Exact Differential Equations

In Chapter 6, you studied applications of differential equations to growth and decay problems. You also learned more about the basic ideas of differential equations and studied the solution technique known as separation of variables. In this appendix, you will learn more about solving differential equations and using them in real-life applications. This section introduces you to a method for solving the first-order differential equation

\[ M(x, y)\, dx + N(x, y)\, dy = 0 \]

for the special case in which this equation represents the exact differential of a function \( z = f(x, y) \).

**Definition of an Exact Differential Equation**

The equation \( M(x, y)\, dx + N(x, y)\, dy = 0 \) is an exact differential equation if there exists a function \( f \) of two variables \( x \) and \( y \) having continuous partial derivatives such that

\[ f_x(x, y) = M(x, y) \quad \text{and} \quad f_y(x, y) = N(x, y). \]

The general solution of the equation is \( f(x, y) = C \).

If \( f \) has continuous second partials, then

\[ \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} \]

This suggests the following test for exactness.

**THEOREM C.1 Test for Exactness**

Let \( M \) and \( N \) have continuous partial derivatives on an open disk \( R \). The differential equation \( M(x, y)\, dx + N(x, y)\, dy = 0 \) is exact if and only if

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

Exactness is a fragile condition in the sense that seemingly minor alterations in an exact equation can destroy its exactness. This is demonstrated in the following example.
NOTE Every differential equation of the form
\[ M(x) \, dx + N(y) \, dy = 0 \]
is exact. In other words, a separable differential equation is actually a special type of an exact equation.

### Example 1 Testing for Exactness

**a.** The differential equation \((xy^2 + x) \, dx + yx^2 \, dy = 0\) is exact because
\[
\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [xy^2 + x] = 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [yx^2] = 2xy.
\]

But the equation \((y^2 + 1) \, dx + xy \, dy = 0\) is not exact, even though it is obtained by dividing each side of the first equation by \(x\).

**b.** The differential equation \(\cos y \, dx + (y^2 - x \sin y) \, dy = 0\) is exact because
\[
\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [\cos y] = -\sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [y^2 - x \sin y] = -\sin y.
\]

But the equation \(\cos y \, dx + (y^2 + x \sin y) \, dy = 0\) is not exact, even though it differs from the first equation only by a single sign.

Note that the test for exactness of \(M(x, y) \, dx + N(x, y) \, dy = 0\) is the same as the test for determining whether \(F(x, y) = M(x, y)i + N(x, y)j\) is the gradient of a potential function. This means that a general solution \(f(x, y) = C\) to an exact differential equation can be found by the method used to find a potential function for a conservative vector field.

### Example 2 Solving an Exact Differential Equation

Solve the differential equation \((2xy - 3x^2) \, dx + (x^2 - 2y) \, dy = 0\).

**Solution** The given differential equation is exact because
\[
\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [2xy - 3x^2] = 2x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [x^2 - 2y].
\]
The general solution, \(f(x, y) = C\), is given by
\[
f(x, y) = \int M(x, y) \, dx = \int (2xy - 3x^2) \, dx = x^2y - x^3 + g(y).
\]

You can determine \(g(y)\) by integrating \(N(x, y)\) with respect to \(y\) and reconciling the two expressions for \(f(x, y)\). An alternative method is to partially differentiate this version of \(f(x, y)\) with respect to \(y\) and compare the result with \(N(x, y)\). In other words,
\[
f_y(x, y) = \frac{\partial}{\partial y} [x^2y - x^3 + g(y)] = x^2 + g'(y) = x^2 - 2y.
\]

So, \(g'(y) = -2y\), and it follows that \(g(y) = -y^2 + C_1\). Therefore,
\[
f(x, y) = x^2y - x^3 - y^2 + C_1
\]
and the general solution is \(x^2y - x^3 - y^2 = C\). Figure C.1 shows the solution curves that correspond to \(C = 1, 10, 100, \text{and} 1000\).
EXAMPLE 3  Solving an Exact Differential Equation

Find the particular solution of

\[(\cos x - x \sin x + y^2)dx + 2xy
dy = 0\]

that satisfies the initial condition \(y = 1\) when \(x = \pi\).

**Solution**  The differential equation is exact because

\[
\frac{\partial M}{\partial y} = \cos x - x \sin x + 2y = \frac{\partial N}{\partial x}.
\]

Because \(N(x, y)\) is simpler than \(M(x, y)\), it is better to begin by integrating \(N(x, y)\).

\[
f(x, y) = \int N(x, y)\ dy = \int 2xy
dy = xy^2 + g(x)
\]

\[
f_y(x, y) = \frac{\partial}{\partial x} [xy^2 + g(x)] = y^2 + g'(x) = \cos x - x \sin x + y^2
\]

\[
g'(x) = \cos x - x \sin x
\]

So, \(g'(x) = \cos x - x \sin x\) and

\[
g(x) = \int (\cos x - x \sin x)\ dx
\]

\[
= x \cos x + C_1
\]

which implies that \(f(x, y) = xy^2 + x \cos x + C_2\), and the general solution is

\[xy^2 + x \cos x = C.\]  
**General solution**

Applying the given initial condition produces

\[\pi(1)^2 + \pi \cos \pi = C\]

which implies that \(C = 0\). So, the particular solution is

\[xy^2 + x \cos x = 0.\]  
**Particular solution**

The graph of the particular solution is shown in Figure C.2. Notice that the graph consists of two parts: the ovals are given by \(y^2 + \cos x = 0\), and the \(y\)-axis is given by \(x = 0\).

In Example 3, note that if \(z = f(x, y) = xy^2 + x \cos x\), the total differential of \(z\) is given by

\[
dz = f_x(x, y)\ dx + f_y(x, y)\ dy
\]

\[
= (\cos x - x \sin x + y^2)\ dx + 2xy\ dy
\]

\[
= M(x, y)\ dx + N(x, y)\ dy.
\]

In other words, \(M\ dx + N\ dy = 0\) is called an *exact* differential equation because \(M\ dx + N\ dy\) is exactly the differential of \(f(x, y)\).
**Integrating Factors**

If the differential equation \( M(x, y) \, dx + N(x, y) \, dy = 0 \) is not exact, it may be possible to make it exact by multiplying by an appropriate factor \( u(x, y) \), which is called an **integrating factor** for the differential equation.

**EXAMPLE 4  Multiplying by an Integrating Factor**

a. If the differential equation

\[
2y \, dx + x \, dy = 0
\]

is multiplied by the integrating factor \( u(x, y) = x \), the resulting equation

\[
2xy \, dx + x^2 \, dy = 0
\]

is exact—the left side is the total differential of \( x^2y \).

b. If the equation

\[
y \, dx - x \, dy = 0
\]

is multiplied by the integrating factor \( u(x, y) = 1/y^2 \), the resulting equation

\[
\frac{1}{y} \, dx - \frac{x}{y^2} \, dy = 0
\]

is exact—the left side is the total differential of \( x/y \).

Finding an integrating factor can be difficult. However, there are two classes of differential equations whose integrating factors can be found routinely—namely, those that possess integrating factors that are functions of either \( x \) alone or \( y \) alone. The following theorem, which is presented without proof, outlines a procedure for finding these two special categories of integrating factors.

**THEOREM C.2  Integrating Factors**

Consider the differential equation \( M(x, y) \, dx + N(x, y) \, dy = 0 \).

1. If

\[
\frac{1}{N(x, y)} \left[ M_y(x, y) - N_x(x, y) \right] = h(x)
\]

is a function of \( x \) alone, then \( e^{\int h(x) \, dx} \) is an integrating factor.

2. If

\[
\frac{1}{M(x, y)} \left[ N_y(x, y) - M_x(x, y) \right] = k(y)
\]

is a function of \( y \) alone, then \( e^{\int k(y) \, dy} \) is an integrating factor.

**STUDY TIP** If either \( h(x) \) or \( k(y) \) is constant, Theorem C.2 still applies. As an aid to remembering these formulas, note that the subtracted partial derivative identifies both the denominator and the variable for the integrating factor.
EXAMPLE 5 Finding an Integrating Factor

Solve the differential equation \((y^2 - x)\,dx + 2y\,dy = 0\).

Solution The given equation is not exact because \(M_y(x, y) = 2y\) and \(N_x(x, y) = 0\). However, because
\[
\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{2y - 0}{2y} = 1 = h(x)
\]
it follows that \(e^{h(x)}\,dx = e^x\,dx = e^x\) is an integrating factor. Multiplying the given differential equation by \(e^x\) produces the exact differential equation
\[
(y^2e^x - xe^x)\,dx + 2ye^x\,dy = 0
\]
whose solution is obtained as follows.
\[
f(x, y) = \int N(x, y)\,dy = \int 2ye^x\,dy = y^2e^x + g(x)
\]
\[
f_x(x, y) = y^2e^x + g'(x) = y^2e^x - xe^x
\]
\[
g'(x) = -xe^x
\]
Therefore, \(g'(x) = -xe^x\) and \(g(x) = -xe^x + e^x + C_1\), which implies that
\[
f(x, y) = y^2e^x - xe^x + e^x + C_1.
\]
The general solution is \(y^2e^x - xe^x + e^x = C\), or \(y^2 - x + 1 = Ce^{-x}\).

The next example shows how a differential equation can help in sketching a force field given by \(\mathbf{F}(x, y) = M(x, y)i + N(x, y)j\).

EXAMPLE 6 An Application to Force Fields

Sketch the force field given by
\[
\mathbf{F}(x, y) = \frac{2y}{\sqrt{x^2 + y^2}}i - \frac{y^2 - x}{\sqrt{x^2 + y^2}}j
\]
by finding and sketching the family of curves tangent to \(\mathbf{F}\).

Solution At the point \((x, y)\) in the plane, the vector \(\mathbf{F}(x, y)\) has a slope of
\[
\frac{dy}{dx} = \frac{-(y^2 - x)/\sqrt{x^2 + y^2}}{2y/\sqrt{x^2 + y^2}} = -\frac{(y^2 - x)}{2y}
\]
which, in differential form, is
\[
2y\,dy = -(y^2 - x)\,dx
\]
\[
(y^2 - x)\,dx + 2y\,dy = 0.
\]
From Example 5, you know that the general solution of this differential equation is \(y^2 - x + 1 = Ce^{-x}\), or \(y^2 = x - 1 + Ce^{-x}\). Figure C.3 shows several representative curves from this family. Note that the force vector at \((x, y)\) is tangent to the curve passing through \((x, y)\).
EXERCISES FOR APPENDIX C.1

In Exercises 1–10, determine whether the differential equation is exact. If it is, find the general solution.

1. \((2x - 3y) \, dx + (2y - 3x) \, dy = 0\)
2. \(y \, e^x \, dx + e^x \, dy = 0\)
3. \((3y^2 + 10xy^2) \, dx + (6xy - 2 + 10x^2) \, dy = 0\)
4. \(2 \cos(2x - y) \, dx - \cos(2x - y) \, dy = 0\)
5. \((4x^3 - 6xy^2) \, dx + (4y^3 - 6xy) \, dy = 0\)
6. \(2y^2e^{y^2} \, dx + 2xye^{y^2} \, dy = 0\)
7. \(\frac{1}{x^2 + y^2} (x \, dy - y \, dx) = 0\)
8. \(e^{-(x+y)^2} (x \, dx + y \, dy) = 0\)
9. \(\frac{1}{(x-y)^2} (y^2 \, dx + x^2 \, dy) = 0\)
10. \(e^x \cos xy \left[ y \, dx + (x + \tan xy) \, dy \right] = 0\)

In Exercises 11 and 12, (a) sketch an approximate solution of the differential equation satisfying the initial condition on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the sketch in part (a).

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2x \tan y + 5) , dx + (x^2 \sec^2 y) , dy = 0)</td>
<td>(\left(\frac{1}{2}, \frac{\pi}{4}\right))</td>
</tr>
<tr>
<td>(\frac{1}{\sqrt{x^2 + y^2}} (x , dx + y , dy) = 0)</td>
<td>(4, 3)</td>
</tr>
</tbody>
</table>

In Exercises 17–26, find the integrating factor that is a function of \(x\) or \(y\) alone and use it to find the general solution of the differential equation.

17. \(y \, dx - (x + 6y^2) \, dy = 0\)
18. \((2x^3 + y) \, dx - x \, dy = 0\)
19. \((5x^2 - y) \, dx + x \, dy = 0\)
20. \((5x^2 - y^2) \, dx + 2y \, dy = 0\)
21. \((x + y) \, dx + \tan x \, dy = 0\)
22. \((2x^2 y - 1) \, dx + x^3 \, dy = 0\)
23. \(y^2 \, dx + (xy - 1) \, dy = 0\)
24. \((x^2 + 2x + y) \, dx + 2 \, dy = 0\)
25. \(2y \, dx + (x - \sin \sqrt{y}) \, dy = 0\)
26. \((-2y^3 + 1) \, dx + (3xy^2 + x^3) \, dy = 0\)

In Exercises 27–30, use the integrating factor to find the general solution of the differential equation.

<table>
<thead>
<tr>
<th>Integrating Factor</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a(x, y) = xy^2)</td>
<td>((4x^2y + 2y^2) , dx + (3x^3 + 4xy) , dy = 0)</td>
</tr>
<tr>
<td>(a(x, y) = x^2 y)</td>
<td>((3y^2 + 5x^2y) , dx + (3xy + 2x^3) , dy = 0)</td>
</tr>
<tr>
<td>(a(x, y) = x^{-2}y^{-3})</td>
<td>((-y^3 + x^2y) , dx + (2xy^4 - 2x^3) , dy = 0)</td>
</tr>
<tr>
<td>(a(x, y) = x^{-2}y^{-2})</td>
<td>(-y^3 , dx + (xy^2 - x^2) , dy = 0)</td>
</tr>
</tbody>
</table>

31. Show that each expression is an integrating factor for the differential equation \(y \, dx - x \, dy = 0\).
   (a) \(\frac{1}{x^2}\)  (b) \(\frac{1}{y^2}\)  (c) \(\frac{1}{xy}\)  (d) \(\frac{1}{x^2 + y^2}\)

32. Show that the differential equation \((axy^2 + by) \, dx + (bx^2y + ax) \, dy = 0\) is exact only if \(a = b\). If \(a \neq b\), show that \(x^a y^n\) is an integrating factor, where \(m = \frac{2b + a}{a + b^n}\) and \(n = \frac{-2a + b}{a + b^n}\).

In Exercises 33–36, use a graphing utility to graph the family of curves tangent to the given force field.

33. \(F(x, y) = \frac{y}{\sqrt{x^2 + y^2}} i - \frac{x}{\sqrt{x^2 + y^2}} j\)
34. \(F(x, y) = \frac{x}{\sqrt{x^2 + y^2}} i - \frac{y}{\sqrt{x^2 + y^2}} j\)
35. \(F(x, y) = 4x^2 y i - \left(2xy^2 + \frac{x}{y^2}\right) j\)
36. \(F(x, y) = (1 + x^2) i - 2xy j\)
In Exercises 37 and 38, find an equation for the curve with the specified slope passing through the given point.

<table>
<thead>
<tr>
<th>Slope</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>(dy \over dx)</td>
<td>(y - x \over 3y - x)</td>
</tr>
<tr>
<td>(dy \over dx)</td>
<td>(-2xy \over x^2 + y^2)</td>
</tr>
</tbody>
</table>

39. **Cost** If \(y = C(x)\) represents the cost of producing \(x\) units in a manufacturing process, the **elasticity of cost** is defined as

\[
E(x) = \text{marginal cost} \over \text{average cost} = C'(x) \over C(x) = x \over dy \over dx.
\]

Find the cost function if the elasticity function is

\[
E(x) = \frac{20x - y}{2y - 10x}
\]

where \(C(100) = 500\) and \(x \geq 100\).

40. **Euler’s Method** Consider the differential equation \(y' = F(x, y)\) with the initial condition \(y(x_0) = y_0\). At any point \((x_k, y_k)\) in the domain of \(F\), \(F(x_k, y_k)\) yields the slope of the solution at that point. Euler’s Method gives a discrete set of estimates of the \(y\) values of a solution of the differential equation using the iterative formula

\[
y_{k+1} = y_k + F(x_k, y_k) \Delta x
\]

where \(\Delta x = x_{k+1} - x_k\).

(a) Write a short paragraph describing the general idea of how Euler’s Method works.

(b) How will decreasing the magnitude of \(\Delta x\) affect the accuracy of Euler’s Method?

41. **Euler’s Method** Use Euler’s Method (see Exercise 40) to approximate \(y(1)\) for the values of \(\Delta x\) given in the table if \(y' = x + \sqrt{y}\) and \(y(0) = 2\). (Note that the number of iterations increases as \(\Delta x\) decreases.) Sketch a graph of the approximate solution on the direction field in the figure.

<table>
<thead>
<tr>
<th>(\Delta x)</th>
<th>0.50</th>
<th>0.25</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate of (y(1))</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value of \(y(1)\), accurate to three decimal places, is 4.213.

42. **Programming** Write a program for a graphing utility or computer that will perform the calculations of Euler’s Method for a specified differential equation, interval, \(\Delta x\), and initial condition. The output should be a graph of the discrete points approximating the solution.

43. **Euler’s Method** In Exercises 43–46, (a) use the program you wrote in Exercise 42 to approximate the solution of the differential equation over the indicated interval with the specified value of \(\Delta x\) and the initial condition, (b) solve the differential equation analytically, and (c) use a graphing utility to graph the particular solution and compare the result with the graph of part (a).

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Interval</th>
<th>(\Delta x)</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y' = x^{3/2})</td>
<td>[1, 2]</td>
<td>0.01</td>
<td>(y(1) = 1)</td>
</tr>
<tr>
<td>(y' = \frac{\pi}{4}(y^2 + 1))</td>
<td>[−1, 1]</td>
<td>0.1</td>
<td>(y(-1) = -1)</td>
</tr>
<tr>
<td>(y' = -\frac{xy}{x^2 + y^2})</td>
<td>[2, 4]</td>
<td>0.05</td>
<td>(y(2) = 1)</td>
</tr>
<tr>
<td>(y' = \frac{6x + y^2}{y(3y - 2x)})</td>
<td>[0, 5]</td>
<td>0.2</td>
<td>(y(0) = 1)</td>
</tr>
</tbody>
</table>

44. **Euler’s Method** Repeat Exercise 45 for \(\Delta x = 1\) and discuss how the accuracy of the result changes.

45. **Euler’s Method** Repeat Exercise 46 for \(\Delta x = 0.5\) and discuss how the accuracy of the result changes.

**True or False?** In Exercises 49–52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

49. The differential equation \(2xy dx + (y^2 - x^2) dy = 0\) is exact.

50. If \(M dx + N dy = 0\) is exact, then \(xM dx + xN dy = 0\) is also exact.

51. If \(M dx + N dy = 0\) is exact, then \([f(x) + M] dx + [g(y) + N] dy = 0\) is also exact.

52. The differential equation \(f(x) dx + g(y) dy = 0\) is exact.