

**Nonhomogeneous Equations**

In the preceding section, we represented damped oscillations of a spring by the *homogeneous* second-order linear equation

\[ \frac{d^2y}{dt^2} + \frac{p}{m} \frac{dy}{dt} + \frac{k}{m} y = 0. \]

**Free motion**

This type of oscillation is called *free* because it is determined solely by the spring and gravity and is free of the action of other external forces. If such a system is also subject to an external periodic force such as caused by vibrations at the opposite end of the spring, the motion is called *forced*, and it is characterized by the *nonhomogeneous* equation

\[ \frac{d^2y}{dt^2} + \frac{p}{m} \frac{dy}{dt} + \frac{k}{m} y = a \sin bt. \]

**Forced motion**

In this section, you will study two methods for finding the general solution of a nonhomogeneous linear differential equation. In both methods, the first step is to find the general solution of the corresponding homogeneous equation.

\[ y = y_h \]

**General solution of homogeneous equation**

Having done this, you try to find a particular solution of the nonhomogeneous equation.

\[ y = y_p \]

**Particular solution of nonhomogeneous equation**

By combining these two results, you can conclude that the general solution of the nonhomogeneous equation is \( y = y_h + y_p \), as stated in the following theorem.

**THEOREM 15.6 Solution of Nonhomogeneous Linear Equation**

Let

\[ y'' + ay' + by = F(x) \]

be a second-order nonhomogeneous linear differential equation. If \( y_p \) is a particular solution of this equation and \( y_h \) is the general solution of the corresponding homogeneous equation, then

\[ y = y_h + y_p \]

is the general solution of the nonhomogeneous equation.
Method of Undetermined Coefficients

You already know how to find the solution \( y_h \) of a linear homogeneous differential equation. The remainder of this section looks at ways to find the particular solution \( y_p \). If \( F(x) \) in
\[
y'' + ay' + by = F(x)
\]
consists of sums or products of \( x^n, e^{mx}, \cos \beta x, \) or \( \sin \beta x \), you can find a particular solution \( y_p \) by the method of **undetermined coefficients**. The gist of this method is to guess that the solution \( y_p \) is a generalized form of Here are some examples.

1. If \( F(x) = 3x^2 \), choose \( y_p = Ax^2 + Bx + C \).
2. If \( F(x) = 4xe^x \), choose \( y_p = Ax^2 + Be^x \).
3. If \( F(x) = x + \sin 2x \), choose \( y_p = (Ax + B) + C \sin 2x + D \cos 2x \).

Then, by substitution, determine the coefficients for the generalized solution.

**EXAMPLE 1  Method of Undetermined Coefficients**

Find the general solution of the equation
\[
y'' - 2y' - 3y = 2 \sin x.
\]

**Solution**  To find \( y_h \), solve the characteristic equation.
\[
m^2 - 2m - 3 = 0
\]
\[
(m + 1)(m - 3) = 0
\]
\[
m = -1 \quad \text{or} \quad m = 3
\]
Thus, \( y_h = C_1 e^{-x} + C_2 e^{3x} \). Next, let \( y_p \) be a generalized form of \( 2 \sin x \).
\[
y_p = A \cos x + B \sin x
\]
\[
y_p' = -A \sin x + B \cos x
\]
\[
y_p'' = -A \cos x - B \sin x
\]
Substitution into the original differential equation yields
\[
y'' - 2y' - 3y = 2 \sin x
\]
\[
-A \cos x - B \sin x + 2A \sin x - 2B \cos x - 3A \cos x - 3B \sin x = 2 \sin x
\]
\[
(-4A - 2B) \cos x + (2A - 4B) \sin x = 2 \sin x.
\]
By equating coefficients of like terms, you obtain
\[
-4A - 2B = 0 \quad \text{and} \quad 2A - 4B = 2
\]
with solutions \( A = \frac{1}{2} \) and \( B = -\frac{1}{2} \). Therefore,
\[
y_p = \frac{1}{2} \cos x - \frac{2}{5} \sin x
\]
and the general solution is
\[
y = y_h + y_p
\]
\[
= C_1 e^{-x} + C_2 e^{3x} + \frac{1}{2} \cos x - \frac{2}{5} \sin x.
\]
In Example 1, the form of the homogeneous solution

\[ y_h = C_1 e^{-x} + C_2 e^{3x} \]

has no overlap with the function \( F(x) \) in the equation

\[ y'' + ay' + by = F(x) \]

However, suppose the given differential equation in Example 1 were of the form

\[ y'' - 2y' - 3y = e^{-x}. \]

Now, it would make no sense to guess that the particular solution were \( y = Ae^{-x} \), because you know that this solution would yield 0. In such cases, you should alter your guess by multiplying by the lowest power of \( x \) that removes the duplication. For this particular problem, you would guess

\[ y_p = Axe^{-x}. \]

**EXAMPLE 2 Method of Undetermined Coefficients**

Find the general solution of

\[ y'' - 2y' = x + 2e^x. \]

**Solution** The characteristic equation \( m^2 - 2m = 0 \) has solutions \( m = 0 \) and \( m = 2 \). Thus,

\[ y_h = C_1 + C_2 e^{2x}. \]

Because \( F(x) = x + 2e^x \), your first choice for \( y_p \) would be \( (A + Bx) + Ce^x \). However, because \( y_h \) already contains a constant term \( C_1 \), you should multiply the polynomial part by \( x \) and use

\[
\begin{align*}
y_p &= Ax + Bx^2 + Ce^x \\
y_p' &= A + 2Bx + Ce^x \\
y_p'' &= 2B + Ce^x.
\end{align*}
\]

Substitution into the differential equation produces

\[
y'' - 2y' = x + 2e^x \]

\[
(2B + Ce^x) - 2(A + 2Bx + Ce^x) = x + 2e^x \\
(2B - 2A) - 4Bx - Ce^x = x + 2e^x.
\]

Equating coefficients of like terms yields the system

\[
2B - 2A = 0, \quad -4B = 1, \quad -C = 2
\]

with solutions \( A = B = -\frac{1}{4} \) and \( C = -2 \). Therefore,

\[ y_p = -\frac{1}{4} x - \frac{1}{4} x^2 - 2e^x \]

and the general solution is

\[
y = y_h + y_p = C_1 + C_2 e^{2x} - \frac{1}{4} x - \frac{1}{4} x^2 - 2e^x.
\]
In Example 2, the polynomial part of the initial guess
\[(A + Bx) + Ce^x\]
for \(y_p\) overlapped by a constant term with \(y_h = C_1 + C_2e^{2x}\), and it was necessary to multiply the polynomial part by a power of \(x\) that removed the overlap. The next example further illustrates some choices for \(y_p\) that eliminate overlap with \(y_h\). Remember that in all cases the first guess for \(y_p\) should match the types of functions occurring in \(F(x)\).

EXAMPLE 3  Choosing the Form of the Particular Solution

Determine a suitable choice for \(y_p\) for each of the following.

\[
\begin{align*}
\frac{d^2y}{dx^2} + ay' + by &= F(x) & y_h &= C_1 + C_2x \\
a. \quad y'' &= x^2 & \quad \text{for } y_h = C_1 + C_2x \\
b. \quad y'' + 2y' + 10y &= 4\sin 3x & \quad y_h &= C_1e^{-x}\cos 3x + C_2e^{-x}\sin 3x \\
c. \quad y'' - 4y' + 4 &= e^{2x} & \quad y_h &= C_1e^{2x} + C_2xe^{2x}
\end{align*}
\]

Solution

a. Because \(F(x) = x^2\), the normal choice for \(y_p\) would be \(A + Bx + Cx^2\). However, because \(y_h = C_1 + C_2x\) already contains a linear term, you should multiply by \(x^2\) to obtain
\[y_p = Ax^2 + Bx^3 + Cx^4.\]

b. Because \(F(x) = 4\sin 3x\) and each term in \(y_h\) contains a factor of \(e^{-x}\), you can simply let
\[y_p = A\cos 3x + B\sin 3x.\]

c. Because \(F(x) = e^{2x}\), the normal choice for \(y_p\) would be \(Ae^{2x}\). However, because \(y_h = C_1e^{2x} + C_2xe^{2x}\) already contains an \(xe^{2x}\) term, you should multiply by \(x^2\) to get
\[y_p = Ax^2e^{2x}.\]

EXAMPLE 4  Solving a Third-Order Equation

Find the general solution of
\[y''' + 3y'' + 3y' + y = x.\]

Solution  From Example 6 in the preceding section, you know that the homogeneous solution is
\[y_h = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x}.\]

Because \(F(x) = x\), let \(y_p = A + Bx\) and obtain \(y_p' = B\) and \(y_p'' = 0\). Thus, by substitution, you have
\[0 + 3(0) + 3(B) + (A + Bx) = (3B + A) + Bx = x.\]

Thus, \(B = 1\) and \(A = -3\), which implies that \(y_p = -3 + x\). Therefore, the general solution is
\[y = y_h + y_p = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} - 3 + x.\]
Variation of Parameters

The method of undetermined coefficients works well if \( F(x) \) is made up of polynomials or functions whose successive derivatives have a cyclic pattern. For functions such as \( 1/x \) and \( \tan \, x \), which do not have such characteristics, it is better to use a more general method called variation of parameters. In this method, you assume that \( y_p \) has the same form as \( y_h \), except that the constants in \( y_h \) are replaced by variables.

**Variation of Parameters**

To find the general solution to the equation \( y'' + ay' + by = F(x) \), use the following steps.

1. Find \( y_h = C_1y_1 + C_2y_2 \).
2. Replace the constants by variables to form \( y_p = u_1y_1 + u_2y_2 \).
3. Solve the following system for \( u_1' \) and \( u_2' \).
   \[
   \begin{align*}
   u_1'y_1 + u_2'y_2 &= 0 \\
   u_1'y_1' + u_2'y_2' &= F(x)
   \end{align*}
   \]
4. Integrate to find \( u_1 \) and \( u_2 \). The general solution is \( y = y_h + y_p \).

**EXAMPLE 5 Variation of Parameters**

Solve the differential equation

\[
y'' - 2y' + y = \frac{e^x}{2x}, \quad x > 0.
\]

**Solution** The characteristic equation \( m^2 - 2m + 1 = (m - 1)^2 = 0 \) has one solution, \( m = 1 \). Thus, the homogeneous solution is

\[
y_h = C_1y_1 + C_2y_2 = C_1e^x + C_2xe^x.
\]

Replacing \( C_1 \) and \( C_2 \) by \( u_1 \) and \( u_2 \), produces

\[
y_p = u_1y_1 + u_2y_2 = u_1e^x + u_2xe^x.
\]

The resulting system of equations is

\[
\begin{align*}
   u_1'e^x + u_2'xe^x &= 0 \\
   u_1'e^x + u_2'(xe^x + e^x) &= \frac{e^x}{2x}
\end{align*}
\]

Subtracting the second equation from the first produces \( u_2' = 1/(2x) \). Then, by substitution in the first equation, you have \( u_1' = -\frac{1}{2} \). Finally, integration yields

\[
   u_1 = -\int \frac{1}{2} \, dx = -\frac{x}{2} \quad \text{and} \quad u_2 = \frac{1}{2} \int \frac{1}{x} \, dx = \frac{1}{2} \ln x = \ln \sqrt{x}.
\]

From this result it follows that a particular solution is

\[
y_p = -\frac{1}{2}xe^x + (\ln \sqrt{x})xe^x
\]

and the general solution is

\[
y = C_1e^x + C_2xe^x - \frac{1}{2}xe^x + xe^x \ln \sqrt{x}.
\]
EXAMPLE 6 Variation of Parameters

Solve the differential equation
\[ y'' + y = \tan x. \]

Solution Because the characteristic equation \( m^2 + 1 = 0 \) has solutions \( m = \pm i \), the homogeneous solution is
\[ y_h = C_1 \cos x + C_2 \sin x. \]
Replacing \( C_1 \) and \( C_2 \) by \( u_1 \) and \( u_2 \) produces
\[ y_p = u_1 \cos x + u_2 \sin x. \]
The resulting system of equations is
\[
\begin{align*}
u_1' \cos x + u_2' \sin x &= 0 \\
u_1' \sin x + u_2' \cos x &= \tan x.
\end{align*}
\]
Multiplying the first equation by \( \sin x \) and the second by \( \cos x \) produces
\[
\begin{align*}
u_1' \sin x \cos x + u_2' \sin^2 x &= 0 \\
u_1' \sin x \cos x + u_2' \cos^2 x &= \sin x.
\end{align*}
\]
Adding these two equations produces \( u_2' = \sin x \), which implies that
\[
\begin{align*}
u_1' &= -\frac{\sin^2 x}{\cos x} \\
&= \frac{\cos^2 x - 1}{\cos x} \\
&= \cos x - \sec x.
\end{align*}
\]
Integration yields
\[
\begin{align*}
u_1 &= \int (\cos x - \sec x) \, dx \\
&= \sin x - \ln |\sec x + \tan x|
\end{align*}
\]
and
\[
\begin{align*}
u_2 &= \int \sin x \, dx \\
&= -\cos x
\end{align*}
\]
so that
\[
\begin{align*}y_p &= \sin x \cos x - \cos x \ln |\sec x + \tan x| - \sin x \cos x \\
&= -\cos x \ln |\sec x + \tan x|
\end{align*}
\]
and the general solution is
\[
\begin{align*}y &= y_h + y_p \\
&= C_1 \cos x + C_2 \sin x - \cos x \ln |\sec x + \tan x|.
\end{align*}
\]
In Exercises 1–4, verify the solution of the differential equation.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( y = 2(e^{2t} - \cos x) )</td>
<td>( y'' + y = 10e^{2t} )</td>
</tr>
<tr>
<td>2. ( y = (2 + \frac{1}{2}x)\sin x )</td>
<td>( y'' + y = \cos x )</td>
</tr>
<tr>
<td>3. ( y = 3 \sin x - \cos x \ln</td>
<td>\sec x + \tan x</td>
</tr>
<tr>
<td>4. ( y = (5 - \ln</td>
<td>\sin x</td>
</tr>
</tbody>
</table>

In Exercises 5–20, solve the differential equation by the method of undetermined coefficients.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. ( y'' - 3y' + 2y = 2x )</td>
<td>6. ( y'' - 2y' - 3y = x^2 - 1 )</td>
</tr>
<tr>
<td>7. ( y'' + y = x^3 )</td>
<td>8. ( y'' + 4y = 4 )</td>
</tr>
<tr>
<td>9. ( y'' + 2y' = 2e^t )</td>
<td>10. ( y'' - 9y = 5e^{3t} )</td>
</tr>
<tr>
<td>11. ( y'' - 10y' + 25y = 5 + 6e^t )</td>
<td>12. ( 16y'' - 8y' + y = 4(x + e^t) )</td>
</tr>
<tr>
<td>13. ( y'' + y' = 2 \sin x )</td>
<td>14. ( y'' + y' - 2y = 3 \cos 2x )</td>
</tr>
<tr>
<td></td>
<td>15. ( y'' + 9y = \sin 3x )</td>
</tr>
<tr>
<td>16. ( y'' + 4y' + 5y = \sin x + \cos x )</td>
<td>17. ( y'' - 3y' + 2y = 2e^{-2t} )</td>
</tr>
<tr>
<td>18. ( y'' - y'' = 4x^2 )</td>
<td>19. ( y' - 4y = xe^t - xe^{4t} )</td>
</tr>
<tr>
<td></td>
<td>( y(0) = \frac{1}{3} )</td>
</tr>
<tr>
<td>20. ( y'' + 2y = \sin x )</td>
<td>( y(\frac{\pi}{2}) = \frac{2}{5} )</td>
</tr>
</tbody>
</table>

21. **Think About It**

   (a) Explain how, by observation, you know that a particular solution of the differential equation \( y'' + 3y = 12 \) is \( y_p = 4 \).

   (b) Use the explanation of part (a) to give a particular solution of the differential equation \( y'' + 5y = 10 \).

   (c) Use the explanation of part (a) to give a particular solution of the differential equation \( y'' + 2y' + 2y = 8 \).

22. **Think About It**

   (a) Explain how, by observation, you know that a form of a particular solution of the differential equation \( y'' + 3y = 12 \sin x \) is \( y_p = A \sin x \).

   (b) Use the explanation of part (a) to find a particular solution of the differential equation \( y'' + 5y = 10 \cos x \).

   (c) Compare the algebra required to find particular solutions in parts (a) and (b) with that required if the form of the particular solution were \( y_p = A \cos x + B \sin x \).

In Exercises 23–28, solve the differential equation by the method of variation of parameters.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>23. ( y'' + y = \sec x )</td>
<td>24. ( y'' + y = \sec x \tan x )</td>
</tr>
<tr>
<td>25. ( y'' + 4y = \csc 2x )</td>
<td>26. ( y'' - 4y' + 4y = x^2e^{2x} )</td>
</tr>
<tr>
<td>27. ( y'' - 2y' + y = e^x \ln x )</td>
<td>28. ( y'' - 4y' + 4y = \frac{e^{2x}}{x} )</td>
</tr>
</tbody>
</table>

### Electrical Circuits

In Exercises 29 and 30, use the electrical circuit differential equation

\[
\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \left( \frac{1}{LC} \right) q = \left( \frac{1}{L} \right) E(t)
\]

where \( R \) is the resistance (in ohms), \( C \) is the capacitance (in farads), \( L \) is the inductance (in henrys), \( E(t) \) is the electromotive force (in volts), and \( q \) is the charge on the capacitor (in coulombs). Find the charge \( q \) as a function of time for the electrical circuit described. Assume that \( q(0) = 0 \) and \( q'(0) = 0 \).

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>29. ( R = 20, C = 0.02, L = 2 )</td>
<td>( E(t) = 12 \sin 5t )</td>
</tr>
<tr>
<td>30. ( R = 20, C = 0.02, L = 1 )</td>
<td>( E(t) = 10 \sin 5t )</td>
</tr>
</tbody>
</table>

### Vibrating Spring

In Exercises 31–34, find the particular solution of the differential equation

\[
\frac{w}{g} y''(t) + by'(t) + ky(t) = \frac{w}{g} F(t)
\]

for the oscillating motion of an object on the end of a spring. Use a graphing utility to graph the solution. In the equation, \( y \) is the displacement from equilibrium (positive direction is downward) measured in feet, and \( t \) is time in seconds (see figure). The constant \( w \) is the weight of the object, \( g \) is the acceleration due to gravity, \( b \) is the magnitude of the resistance to the motion, \( k \) is the spring constant from Hooke’s Law, and \( F(t) \) is the acceleration imposed on the system.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>31. ( \frac{24}{52} y'' + 48y = \frac{24}{52}(48 \sin 4t) )</td>
<td>( y(0) = \frac{1}{4}, y'(0) = 0 )</td>
</tr>
<tr>
<td>32. ( \frac{2}{52} y'' + 4y = \frac{2}{52}(4 \sin 8t) )</td>
<td>( y(0) = \frac{1}{4}, y'(0) = 0 )</td>
</tr>
<tr>
<td>33. ( \frac{4}{52} y'' + y' + 4y = \frac{4}{52}(4 \sin 8t) )</td>
<td>( y(0) = \frac{1}{4}, y'(0) = -3 )</td>
</tr>
<tr>
<td>34. ( \frac{4}{52} y'' + \frac{1}{2} y' + \frac{2}{2} y = 0 )</td>
<td>( y(0) = \frac{1}{2}, y'(0) = -4 )</td>
</tr>
</tbody>
</table>
35. **Vibrating Spring**  Rewrite $y_h$ in the solution for Exercise 31 by using the identity

$$a \cos \omega t + b \sin \omega t = \sqrt{a^2 + b^2} \sin(\omega t + \phi)$$

where $\phi = \arctan \frac{a}{b}$.

36. **Vibrating Spring**  The figure shows the particular solution of the differential equation

$$\frac{4}{32}y'' + by' + \frac{25}{2}y = 0$$

$$y(0) = \frac{1}{2}, y'(0) = -4$$

for values of the resistance component $b$ in the interval $[0, 1]$. (Note that when $b = \frac{1}{2}$, the problem is identical to that of Exercise 34.)

(a) If there is no resistance to the motion ($b = 0$), describe the motion.

(b) If $b > 0$, what is the ultimate effect of the retarding force?

(c) Is there a real number $M$ such that there will be no oscillations of the spring if $b > M$? Explain your answer.

37. **Parachute Jump**  The fall of a parachutist is described by the second-order linear differential equation

$$\frac{w}{g} \frac{d^2y}{dt^2} - k \frac{dy}{dt} = w$$

where $w$ is the weight of the parachutist, $y$ is the height at time $t$, $g$ is the acceleration due to gravity, and $k$ is the drag factor of the parachute. If the parachute is opened at 2000 feet, $y(0) = 2000$, and at that time the velocity is $y'(0) = -100$ feet per second, then for a 160-pound parachutist, using $k = 8$, the differential equation is

$$-5y'' - 8y' = 160.$$  

Using the given initial conditions, verify that the solution of the differential equation is

$$y = 1950 + 50e^{-1.6t} - 20t.$$  

38. **Parachute Jump**  Repeat Exercise 37 for a parachutist who weighs 192 pounds and has a parachute with a drag factor of $k = 9$.

39. Solve the differential equation

$$x^2y'' - xy' + y = 4x \ln x$$

given that $y_1 = x$ and $y_2 = x \ln x$ are solutions of the corresponding homogeneous equation.

40. **True or False?**  $y_p = -e^{2t} \cos e^{-t}$ is a particular solution of the differential equation

$$y'' - 3y' + 2y = \cos e^{-t}.$$