc 2\pi x \)
59. \( y = \sin(x + \pi) \)  
60. \( y = \cos \left(x - \frac{\pi}{3}\right) \)
61. \( y = 1 + \cos \left(x - \frac{\pi}{2}\right) \)  
62. \( y = 1 + \sin \left(x + \frac{\pi}{2}\right) \)

**Graphical Reasoning** In Exercises 63 and 64, find \( a \), \( b \), and \( c \) such that the graph of the function matches the graph in the figure.

63. \( y = a \cos(bx - c) \)  
64. \( y = a \sin(bx - c) \)

65. **Think About It** Sketch the graphs of \( f(x) = \sin x \), \( g(x) = |\sin x| \), and \( h(x) = \sin(|x|) \). In general, how are the graphs of \( |f(x)| \) and \( f(|x|) \) related to the graph of \( f \)?

66. **Think About It** The model for the height \( h \) of a Ferris wheel car is
\[
h = 51 + 50 \sin 8\pi t
\]
where \( t \) is measured in minutes. (The Ferris wheel has a radius of 50 feet.) This model yields a height of 51 feet when \( t = 0 \). Alter the model so that the height of the car is 1 foot when \( t = 0 \).

67. **Sales** The monthly sales \( S \) (in thousands of units) of a seasonal product are modeled by
\[
S = 58.3 + 32.5 \cos \frac{\pi t}{6}
\]
where \( t \) is the time (in months) with \( t = 1 \) corresponding to January. Use a graphing utility to graph the model for \( S \) and determine the months when sales exceed 75,000 units.

68. **Investigation** Two trigonometric functions \( f \) and \( g \) have a period of 2, and their graphs intersect at \( x = 5.35 \).
   (a) Give one smaller and one larger positive value of \( x \) where the functions have the same value.
   (b) Determine one negative value of \( x \) where the graphs intersect.
   (c) Is it true that \( f(13.35) = g(-4.65) \)? Give a reason for your answer.

69. **Pattern Recognition** In Exercises 69 and 70, use a graphing utility to compare the graph of \( f \) with the given graph. Try to improve the approximation by adding a term to \( f(x) \). Use a graphing utility to verify that your new approximation is better than the original. Can you find other terms to add to make the approximation even better? What is the pattern? (In Exercise 69, sine terms can be used to improve the approximation and in Exercise 70, cosine terms can be used.)

69. \( f(x) = \frac{4}{\pi} \left( \sin \frac{x}{\pi} + \frac{1}{3} \sin 3\pi x \right) \)

70. \( f(x) = \frac{1}{2} - \frac{4}{\pi^2} \left( \cos \frac{x}{\pi} + \frac{1}{9} \cos 3\pi x \right) \)