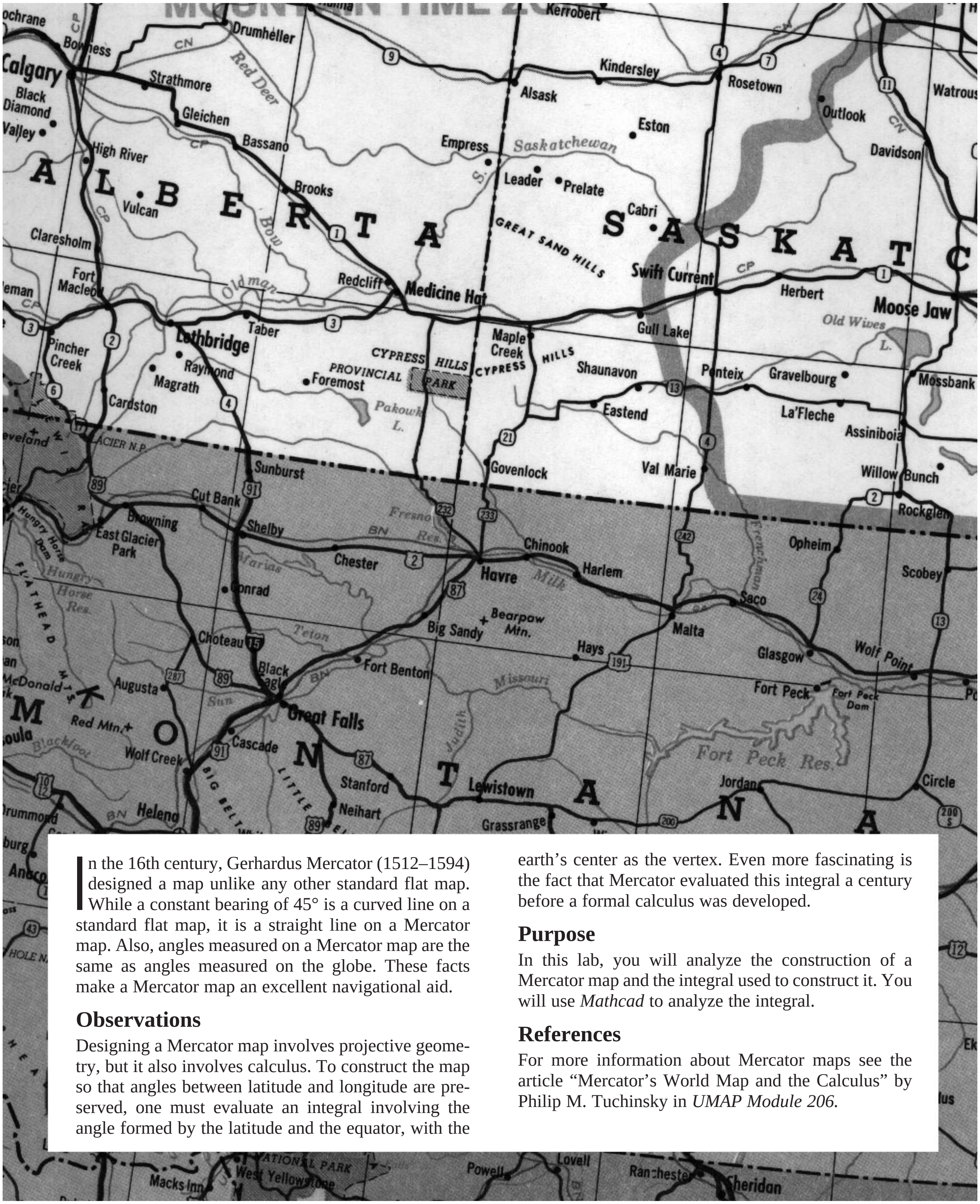
LAB 10

CALCULUS OF THE MERCATOR MAP

Integration



In the 16th century, Gerhardus Mercator (1512–1594) designed a map unlike any other standard flat map. While a constant bearing of 45° is a curved line on a standard flat map, it is a straight line on a Mercator map. Also, angles measured on a Mercator map are the same as angles measured on the globe. These facts make a Mercator map an excellent navigational aid.

Observations

Designing a Mercator map involves projective geometry, but it also involves calculus. To construct the map so that angles between latitude and longitude are preserved, one must evaluate an integral involving the angle formed by the latitude and the equator, with the earth's center as the vertex. Even more fascinating is the fact that Mercator evaluated this integral a century before a formal calculus was developed.

Purpose

In this lab, you will analyze the construction of a Mercator map and the integral used to construct it. You will use *Mathcad* to analyze the integral.

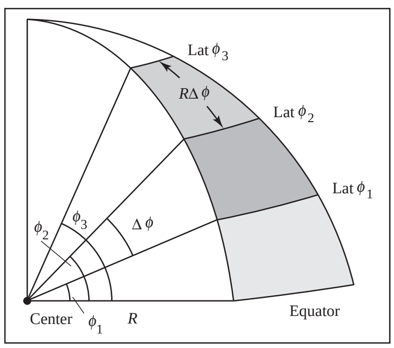
References

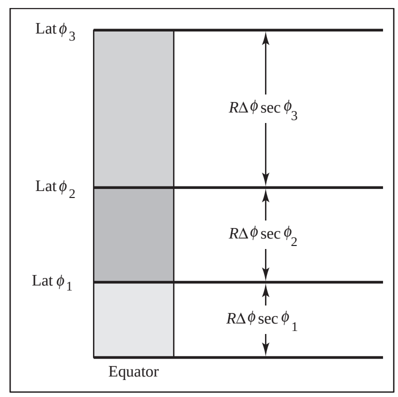
For more information about Mercator maps see the article "Mercator's World Map and the Calculus" by Philip M. Tuchinsky in *UMAP Module 206.*

Data

For lines to appear as straight lines on a flat map, Mercator realized that latitude lines must be stretched horizontally by a scaling factor of sec ϕ, where ϕ is the angle (in radians) of the latitude line. In order for the map to preserve the angles between latitude and longitude lines, the vertical lengths of longitude lines must also be stretched by a scaling factor of sec ϕ at latitude ϕ.

To calculate these vertical lengths, imagine a globe with latitude lines marked at angles of every $\Delta\phi$ radians, with $\Delta\phi = \phi_i - \phi_{i-1}$. The arc length of consecutive latitude lines is $R\Delta\phi$. Therefore, the vertical distance between the first latitude line and the equator is $R\Delta\phi \sec \phi_1$. The vertical distance between the second latitude line and the first latitude line is $R\Delta\phi \sec \phi_2$, and so on, as shown below.





On a globe, the angle between consecutive latitude lines is $\Delta\phi$, and the arc length between them is $R\Delta\phi$ (see the figure on the left above). Therefore, the vertical distance between latitude line i and latitude line $i - 1$ is $R\Delta\phi \sec \phi_i$, and the distance from the equator to the ith latitude line is approximately

$$R\Delta\phi \sec \phi_1 + R\Delta\phi \sec \phi_2 + \cdots + R\Delta\phi \sec \phi_i.$$

Exercises

Name ______________________________

Date ____________________ Class ____________

Instructor ______________________________

1. ***Using Summation Notation and Calculus.*** Use summation notation to write how far from the equator to draw the line representing latitude ϕ_n. As the value for $\Delta\phi$ gets smaller and smaller, the approximations of the distance from the equator to a latitude line get better and better. Use this observation and your knowledge of calculus to calculate the total vertical distance of each latitude line from the equator. Use the result to complete the following table. (Use a globe radius of $R = 6$ inches.)

Latitude Line (in degrees north of the equator)	Distance From Equator
15°	
30°	
45°	
60°	
75°	
90°	

2. ***Bad Latitude?*** In Exercise 1, what problem do you encounter when you attempt to calculate the distance from the equator to the latitude line 90° north of the equator? What does this latitude represent on the map?

3. ***Proof.*** In the calculation of the total vertical distance of a latitude line from the equator on a Mercator map, you need to evaluate the integral

$$\int \sec \phi \, d\phi.$$

Prove that the following integration formula is true.

$$\int \sec \phi \, d\phi = \ln|\sec \phi + \tan \phi| + C$$

4. ***Integrating With Technology.*** Use *Mathcad* to evaluate

$$\int \sec \phi \, d\phi.$$

Does the result given by *Mathcad* agree with the formula given in Exercise 3? If not, are the two results equivalent? Explain.

5. ***Are the Results Valid?*** In Exercise 3, you showed that

$$\int \sec \phi \, d\phi = \ln|\sec \phi + \tan \phi| + C.$$

Use differentiation to show that the following results are also valid.

(a) $\displaystyle\int \sec \phi \, d\phi = -\ln\left|\tan\left[\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)\right]\right| + C$

(b) $\displaystyle\int \sec \phi \, d\phi = \ln\left|\tan\left[\frac{1}{2}\left(\frac{\pi}{2} + \phi\right)\right]\right| + C$

6. ***Radians to Degrees.*** The angle ϕ for the integral

$$\int \sec \phi \, d\phi$$

is measured in radians. How would you write the integral if the angle ϕ is measured in degrees? Evaluate the indefinite integral and compare the result to those of Exercise 5.

7. ***The Same Distance?*** On a Mercator map, does one inch along the latitude at 30° north of the equator represent the same earth-distance as one inch along the latitude at 60° north of the equator? Explain your reasoning.

8. ***Area and Distance.*** You know a Mercator map is useful as a navigational aid, but can you use a Mercator map to compare the areas of two regions? Use the Mercator map and the table of areas to answer the question. What happens to the area of a region as you move farther from the equator? Do you think it would be easy to determine distance between two cities on a Mercator map? Explain your reasoning.
(Source: The Universal Almanac)

Region	Area (in square miles)
Africa	11,687,188
Antarctica	5,100,023
Asia	17,176,102
Australia	3,035,651
Europe	4,065,945
North America	9,357,294
South America	6,880,638