Logic

L.1  Statements and Connectives
L.2  Truth Tables
L.3  Implications
L.4  Valid Arguments
L.5  Quantifiers and Euler Diagrams
Grammar, Logic, and Rhetoric

Grammar, logic, and rhetoric are the first three of the seven liberal arts that dominated education from Roman times to the past century. Rhetoric uses every possible tool for persuasion, and the following are examples of several invalid forms of rhetorical reasoning that are continuing traps for the unwary.

**Fallacy of Ambiguity.** “Tom was mad when he broke the vase. Mad people should be locked away in lunatic asylums. Therefore, Tom should be confined to an asylum.” This is not a valid argument because it confuses two different meanings of “mad”—“angry” and “crazy.”

**Fallacy of Composition.** “Every member of the committee is wise. Therefore, the committee will act wisely.” This confuses a characteristic of each part with that of the whole: If every part of a television costs less than $25, does the TV itself cost less than $25?

**Fallacy of the Complex Question.** “Have you stopped partying and gotten down to work yet?” This “loaded” question includes an assumption so that neither a “yes” nor a “no” answer will be accurate for someone who has been working all along.

**Fallacy of the non sequitur.** “Everyone wants lower taxes. I know about complicated finance because I’m a banker. Therefore, electing me will restore responsible government.” The final sentence literally “does not follow” from the others: Each separately seems acceptable, but taken together, the first and second do not lead to the third.

**Fallacy of the False Cause.** “He said that the stock market would rise yesterday. It did. Therefore, he can predict the market.” But did he really know or was he just guessing? Implying a connection is not the same as demonstrating a cause-and-effect relationship.

In this chapter we will explore some of the structures that arguments must have in order to be valid.

**Introduction**

Although language is useful for warning of impending disaster or asking directions, one of its most powerful aspects is its potential for convincing others of the validity of your position. Persuasion requires...
tact and perseverance, but its effect may be lost if the justifications contain even the subtlest fallacy. We begin our discussion of symbolic logic as “the mathematics of correct reasoning” by introducing several fundamental ideas and notations.

**Statements**

A *statement* is a declarative sentence that is either true or false. Commands (“Don’t interrupt”), opinions (“Chocolate tastes good”), questions (“Is today Tuesday?”), and paradoxes (“This sentence is false”) are not statements. We do not need to know whether a particular statement is true or false—we require only that the statement has the property that it must be one or the other (“There are exactly 1783 pennies in this jar”).

We will use lowercase letters (\(p, q\), and so on) for statements and the capital letters \(T\) for “true” and \(F\) for “false.” A *tautology* is a statement that is always true and we reserve the lowercase letter \(t\) for such a statement. Similarly, \(f\) will represent a *contradiction*, which is a statement that is always false.

A *compound statement* is a combination of statements using one or more of the following logical connectives:

<table>
<thead>
<tr>
<th>Word(s)</th>
<th>Type</th>
<th>Symbol</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>not</td>
<td>negation</td>
<td>(\sim)</td>
<td>(\sim p)</td>
</tr>
<tr>
<td>and</td>
<td>conjunction</td>
<td>(\land)</td>
<td>(p \land q)</td>
</tr>
<tr>
<td>or</td>
<td>disjunction</td>
<td>(\lor)</td>
<td>(p \lor q)</td>
</tr>
<tr>
<td>if . . . , then . . .</td>
<td>implication</td>
<td>(\rightarrow)</td>
<td>(p \rightarrow q)</td>
</tr>
</tbody>
</table>

We will discuss *implications* in more detail in Section 3 and *universal quantifiers* (“all,” “every”) and *existential quantifiers* (“there is,” “for at least one”) in Section 5.

The presence of one of these words in a sentence does not necessarily mean that the sentence expresses a compound statement: “‘Mocha madness’ is a flavor of Tom and Jerry’s ice cream” is a statement, but the “and” is not used as a logical connective.

The *order of operations* for logical connectives is \(\sim, \land, \lor,\) and then \(\rightarrow\), so that negation (“not”) is carried out first, then conjunction (“and”), then disjunction (“or”), and finally implication (“if . . . , then . . . ”). We shall use parentheses when necessary or just to clarify the meaning. Thus, according to the order of operations,

\[
\sim p \lor q \text{ means } (\sim p) \lor q \text{ rather than } \sim (p \lor q)
\]

Since \(\sim\) is done before \(\lor\).
and

$p \rightarrow q \land r$ means $p \rightarrow (q \land r)$ rather than $(p \rightarrow q) \land r$.

Since $\land$ is done before $\rightarrow$

**EXAMPLE 1**

**TRANSLATING SYMBOLS INTO ENGLISH**

Let $p$ represent the statement “the home team won” and let $q$ represent the statement “the celebration went on beyond 1 o’clock.” Express each symbolic statement as a sentence.

a. $\sim p$  
   b. $p \land q$  
   c. $p \lor \sim q$  
   d. $p \rightarrow q$

**Solution**

a. We could translate $\sim p$ as “it is not true that the home team won,” but we prefer the more natural “the home team lost.”

b. “The home team won and the celebration went on beyond 1 o’clock.”

c. “The home team won or the celebration ended by 1 o’clock.”

d. “If the home team won, then the celebration went on beyond 1 o’clock.”

**Practice Problem 1**

(Continuation of Example 1) Translate $\sim q \rightarrow \sim p$.

**Solution on page 5**

**EXAMPLE 2**

**TRANSLATING FROM ENGLISH INTO SYMBOLS**

Let $p$ represent the statement “John ordered codfish,” and let $q$ represent the statement “John ordered ice cream.” Represent each sentence symbolically using $p$, $q$, and logical connectives.

a. John did not order ice cream.

b. John ordered codfish or ice cream.

c. John ordered codfish but not ice cream.

d. If John did not order codfish, then he ordered ice cream.

**Solution**

a. $\sim q$

b. $p \lor q$

c. $p \land \sim q$

d. $\sim p \rightarrow q$
Truth Values

The truth value of a compound statement is either true (T) or false (F), depending on the truth or falsity of its components. For instance, the compound statement “the ink is blue and the paper is white” is false if the pen has red ink even if the paper actually is white (because both parts of a conjunction must be true for the conjunction to be true). Although the calculation of truth values for a compound statement may be complicated and will be explored further in the next section, we can make several general observations about truth values. Negation reverses truth and falsity, so we immediately have:

\[
\sim t \quad \text{is the same as} \quad f \\
\sim f \quad \text{is the same as} \quad t \\
\sim (\sim p) \quad \text{is the same as} \quad p
\]

As observed above, for a conjunction to be true, both parts of must be true, so we have:

\[
p \land t \quad \text{is the same as} \quad p \\
p \land f \quad \text{is the same as} \quad f
\]

For a disjunction, either part being true makes it true, so we have:

\[
p \lor t \quad \text{is the same as} \quad t \\
p \lor f \quad \text{is the same as} \quad p
\]

Practice Problem 2 (Continuation of Example 2) Translate into symbols: “John ordered neither codfish nor ice cream.”

Practice Problem 3 Express “I ain’t not going to bed” without the double negative.
**Section Summary**

A statement is either true (T) or false (F). A compound statement is a combination of statements using logical connectives and parentheses (as necessary).

\[
\begin{align*}
\sim p & \quad \text{not } p \\
p \land q & \quad p \text{ and } q \\
p \lor q & \quad p \text{ or } q \\
p \rightarrow q & \quad \text{if } p, \text{ then } q \\
\end{align*}
\]

A tautology \( t \) is a statement that is always true, while a contradiction \( f \) is always false.

**Solutions to Practice Problems**

1. “If the celebration ended by 1 o’clock, then the home team lost.”
2. This could be translated as either \( \sim (p \lor q) \) or \( \sim p \land \sim q \).
3. “I ain’t not going to bed” is really \( \sim (\sim p) \), which is just “I am going to bed.”

**Exercises**

Let \( p \) represent the statement “the yard is large,” and let \( q \) represent “the house is small.” Express each symbolic statement as a sentence.

1. \( \sim p \)
2. \( \sim q \)
3. \( p \land q \)
4. \( \sim p \lor q \)
5. \( p \rightarrow q \)
6. \( \sim q \rightarrow \sim p \)

Let \( p \) represent the statement “the cat is purring,” and let \( q \) represent “the dog is barking.” Express each symbolic statement as a sentence.

7. \( p \lor q \)
8. \( p \land \sim q \)
9. \( \sim p \land q \)
10. \( \sim p \lor \sim q \)
11. \( q \rightarrow p \)
12. \( \sim p \rightarrow \sim q \)

Let \( p \) represent the statement “the pizza is ready,” let \( q \) represent “the pool is cold,” and let \( r \) represent “the music is loud.” Express each sentence in symbolic form.

13. The pizza isn’t ready.
14. The pool isn’t cold or the music is loud.
15. The pizza is ready and the pool is cold.
16. The pizza is not ready and the music is loud.
17. The pool is cold or the music isn’t loud.
18. The music is loud and the pool is cold, or the pizza is ready.
19. If the music isn’t loud, then the pizza is ready.
20. If it is not the case that the pizza is ready and the pool is cold, then the music is loud.

Explorations and Excursions
The following problems extend and augment the material presented in the text.

Unintended Meanings
Many common sentences do not express what was probably intended. Reformulate each sentence to express the intended statement correctly.

21. This door must remain closed at all times.
22. Let us now review some facts that may be forgotten.
23. All applicants were not selected.
24. An epilepsy support group is forming for persons considering brain surgery on the second Tuesday of each month.
25. I tried to find a book about solving logic problems without success.
Introduction

Symbolic logic has many similarities to the ordinary algebra of real numbers. We have already mentioned that \( \neg (\neg p) \) is the same as \( p \), and this bears a striking resemblance to the usual rule for signed numbers that \( -(-x) = x \). But before we learned to manipulate algebraic expressions, such as factoring \( x^2 - 3x - 4 \) into \((x-4)(x+1)\), we learned how to evaluate such expressions. In this section, we will “evaluate” compound statements for various truth “values” of their components.

Truth Tables for \( \neg \), \( \land \), and \( \lor \)

A truth table for a compound statement is a list of the truth or falsity of the statement for every possible combination of truth and falsity of its components. We begin with negation:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

“Not true” is “false”

“Not false” is “true”

Because this truth table completely explains the result of negating a statement, it may be taken as the definition of negation.

The truth tables for conjunction (“and”) and disjunction (“or”) require more rows of possibilities because they involve two statements, each of which may be true or false:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

True only if both are true

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \lor q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

True if either is true

The alternatives “closed/open” and “on/off” play the same role for circuits that “true/false” plays in logic, and in this section we will make “truth tables” for logical expressions similar to those above. The observation that logical calculations can be carried out by electrical circuits is the basis for today’s electronic computers, whose switches are transistors in integrated circuits etched on silicon wafers. Recent chemical advances indicate that synthetic molecular “switches” may work 100 billion times faster than today’s silicon circuits, suggesting the possibility of enormous increases in computing speeds.
The disjunction $p \lor q$ is sometimes called the “inclusive disjunction” since it is true when either or both of $p$ and $q$ are true. This agrees with the interpretation of “or” in statements such as “Call a doctor if you are in pain or have a temperature,” in which the intention is clearly to include the case where both are true.*

We will give the truth table for the implication $p \rightarrow q$ in the next section.

**Calculating Truth Tables**

When making a truth table for a compound statement, we use the “order of operations” to evaluate the various parts of the expression in the correct order. One systematic way of carrying this out is demonstrated in the following example.

**CONSTRUCTING A TRUTH TABLE**

Construct a truth table for the compound statement $p \lor (\sim p \land q)$.

**Solution**

Each $p$ and $q$ may be either of two values, so we need $2 \times 2 = 4$ rows in our table. We divide the region under the formula for our compound statement into columns corresponding to the parts we must successively calculate.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor (\sim p \land q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

The columns under the connectives $\land$ and $\lor$ stand for the conjunction and disjunction, respectively, of the expressions on the two sides of that connective. For example, the column headed $\land$ stands for the expression $(\sim p \land q)$. Working from the inside of the parentheses outward, the order of operations indicates that we should calculate the columns in the order shown on the next page, completing one new column in each numbered step.

* The exclusive disjunction, as in “Each dinner comes with rice or potatoes,” is discussed in the Explorations and Excursions at the end of this section.
### L.2 Truth Tables

Before continuing, be sure that you understand the steps in the construction of this truth table, because from now on we may omit the intermediate steps.

If the truth values in the final column were all T’s, the statement would be a tautology, while if they were all F’s, it would be a contradiction. The statement in Example 1 is neither a tautology nor a contradiction.

Keeping only the final column from Example 1, we see that the truth table for the compound statement \( p \lor (\sim p \land q) \) is the same as that for the disjunction \( p \lor q \):

<table>
<thead>
<tr>
<th>1. ( p )</th>
<th>( q )</th>
<th>( p \lor (\sim p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Negating the \( p \) column

<table>
<thead>
<tr>
<th>2. ( p )</th>
<th>( q )</th>
<th>( p \lor (\sim p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Copying the \( q \) column

<table>
<thead>
<tr>
<th>3. ( p )</th>
<th>( q )</th>
<th>( p \lor (\sim p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

\( \land \) of the \( \sim p \) and \( q \) columns

<table>
<thead>
<tr>
<th>4. ( p )</th>
<th>( q )</th>
<th>( p \lor (\sim p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

\( \lor \) of the \( \sim p \) and \( \land \) of the \( q \) columns

<table>
<thead>
<tr>
<th>5. ( p )</th>
<th>( q )</th>
<th>( p \lor (\sim p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The red column is the \( \lor \) of the blue columns

The fifth table is the completed truth table, with the truth values for the entire statement shown in the “final” column under the \( \lor \).
Substitution Principle

For any statements \(a\) and \(b\), let \(P(a)\) be a compound statement containing statement \(a\), and let \(P(b)\) be the same compound statement but with each instance of statement \(a\) replaced by statement \(b\). If \(a\) and \(b\) are logically equivalent, then \(P(a)\) and \(P(b)\) are also logically equivalent.

\[
\begin{array}{c|c|c}
 p & q & p \lor \neg (\neg p \land q) \\
 T & T & T \\
 T & F & T \\
 F & T & T \\
 F & F & F \\
\end{array}
\]

Different statements can have the same truth tables.

When we say that two statements have the same truth table, we are referring only to their final columns.

Practice Problem

Construct a truth table for the compound statement \(p \lor \neg q\).

Solution on page 17

Logical Equivalence

Two statements \(a\) and \(b\) are logically equivalent, written \(a \equiv b\), if they have the same truth table. Since we saw from Example 1 that the statement \(p \lor (\neg p \land q)\) has the same truth table as the disjunction \(p \lor q\), these statements are logically equivalent:

\[
p \lor (\neg p \land q) \equiv p \lor q
\]

Therefore, whenever we encounter the statement \(p \lor (\neg p \land q)\), we may replace it by the simpler statement \(p \lor q\). This logical simplification of statements is similar to the algebraic simplification of expressions, such as replacing \(2x - (x - 1)\) by the shorter and simpler \(x + 1\).

We may even replace one statement by a logically equivalent one within a more complicated statement, as formalized by the following substitution principle:

Substitution Principle

For any statements \(a\) and \(b\), let \(P(a)\) be a compound statement containing statement \(a\), and let \(P(b)\) be the same compound statement but with each instance of statement \(a\) replaced by statement \(b\). If \(a\) and \(b\) are logically equivalent, then \(P(a)\) and \(P(b)\); that is,

\[
\text{if } a = b, \quad \text{then } P(a) = P(b)
\]

We now list some of the basic properties of logical equivalences, each of which can be shown by a truth table.

An identity is a quantity that leaves every quantity unchanged when combined with it under a given operation. For real numbers, 0 is
the additive identity since \( 0 + x = x \) for every \( x \), while 1 is the multiplicative identity since \( 1 \cdot x = x \) for every \( x \). In symbolic logic, the tautology \( t \) (which is always true) and the contradiction \( f \) (which is always false) act as identities for conjunction and disjunction, respectively, as shown in laws 1 and 5 below.

<table>
<thead>
<tr>
<th>Identity Laws</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( p \land t = p )</td>
</tr>
<tr>
<td>2. ( p \land f = f )</td>
</tr>
<tr>
<td>3. ( p \land \sim p = f )</td>
</tr>
<tr>
<td>4. ( p \lor t = t )</td>
</tr>
<tr>
<td>5. ( p \lor f = p )</td>
</tr>
<tr>
<td>6. ( p \lor \sim p = t )</td>
</tr>
</tbody>
</table>

Identity laws 1, 2, 4, and 5 were discussed on page 4, and laws 3 and 6 are true since exactly one of \( p \) and \( \sim p \) can be true, so their conjunction is false and their disjunction is true. We also remarked on page 4 that the negation of a negation is the same as the original statement.

<table>
<thead>
<tr>
<th>Double Negation Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sim (\sim p) = p )</td>
</tr>
</tbody>
</table>

The negation of a negation is the original statement.

An *idempotent* quantity is one that is unchanged when combined with itself. For real numbers, 0 is the only additive idempotent since \( 0 + 0 = 0 \), and 1 is the only multiplicative idempotent since \( 1 \cdot 1 = 1 \). In symbolic logic, every statement is both a conjunctive and a disjunctive idempotent because repeating the same statement a second time does not alter its meaning.

<table>
<thead>
<tr>
<th>Idempotent Laws</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( p \land p = p ) A statement “and” itself is just the original statement</td>
</tr>
<tr>
<td>2. ( p \lor p = p ) A statement “or” itself is just the original statement</td>
</tr>
</tbody>
</table>

The order of the statements make no difference to their combined truth (“the tea is hot and the muffins are buttered” is the same as “the muffins are buttered and the tea is hot”). Such statements are said to “commute.”

* From the Latin words *idem* for “same” and *potens* for “strength,” suggesting that repetitions do not increase the meaning.
† From the Latin word *commutare* meaning “to change with.”
Commutative Laws

1. \( p \land q = q \land p \)  
   In an “and” statement, the order may be reversed

2. \( p \lor q = q \lor p \)  
   In an “or” statement, the order may be reversed

Similarly, a string of conjunctions or disjunctions may be combined (or “associated”) in any order.

Associative Laws

1. \( p \land (q \land r) = (p \land q) \land r \)  
   A series of “and”s may be grouped in any order

2. \( p \lor (q \lor r) = (p \lor q) \lor r \)  
   A series of “or”s may be grouped in any order

While ordinary multiplication distributes over addition [for instance, \( 2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4 \)], in symbolic logic both conjunction and disjunction distribute over the other (“dessert” and “coffee or tea” means “dessert and coffee” or “dessert and tea”).

Distributive Laws

1. \( p \land (q \lor r) = (p \land q) \lor (p \land r) \)  
   An “and” applied to an “or” is two “and”s joined by an “or”

2. \( p \lor (q \land r) = (p \lor q) \land (p \lor r) \)  
   An “or” applied to an “and” is two “or”s joined by an “and”

Unlike ordinary algebra, symbolic logic includes a special fact that essentially says that having redundant information does not provide anything new.

Absorption Laws

1. \( p \land (p \lor q) = p \)  
   Since (for each law) if \( p \) is true, then both sides are true, and if \( p \) is false, then both sides are false

2. \( p \lor (p \land q) = p \)

**EXAMPLE 2**

**VERIFYING THE FIRST DISTRIBUTIVE LAW**

Use truth tables to establish the first distributive law:

\[ p \land (q \lor r) = (p \land q) \lor (p \land r). \]
### Solution

Since each of $p$, $q$, and $r$ may be either of two values, we need $2 \times 2 \times 2 = 8$ rows in our table. Rather than building two separate tables (one for each side of the $\equiv$ sign), we shall make one large table that first finds $p \land (q \lor r)$ and then $(p \land q) \lor (p \land r)$. If the final columns for these compound statements are the same, then their logical equivalence will be established. The completed table is shown below.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$p \land (q \lor r)$</th>
<th>$(p \land q) \lor (p \land r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
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<td>$T$ $T$ $T$ $T$</td>
<td>$T$ $T$ $T$ $T$</td>
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<tr>
<td>$T$</td>
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<td>$F$</td>
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<tr>
<td>$T$</td>
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<td>$T$ $F$ $T$ $T$</td>
<td>$T$ $F$ $T$ $T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$T$ $F$ $F$ $F$</td>
<td>$T$ $F$ $F$ $F$</td>
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<tr>
<td>$F$</td>
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<td>$F$ $F$ $T$ $T$</td>
<td>$F$ $T$ $F$ $F$</td>
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<td>$F$</td>
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<tr>
<td>$F$</td>
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<td>$F$</td>
<td>$F$ $F$ $F$ $F$</td>
<td>$F$ $F$ $F$ $F$</td>
</tr>
</tbody>
</table>

Since both final columns are the same, we have proved that $\land$ distributes over $\lor$: $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$.

### Practice Problem 2

Verify the second “absorption law” by showing that $p \lor (p \land q) \equiv p$.

➤ Solution on page 17

### Logical Equivalence by Symbolic Manipulation

We now have a second way, besides truth tables, of establishing the logical equivalency of two statements: If one can be transformed into the other using the laws of symbolic logic, then they are equivalent.

### Using Symbolic Manipulation

Use symbolic manipulation to show that $p \lor (\sim p \land q)$ is logically equivalent to $p \lor q$. 
Notice that symbolic manipulation is very similar to the familiar manipulation of variables that you learned in algebra. By the comment on pages 9–10 (after Example 1), we have already found this same equivalence by truth tables. Both methods will be important in the next sections.

**Solution**

We begin with the more complicated side, \( p \lor (\neg p \land q) \), and simplify it by successive logical equivalences until we obtain the other side, \( p \lor q \).

\[
\begin{align*}
  p \lor (\neg p \land q) &= (p \lor \neg p) \land (p \lor q) & \text{Distributive law 2} \\
  &= (t) \land (p \lor q) & \text{Identity law 6} \\
  &= p \lor q & \text{Commutative law 1 and identity law 1}
\end{align*}
\]

Therefore, \( p \lor (\neg p \land q) = p \lor q \).

Notice that symbolic manipulation is very similar to the familiar manipulation of variables that you learned in algebra. By the comment on pages 9–10 (after Example 1), we have already found this same equivalence by truth tables. Both methods will be important in the next sections.

**Practice Problem 3**

Use the laws of symbolic logic to show that \( \neg p \land (p \lor q) \) is logically equivalent to \( \neg p \lor q \).  

**Solution on page 17**

**De Morgan’s Laws**

Suppose that someone claims he is rich and famous. To prove him wrong, you only need to show either that he is not rich or that he is not famous (since he is claiming both). In other words, the negation of an “and” is the “or” of the negations, which may be written in symbolic form as:

\[
\neg (p \land q) = \neg p \lor \neg q
\]

Similarly, if a more modest person claims that she is rich or famous, to prove her wrong you need to show that she is both not rich and not famous. That is, the negation of an “or” is the “and” of the negations:

\[
\neg (p \lor q) = \neg p \land \neg q
\]

These two observations are known as De Morgan’s laws.*

**De Morgan’s Laws**

1. \( \neg (p \land q) = \neg p \lor \neg q \)  
   To negate an “and,” change it to “or” and negate each part

2. \( \neg (p \lor q) = \neg p \land \neg q \)  
   To negate an “or,” change it to “and” and negate each part

* After Augustus de Morgan (1806–1871), British mathematician and logician.
De Morgan’s laws can be verified by truth tables (see Exercises 39 and 40), and the following exploration shows how a truth table can be done with a graphing calculator.

### Graphing Calculator Exploration

Truth tables can be constructed on most graphing calculators by using 1 for \( T \) and 0 for \( F \). We verify the second of De Morgan’s laws, \( \sim (p \lor q) \equiv \sim p \land \sim q \), as follows: Enter the \( T F F \) values for \( p \) in the list \( L_1 \) as \( \{1,1,0,0\} \) and the \( T F T \) values for \( q \) in the list \( L_2 \) as \( \{1,0,1,0\} \). Evaluate \( \sim (p \lor q) \) as \( \text{not}(L_1 \lor L_2) \) and \( \sim p \land \sim q \) as \( \text{not}(L_1) \text{ and not}(L_2) \).

<table>
<thead>
<tr>
<th>( L_1 )</th>
<th>( L_2 )</th>
<th>( \sim (p \lor q) )</th>
<th>( \sim p \land \sim q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \ 1 \ 0 \ 0 )</td>
<td>( 1 \ 0 \ 1 \ 0 )</td>
<td>( 0 \ 0 \ 0 \ 1 )</td>
<td>( 0 \ 0 \ 0 \ 1 )</td>
</tr>
</tbody>
</table>

Since the resulting truth values are the same, the two expressions are logically equivalent and we have shown that \( \sim (p \lor q) \equiv \sim p \land \sim q \).

### Example 4

Negate the compound statement “The cup is blue and the plate is green.”

**Solution**

Although a correct solution could be

It is not true that “the cup is blue and the plate is green.”

we can use De Morgan’s law that \( \sim (p \land q) \equiv \sim p \lor \sim q \) to obtain a better sentence for the negation of this conjunction:

The cup is not blue or the plate is not green.

### Practice Problem 4

Use De Morgan’s laws to negate the compound statement “The spoon is silver or the fork is not gold.”

➤ Solution on page 17
Are Both “And” and “Or” Necessary?

Do we really need both words “and” and “or,” or can one be expressed using the other? The answer is that either one can be expressed using the other together with negations, as the following equivalences show.

\[
p \land q = (\sim p \lor \sim q) \quad \text{From negating both sides of } \sim (p \land q) = \sim p \lor \sim q
\]

\[
p \lor q = (\sim p \land \sim q) \quad \text{From negating both sides of } \sim (p \lor q) = \sim p \land \sim q
\]

The first equivalence shows that it is logically possible to eliminate the word “and” because it can be expressed in terms of “or” and “not.” However, this would mean replacing the statement “he is rich and famous” by the statement “it is not true that he is either not rich or not famous.” Both statements are logically equivalent, but it is clearly convenient to retain both “and” and “or” for simplicity and clarity.

### Section Summary

A truth table is a complete list of the truth values of a statement for every possible combination of the truth and falsity of its components. Two statements are logically equivalent if they have the same truth table, and logically equivalent statements may be used in place of each other by the substitution principle. Using the laws of symbolic logic, a statement can be transformed into other logically equivalent forms.

<table>
<thead>
<tr>
<th>Identity laws</th>
<th>1. ( p \land t = p )</th>
<th>4. ( p \lor t = t )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2. ( p \land f = f )</td>
<td>5. ( p \lor f = p )</td>
</tr>
<tr>
<td></td>
<td>3. ( p \land \sim p = f )</td>
<td>6. ( p \lor \sim p = t )</td>
</tr>
<tr>
<td>Double negation</td>
<td>( \sim (\sim p) = p )</td>
<td></td>
</tr>
<tr>
<td>Idempotent laws</td>
<td>1. ( p \land p = p )</td>
<td>2. ( p \lor p = p )</td>
</tr>
<tr>
<td>Commutative laws</td>
<td>1. ( p \land (q \land r) = (p \land q) \land r )</td>
<td>2. ( p \lor (q \lor r) = (p \lor q) \lor r )</td>
</tr>
<tr>
<td>Associative laws</td>
<td>1. ( p \land (q \lor r) = (p \land q) \lor (p \land r) )</td>
<td>2. ( p \lor (q \land r) = (p \lor q) \land (p \lor r) )</td>
</tr>
<tr>
<td>Distributive laws</td>
<td>1. ( p \land (p \lor q) = p )</td>
<td>2. ( p \lor (p \land q) = p )</td>
</tr>
<tr>
<td>Absorption laws</td>
<td>1. ( \sim (p \land q) = \sim p \lor \sim q )</td>
<td>2. ( \sim (p \lor q) = \sim p \land \sim q )</td>
</tr>
<tr>
<td>De Morgan’s laws</td>
<td>1. ( \sim (p \lor q) = \sim p \land \sim q )</td>
<td>2. ( \sim (p \land q) = \sim p \lor \sim q )</td>
</tr>
</tbody>
</table>
### Solutions to Practice Problems

1. (1) $p \; q \; p \lor \neg q$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & F 
   \end{array} \]  

   (2) $p \; q \; p \lor \neg q$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & F 
   \end{array} \]

   (3) $p \; q \; p \lor \neg q$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & T 
   \end{array} \]  

   (4) $p \; q \; p \lor \neg q$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & F 
   \end{array} \]

2. (1) $p \; q \; p \lor (p \land q)$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & T 
   \end{array} \]  

   (2) $p \; q \; p \lor (p \land q)$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & T 
   \end{array} \]  

   (3) $p \; q \; p \lor (p \land q)$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & T 
   \end{array} \]  

   (4) $p \; q \; p \lor (p \land q)$  
   \[ \begin{array}{ccc} 
   T & T & T \\
   T & F & T \\
   F & T & F \\
   F & F & T 
   \end{array} \]

The column under $\lor$ is the same as the column for $p$, so $p \lor (p \land q) = p$.

3. \[ \begin{array}{c} 
   \neg p \land (p \lor q) = (\neg p \land p) \lor (\neg p \land q) \\
   = (f) \lor (\neg p \land q) \\
   = \neg p \land q \\
   \end{array} \]

   - Distributive law 1
   - Commutative law 1
   - and Identity Law 3
   - Commutative law 2
   - and Identity Law 5

4. “The spoon is not silver and the fork is gold” by using De Morgan’s law that \( \neg (p \lor q) \equiv \neg p \land \neg q \) and the double negation \( \neg (\neg q) \equiv q \).
**L.2 Exercises**

For each compound statement:

a. Find its truth value given that \( p \) is \( T \), \( q \) is \( F \), and \( r \) is \( T \).

b. Check your answer using a graphing calculator.

1. \( \sim p \)
2. \( \sim q \)
3. \( p \land q \)
4. \( p \lor q \)
5. \( \sim p \lor r \)
6. \( q \land \sim r \)
7. \( p \land (q \lor r) \)
8. \( p \lor (q \land r) \)
9. \( \sim p \lor (q \land \sim r) \)
10. \( \sim p \lor (\sim q \land r) \)

For each compound statement:

a. Construct a truth table. Identify any tautologies and contradictions.

b. Check your answer using a graphing calculator.

11. \( p \land \sim q \)
12. \( \sim p \lor q \)
13. \( (\sim p \lor q) \land \sim q \)
14. \( \sim p \lor (p \land \sim q) \)
15. \( (p \lor \sim q) \lor (p \land q) \)
16. \( (\sim p \lor \sim q) \land (p \land q) \)
17. \( \sim (p \lor r) \land (\sim q \land r) \)
18. \( (p \lor q) \lor (\sim q \land r) \)

Simplify each statement by symbolic manipulation using the laws of symbolic logic. Identify any tautologies and contradictions.

19. \( p \land (\sim p \lor q) \)
20. \( p \lor (\sim p \land q) \)
21. \( (p \land q) \lor (\sim p \land q) \)
22. \( (p \land \sim q) \lor (p \land q) \)
23. \( \sim (p \land \sim q) \lor p \)
24. \( p \land \sim (p \lor \sim q) \)

Show that the first absorption law and De Morgan’s laws lead to the second absorption law.

Show that the second absorption law and De Morgan’s laws lead to the first absorption law.

Use De Morgan’s laws to negate each statement.

27. The coffee is strong and the donuts are hot.
28. The tea is weak or the muffins are ready.
29. The wolf isn’t hungry or the sheep are safe.
30. The cat is asleep and the birds aren’t in danger.

Verify each of the following properties of logical equivalences by constructing truth tables for both statements and checking that they are the same.

31. \( p \land t = p \)
32. \( p \lor f = p \)
33. \( p \lor q = q \lor p \)
34. \( p \land q = q \land p \)
35. \( p \land (q \land r) = (p \land q) \land r \)
36. \( p \lor (q \lor r) = (p \lor q) \lor r \)
37. \( p \lor (q \lor r) = (p \lor q) \land (p \lor r) \)
38. \( p \land (p \lor q) = p \)
39. \( \sim (p \land q) = \sim p \lor \sim q \)
40. \( \sim (p \lor q) = \sim p \land \sim q \)

**Explorations and Excursions**

The following problems extend and augment the material presented in the text.

**Exclusive Disjunction**

The exclusive disjunction \( p \lor q \) interprets “\( p \) or \( q \)” to mean “either \( p \) or \( q \) but not both” and is thus defined by the truth table

**Exclusive Disjunction**

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \lor q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

Identify each disjunction as “inclusive” or “exclusive.”

41. You may start dinner with soup or salad.
42. The music bonus is a free tape or CD.
43. It’s cold! Shut the window or the door.
44. See a doctor when you have a broken arm or leg.

For each compound statement:

a. Construct a truth table. Identify any tautologies and contradictions.
Because the disjunction of several statements is true whenever at least one is true, if we have a truth table with several rows ending in T’s, we can build statements using \( \lor \) and \( \land \) that are true only for one row and then take their disjunction to obtain a statement for the table. This statement is the “disjunctive normal form” of any statement that has the given truth table.

Find the disjunctive normal form for each truth table.

51. 52. 53. 54.

Disjunctive Normal Form
How can we find a statement that matches a given truth table? For tables with only one row ending in T and all the other rows ending in F’s, we can use negation and conjunction to build a statement that is T in just that one particular case:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>??</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Use \( \neg \) and \( \land \) to find a statement for each truth table.

55. 56. 57. 58. 59. 60.

Because the disjunction of several statements is true whenever at least one is true, if we have a truth table with several rows ending in T’s, we can build statements using \( \neg \) and \( \land \) that are true only for one row and then take their disjunction to obtain a statement for the table. This statement is the “disjunctive normal form” of any statement that has the given truth table.
L.3 IMPLICATIONS

Application Preview

“If . . ., then . . .” as a Computer Instruction

Most computer programming languages control the sequence of calculations with special statements that select the next operation on the basis of the values that are currently known. The simplest of these is the “if . . . then . . . else . . .” statement shown in the graphing calculator screen on the left below. This program takes an input value (X) supplied by the user and responds “HELLO” if the value is positive (X > 0) and “GOODBYE” otherwise.

```
PROGRAM:IFTHEN
:Input "X? ",X
:If X>0
:Then
:Disp "HELLO"
:Else
:Disp "GOODBYE"
:End
```

```
prgmIFTHEN
X? 3
HELLO
X? -2
GOODBYE
```

Do you see how the program on the left gives the output on the right?

Although closely related to this use of “if . . ., then . . .” to control the flow of a program, the logical implication $p \rightarrow q$ (which is read “if $p$, then $q$”) discussed in this section is a connective that forms a new statement from the statements $p$ and $q$.

Introduction

The application of the scientific method to analyze cause-and-effect mechanisms has been a spectacular success during the last few centuries, both in creating new knowledge and in destroying useless superstition. Although seemingly similar in intent, the logical “if . . ., then . . .” statements discussed in this section do not suppose any underlying cause-and-effect relationship. We view implications simply as logical connectives.
Implications

The implication \( p \rightarrow q \) links an antecedent \( p \) to a consequent \( q \):

\[
p \rightarrow q \quad \text{means} \quad \text{if } p, \text{ then } q
\]

but makes no claim as to why such a linkage exists. Both of the statements “if you are wicked, then you will be punished” and “if the sky is clear, then the teacup is broken” are equally acceptable as conditional statements, even though the first seems intuitively more reasonable.

How does the truth or falsity of an implication depend on the truth and falsity of its antecedent and consequent? The implication “if you are wicked, then you will be punished” cannot be true if you are wicked and yet go unpunished. That is, the implication “if \( p \), then \( q \)” must be false if \( p \) is true and yet \( q \) is false, and in all other cases we will say that it is true. This leads to the truth values shown below, which serve as a definition of the implication \( p \rightarrow q \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

For instance, consider the conditional statement

If I wear my lucky hat, then the Yankees win. \hspace{1cm} \text{If } p, \text{ then } q

\hspace{1cm} \text{Antecedent} \hspace{1cm} \text{Consequent}

The only way this could be false is when “I wear my lucky hat” yet “the Yankees lose” (that is, when \( p \) is \( T \) and \( q \) is \( F \)). Notice that if “I don’t wear my lucky hat,” we don’t know how the Yankees might do, and so we accept the conditional statement as true by default (even though it is not applicable), as shown in the last two rows of the preceding truth table.*

Since \( p \rightarrow q \) is false only when \( p \) is true and \( q \) is false, and the same is true for \( \sim p \lor q \), we have an equivalent way to express implications:

\[
p \rightarrow q \equiv \sim p \lor q
\]

* This somewhat arbitrary definition of the implication as true in all cases other than “true” \( \rightarrow \) “false” can also be justified on the basis that experiments in science are designed to disprove theories, so an implication should be considered false only under the most restrictive conditions.
From Practice Problem 1a, we have that the negation of an implication can be expressed as follows.

### Conditional Equivalence

\[ p \rightarrow q \equiv \sim p \lor q \]

An implication can be expressed as an “or” with the antecedent negated.

Thus “if I wear my lucky hat, then the Yankees will win” can also be expressed “I don’t wear my lucky hat or the Yankees win”—both statements have the same logical meaning. When performing symbolic calculations, we will often find it helpful to replace the implication \( p \rightarrow q \) by the logically equivalent \( \sim p \lor q \).

### Practice Problem 1

1. Use symbolic manipulation to show that \( \sim (p \rightarrow q) \equiv p \land \sim q \)

2. Use the equivalence in part (a) to negate the conditional statement “if you are at least eighteen, then you can vote.”  

From Practice Problem 1a, we have that the negation of an implication can be expressed as follows.

### Negating an Implication

\[ \sim (p \rightarrow q) \equiv p \land \sim q \]

The negation of an implication can be expressed as an “and” with the consequent negated.

### Spreadsheet Exploration

The spreadsheet below shows the truth table for \( \sim p \lor q \) using the spreadsheet functions \( \text{NOT}(\ldots) \) and \( \text{OR}(\ldots, \ldots) \) together with the Boolean values \( \text{TRUE} \) and \( \text{FALSE} \).

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>TRUE</td>
<td>TRUE</td>
<td>TRUE</td>
</tr>
<tr>
<td>2</td>
<td>TRUE</td>
<td>FALSE</td>
<td>FALSE</td>
</tr>
<tr>
<td>3</td>
<td>FALSE</td>
<td>TRUE</td>
<td>TRUE</td>
</tr>
<tr>
<td>4</td>
<td>FALSE</td>
<td>FALSE</td>
<td>TRUE</td>
</tr>
</tbody>
</table>

Because the truth tables for \( \sim p \lor q \) and \( p \rightarrow q \) are the same, these statements are logically equivalent. Is \( p \rightarrow q \) also equivalent to the spreadsheet expression \( \text{NOT}(\text{AND}(A1, \text{NOT}(B1))) \)?
There are several other phrases besides “if \( p \), then \( q \)” that are commonly used to express the implication \( p \rightarrow q \).

### Alternative Expressions for “If \( p \), then \( q \)”

<table>
<thead>
<tr>
<th>Form</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q ) if ( p )</td>
<td>The Yankees win if I wear my lucky hat.</td>
</tr>
<tr>
<td>( q ) provided ( p )</td>
<td>The Yankees win provided I wear my lucky hat.</td>
</tr>
<tr>
<td>( q ) when ( p )</td>
<td>The Yankees win when I wear my lucky hat.</td>
</tr>
<tr>
<td>( p ) implies ( q )</td>
<td>Wearing my lucky hat implies that the Yankees win.</td>
</tr>
<tr>
<td>( p ) only if ( q )</td>
<td>I am wearing my lucky hat only if the Yankees win.</td>
</tr>
<tr>
<td>( p ) is sufficient for ( q )</td>
<td>Wearing my lucky hat is sufficient for the Yankees to win.</td>
</tr>
<tr>
<td>( q ) is necessary for ( p )</td>
<td>That the Yankees win is necessary for me to be wearing my hat.</td>
</tr>
</tbody>
</table>

Notice that each of these formulations is wrong only when “I wear my lucky hat” is true and “the Yankees win” is false.

### Practice Problem

Identify the antecedent and the consequent in each conditional statement.

a. You will pass this course if you study well.

b. I go swimming only if it is hot.

➤ Solution on page 26

### Converse, Inverse, and Contrapositive

For a given conditional statement, switching the antecedent with the consequent or negating them produces three related conditional statements:

<table>
<thead>
<tr>
<th>Direct statement</th>
<th>( p \rightarrow q )</th>
<th>Original</th>
</tr>
</thead>
<tbody>
<tr>
<td>Converse</td>
<td>( q \rightarrow p )</td>
<td>Switched</td>
</tr>
<tr>
<td>Inverse</td>
<td>( \sim p \rightarrow \sim q )</td>
<td>Negated</td>
</tr>
<tr>
<td>Contrapositive</td>
<td>( \sim q \rightarrow \sim p )</td>
<td>Switched and negated</td>
</tr>
</tbody>
</table>

Are any of these equivalent to each other? Using the conditional equivalence and the commutative and double negation laws, we have

\[
p \rightarrow q \equiv \sim p \lor q \equiv q \lor \sim p \equiv \sim (\sim q) \lor \sim p \equiv \sim q \rightarrow \sim p
\]
Thus we have established the following logical equivalences:

<table>
<thead>
<tr>
<th>More Conditional Equivalences</th>
<th>The direct is equivalent to the contrapositive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \to q \equiv q \to p )</td>
<td>The converse is equivalent to the inverse</td>
</tr>
<tr>
<td>( q \to p \equiv \sim p \to \sim q )</td>
<td></td>
</tr>
</tbody>
</table>

Note that \( p \to q \) is not equivalent to \( q \to p \). (This mistake is often made: “If you work hard, you will get rich. He is rich, so he must have worked hard.” This is wrong: Perhaps he received an inheritance or won the lottery.)

**EXAMPLE FINDING THE CONVERSE, INVERSE, AND CONTRAPOSITIVE**

State the converse, inverse, and contrapositive of the conditional statement

If Harold drives a Ford, then Tom has a boat. \( p \to q \)

**Solution**

The antecedent \( p \) is “Harold drives a Ford,” and the consequent \( q \) is “Tom has a boat.” Therefore,

- Converse: If Tom has a boat, then Harold drives a Ford. \( q \to p \)
- Inverse: If Harold doesn’t drive a Ford, then Tom doesn’t have a boat. \( \sim p \to \sim q \)
- Contrapositive: If Tom doesn’t have a boat, then Harold doesn’t drive a Ford. \( \sim q \to \sim p \)

**Practice Problem**

Which of the answers to Example 1 is logically equivalent to the direct statement?  ➤ Solution on page 26
**Biconditional**

The biconditional, written $p \leftrightarrow q$ and expressed “$p$ if and only if $q$,” is the double implication that $p \rightarrow q$ and $q \rightarrow p$:

$$p \leftrightarrow q = (p \rightarrow q) \land (q \rightarrow p)$$

That is, $p \leftrightarrow q$ means that $p$ and $q$ are either both true or both false:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

What is the difference between the biconditional $p \leftrightarrow q$ and the equivalence $p = q$? The biconditional is a statement about $p$ and $q$ that can be true or false, but “$p$ is equivalent to $q$” exactly when $p \leftrightarrow q$ is true. To put this another way, the biconditional symbol $\leftrightarrow$ may be thought of as $\equiv$ with a truth value $T$ or $F$ saying whether it holds or not. The biconditional is frequently used to define one idea in terms of another: “A positive integer $n$ is even if and only if $n = 2 \cdot k$ for some positive integer $k$.”

**Biconditional and the Order of Operations**

We evaluate biconditionals last of all of the logical connectives, so the order of operations is $\neg$, $\land$, $\lor$, $\rightarrow$, and $\leftrightarrow$. For example, $p \leftrightarrow q \land r$ means $p \leftrightarrow (q \land r)$.

---

**L.3 Section Summary**

The implication $p \rightarrow q$ links the antecedent $p$ to the consequent $q$ and is logically equivalent to the disjunction $\neg p \lor q$. The negation $\neg (p \rightarrow q)$ is logically equivalent to $p \land \neg q$. The four forms of the conditional implication are:

- Direct statement: $p \rightarrow q$
- Converse: $q \rightarrow p$
- Inverse: $\neg p \rightarrow \neg q$
- Contrapositive: $\neg q \rightarrow \neg p$

The direct and contrapositive are equivalent, as are the converse and inverse.

The biconditional $p \leftrightarrow q$ is the double implication that $p \rightarrow q$ and $q \rightarrow p$. 
Solutions to Practice Problems

1. a. 
   \((p \rightarrow q) = \sim (\sim p \lor q)\)  
   Replacing \(p \rightarrow q\) by the equivalent \(\sim p \lor q\).

   Using De Morgan’s law.

   \(p \land \sim q\)

   b. You are at least eighteen and you cannot vote.

2. a. If you study well, then you will pass this course.

   Antecedent Consequent

   b. If I go swimming, then it is hot.

   Antecedent Consequent

3. The contrapositive, “If Tom doesn’t have a boat, then Harold doesn’t drive a Ford,” is logically equivalent to the direct statement.

L.3 Exercises

Write in sentence form the (a) converse, (b) inverse, (c) contrapositive, and (d) negation of the conditional statement. [Hint: It may be helpful to rewrite the statement in “if . . . , then . . . ” form to identify the antecedent and the consequent.]

1. Justice will be done provided the jury is wise.

2. If mice were men, then cowards could be kings.

3. The tooth will be saved when the dentist is skillful.

4. A graceful melody implies a beautiful song.

5. A full bird feeder is sufficient for there to be many birds in the yard.

6. Frequent oil changes are necessary for your motor to last.

Construct a truth table for each statement. Identify any tautologies and contradictions.

Simplify each statement by replacing the conditional \(a \rightarrow c\) by the equivalent \(\sim a \lor c\) and then using the laws of symbolic logic from Section 2 (page 16). Identify any tautologies and contradictions.

17. \((p \rightarrow q) \rightarrow p\)

18. \((p \rightarrow q) \rightarrow q\)

19. \((p \land q) \rightarrow (p \lor q)\)

20. \((p \rightarrow q) \land (p \land \sim q)\)

Explorations and Excursions

The following problems extend and augment the material presented in the text.

More About the Biconditional

Verify each logical equivalence by showing that the truth tables of both sides are the same.

21. \(p \leftrightarrow q = (p \rightarrow q) \land (q \rightarrow p)\)

22. \(p \leftrightarrow q = (p \land q) \lor (\sim p \land \sim q)\)

(The disjunctive normal form for \(p \leftrightarrow q\))

23. \(\sim (p \leftrightarrow q) = \sim p \leftrightarrow \sim q\)

24. \(p \leftrightarrow q = (p \lor q) \rightarrow (p \land q)\)

25. \(p \leftrightarrow q = (p \rightarrow q) \leftrightarrow (q \rightarrow p)\)

26. \((p \leftrightarrow q) \land (q \leftrightarrow r) = [(p \land q) \leftrightarrow q] \land (p \leftrightarrow r)\)
Use symbolic manipulation together with the results of Exercises 21–26 to establish each logical equivalence.

27. \( p \leftrightarrow q = q \leftrightarrow p \)  
28. \( p \leftrightarrow q = \sim p \leftrightarrow \sim q \)  
29. \( \sim (p \leftrightarrow q) = p \leftrightarrow \sim q \)  
30. \( (p \land q) \leftrightarrow p = p \rightarrow q \)

**Consistent and Contrary Statements**

Two statements are *consistent* if they can both be true about the same object: “the car is fast” and “the car is red” are consistent statements. Two statements are *contrary* if they cannot both be true about the same object: “the car is a Jaguar” and “the car is a Porsche” are contrary statements.

**Application Preview**

**Fuzzy Logic and Artificial Intelligence**

Although people have made many attempts over the centuries to create thinking machines that might be said to possess “artificial intelligence,” many now believe that modern computers with their high speeds and vast memories might succeed in the near future. Unfortunately, statements about the real world are often hard to classify as definitely true or false, but many can be rephrased as “probably true” or “possibly false” with a *probability* or *degree of truth* indicated by a number between 0 and 1, where 1 means “absolutely true” and 0 means “absolutely false.” Such truth values lead to a “fuzzy” version of logic in which the old \( F \)'s and \( T \)'s are replaced by probabilities between 0 and 1. Let us write \( P \) for the probability that the statement \( p \) is true and \( Q \) for the probability that the statement \( q \) is true. Clearly, \( \sim p \) is then true with the “complementary” probability \( 1 - P \). Because the “fuzzy truth” of a conjunction can be no better than the truths of its parts, we have

\[
p \land q \text{ is true with probability } \min(P, Q).
\]

\( \min(P, Q) \) means the minimum of the numbers \( P \) and \( Q \).
Similarly, a disjunction is as true as the better of its parts, so

\[ p \lor q \] is true with probability \( \max(P, Q) \). \( \text{max}(P, Q) \) means the maximum of the numbers \( P \) and \( Q \).

Because \( p \rightarrow q = \neg p \lor q \), we also have

\[ p \rightarrow q \] is true with probability \( \max(1 - P, Q) \).

For example, beginning with the fuzzy truth probabilities \( P = 0.8 \) and \( Q = 0.7 \), we obtain the following “fuzzy truth table” using the above formulas:

<table>
<thead>
<tr>
<th></th>
<th>q</th>
<th>( p \land q )</th>
<th>( p \lor q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.7</td>
<td>0.7</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3</td>
<td>0.3</td>
<td>0.8</td>
<td>0.3</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7</td>
<td>0.2</td>
<td>0.7</td>
<td>0.8</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Because large values approximate \( T \) and small values approximate \( F \), these values should remind you of our previous truth tables using \( T \)'s and \( F \)'s. Fuzzy logic has had many applications, particularly with “expert systems” that concentrate on a particular area of practical knowledge (such as controlling factory assembly lines or suggesting medical diagnoses). Can you imagine how to use fuzzy logic to develop a “fuzzy set” theory?

**Introduction**

Although one sense of the word “argument” is “quarrel,” in this section we take it to mean simply “discourse meant to persuade.” By a “valid argument” we do not demand that the conclusion be a true statement but only that a correct path of reasoning leads from the assumptions to the conclusion. Naturally, a valid argument with true assumptions results in a true conclusion. But a logically valid argument with one or more false assumptions may result in a false conclusion, so the validity of the argument does not guarantee the truth of its conclusion.

**Valid Argument**

An *argument* is a claim that one or more premises, when taken together, justify a conclusion. We do not ask whether the premises or the conclu-
We will write an argument by listing the premises, drawing a horizontal line, and then stating the conclusion. For example:

If Mary is awake, then the sun is up. Premise $P_1$
Jim doesn’t have hiccups or Mary is awake. Premise $P_2$
If the birds are silent, then the sun isn’t up. Premise $P_3$
Jim has hiccups. Premise $P_4$

Therefore: The birds are singing. Conclusion $Q$

One way to determine whether this argument is valid would be to calculate the truth table for $P_1 \land P_2 \land P_3 \land P_4 \rightarrow Q$ and check whether the final column is all T’s. However, this method would be tedious, and it has the unpleasant feature that even a minor variation of the argument would require a whole new truth table. Instead, we will develop a basic collection of valid arguments that can be used to check the validity of other arguments. Then we will return to this particular argument.

**The Simplest Valid Arguments**

As is usual in mathematics, the simplest ideas will be the most fundamental and useful for generalization. Consider first the following argument with just one premise implying the conclusion:

It is raining. Premise $P$
Therefore: It is raining or snowing. Conclusion $Q$

In symbols, $p \rightarrow (p \lor q)$

We could show that this is a valid argument by constructing its truth table and observing that the final column is all T’s (see Exercise 21). Instead, we use symbolic manipulation beginning with the conditional
equivalence from page 22 to show that the argument is equivalent to a tautology $t$:

$$p \rightarrow (p \lor q) \equiv \sim p \lor (p \lor q) = (\sim p \lor p) \lor q = t \lor q = t$$

Since the argument is a tautology, it is valid.

<table>
<thead>
<tr>
<th>Extension</th>
</tr>
</thead>
<tbody>
<tr>
<td>The argument $p \rightarrow (p \lor q)$ is valid.</td>
</tr>
</tbody>
</table>

Notice that reasoning by extension decreases the quality of the information: It trades the stronger premise “it is raining” for the weaker conclusion “it is raining or snowing.”

Now consider the argument

<table>
<thead>
<tr>
<th>It is rainy and dark.</th>
<th>Premise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Therefore: It is rainy.</td>
<td>Conclusion</td>
</tr>
</tbody>
</table>

In symbols, $(p \land q) \rightarrow p$

Symbolic manipulation (including De Morgan’s laws) shows that it is equivalent to a tautology and thus is valid:

$$(p \land q) \rightarrow p = (\sim (p \land q) \lor p = (\sim p \lor \sim q) \lor p$$
$$= (\sim q \lor \sim p) \lor p = \sim q \lor (\sim p \lor p) = \sim q \lor t = t$$

<table>
<thead>
<tr>
<th>Simplification</th>
</tr>
</thead>
<tbody>
<tr>
<td>The argument $(p \land q) \rightarrow p$ is valid.</td>
</tr>
</tbody>
</table>

Reasoning by simplification decreases the amount of information, trading the stronger premise “it is rainy and dark” for the weaker conclusion “it is rainy.”

**Syllogisms**

A *syllogism* (from the Greek word *sullogismo* meaning “a reckoning together”) is an argument with two premises, which are sometimes called the “major” and the “minor” premises. Like other forms of arguments, a syllogism is either valid or a fallacy.
DETERMINING WHETHER A SYLLOGISM IS VALID

Is the following syllogism valid?

If it quacks like a duck, then it is a duck. Premise $P_1$
It quacks like a duck. Premise $P_2$
Therefore: It is a duck. Conclusion $Q$

Solution

Let $p$ be “it quacks like a duck,” and let $q$ be “it is a duck.” Then this syllogism has the form

$$p \rightarrow q$$

$$\begin{array}{c}
p \\
q
\end{array}$$

Instead of reducing this argument to a tautology, we find an equivalent expression for the premises.

$$(p \rightarrow q) \land p = (\sim p \lor q) \land p = (\sim p \land p) \lor (p \land q) = f \lor (p \land q) = (p \land q).$$

By the substitution principle (page 10), this syllogism is equivalent to the “simplification” argument:

$$[(p \rightarrow q) \land p] \rightarrow q = (p \land q) \rightarrow q$$

Since simplification is valid, so is the argument $(p \rightarrow q) \land p \rightarrow q$. “It is a duck” does follow from the premises.

The form of syllogism shown to be valid in Example 1 is called *modus ponens* and is sometimes called the “law of detachment.”

**Modus Ponens**

| The argument | $p \rightarrow q$ | $\begin{array}{c}
p \\
q
\end{array}$ |
|--------------|-------------------|----------------|
| An implication and its antecedent imply its consequent | $p \rightarrow q$ | $\begin{array}{c}
p \\
q
\end{array}$ |

* From the Latin words *modus* meaning “method” and *ponens* from the verb “to put,” suggesting that this method puts in the antecedent so the consequent can be concluded.
Replacing \( p \) and \( q \) by \( \sim q \) and \( \sim p \), respectively, this valid syllogism becomes

\[
\begin{align*}
\sim q & \rightarrow \sim p \\
\sim q & \text{ or equivalently } \sim p
\end{align*}
\]

This variation of modus ponens is known as *modus tollens* and is also called the “law of contraposition” or “indirect reasoning.”

**Modus Tollens**

<table>
<thead>
<tr>
<th>The argument</th>
<th>( p \rightarrow q ) is valid.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sim q )</td>
<td>( \sim q ) is valid.</td>
</tr>
<tr>
<td>( \sim p )</td>
<td>( \sim p ) is valid.</td>
</tr>
</tbody>
</table>

An implication and the negation of its consequent imply the negation of its antecedent.

**Invalid Arguments**

Modus ponens and modus tollens combine \( p \rightarrow q \) with either \( p \) or \( \sim q \). The other two choices give *invalid* arguments.

<table>
<thead>
<tr>
<th>The argument</th>
<th>( p \rightarrow q ) is <em>not</em> valid.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>( q ) is <em>not</em> valid.</td>
</tr>
<tr>
<td>( p )</td>
<td>( p ) is <em>not</em> valid.</td>
</tr>
</tbody>
</table>

An implication and its consequent do *not* imply its antecedent.

This is a fallacy because the premises \( p \rightarrow q \) and \( q \) are equivalent to just \( q \) by absorption. Thus the argument reduces to \( q \rightarrow p \), which is *not* a tautology. This last implication \( q \rightarrow p \) is *false* when \( q \) is true and \( p \) is false, showing that the fallacy of the converse fails to be logically correct when \( p \) is false and \( q \) is true. Similarly,

*Tollens* is from the Latin verb meaning “to take away,” suggesting that the consequent is taken away (by being negated) so that the opposite of the antecedent must be concluded.
Using double negation, we can recognize $p \land q$ as $\neg \neg p \land \neg q$, and this observation establishes the following variation on modus ponens as a valid argument.

<table>
<thead>
<tr>
<th>Fallacy of the Inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \rightarrow q$</td>
</tr>
<tr>
<td>$\neg p$</td>
</tr>
<tr>
<td>$\neg q$</td>
</tr>
<tr>
<td>is not valid.</td>
</tr>
</tbody>
</table>

An implication and the negation of its antecedent do not imply the negation of its consequent.

These invalid arguments show that you must be careful about reasoning “backwards” with an implication: The first claims that the consequent implies the antecedent, and the second claims that the antecedent being false implies that the consequent is false. Both are invalid forms of reasoning.

**EXAMPLE 2**

**DETERMINING THE VALIDITY OF A SYLLOGISM**

Is the following syllogism valid?

If the eggs are fried, then I need toast.  \hspace{1cm} \text{Premise } P_1
The eggs aren’t fried. \hspace{1cm} \text{Premise } P_2

Therefore: I don’t need toast. \hspace{1cm} \text{Conclusion } Q

**Solution**

Let $p$ be “the eggs are fried,” and let $q$ be “I need toast.” This syllogism has the form of the fallacy of the inverse:

$p \rightarrow q$

$\neg p$

$\neg q$

This syllogism is not valid.

**Practice Problem 1**

For what truth values of $p$ and $q$ is the “fallacy of the inverse” syllogism false?  \hspace{1cm} \text{Solution on page 38}

Using double negation, we can recognize $p \lor q$ as $\neg (\neg p) \lor q = \neg p \rightarrow q$, and this observation establishes the following variation on modus ponens as a valid argument.
Our final syllogism links two implications together to form a third. It is called the *hypothetical syllogism* since both the premises and the conclusion are of the hypothetical “if . . . , then . . . ” form. Unlike modus tollens and the disjunctive syllogism, this is not another variant of modus ponens, and it is not reducible to the simplification argument. We defer the proof of its validity until the end of this section.

### Disjunctive Syllogism

| The argument | \[ p \lor q \] | An “or” with the negation of one part implies the other part |

### Hypothetical Syllogism

| The argument | \[ p \rightarrow q \] | Two implications with the consequent of the first being the antecedent of the second may be combined into one |

### Analyzing Longer Arguments

We now use syllogisms to *simplify* a longer argument, reducing it in stages until we arrive at a form whose validity we can recognize. We return to the first argument of this section.

### ANALYZING A LONGER ARGUMENT

Is the following argument valid?

- If Mary is awake, then the sun is up. \( P_1 \)
- Jim doesn’t have hiccups or Mary is awake. \( P_2 \)
- If the birds are silent, then the sun isn’t up. \( P_3 \)
- Jim has hiccups. \( P_4 \)

Therefore: The birds are singing. \( Q \)
Solution

We first assign names to each statement:

\( m \) is “Mary is awake”
\( s \) is “the sun is up”  \( \Rightarrow \) so \( \sim s \) is “the sun isn’t up”
\( b \) is “the birds are singing”  \( \Rightarrow \) so \( \sim b \) is “the birds are silent”
\( j \) is “Jim has hiccups”  \( \Rightarrow \) so \( \sim j \) is “Jim doesn’t have hiccups”

This argument then takes the symbolic form

\[
\begin{align*}
m & \rightarrow s \\
\sim j \lor m & \\
\sim b & \rightarrow \sim s \\
j & \rightarrow \sim s \\
b & \rightarrow \sim s \\
Q &
\end{align*}
\]

We simplify the premises in stages, beginning with the simplest premise, \( j \), and the other premise involving this letter, \( \sim j \lor m \), which can be combined using the disjunctive syllogism [since \( \sim (\sim j) = j \)]:

\[
\begin{align*}
\sim j \lor m & \\
j & \quad \text{which reduces the argument to}
\begin{align*}
m & \rightarrow s \\
\sim b & \rightarrow \sim s \\
b & \rightarrow \sim s \\
Q &
\end{align*}
\end{align*}
\]

In the simplified argument, \( m \) and \( m \rightarrow s \) can be combined by modus ponens:

\[
\begin{align*}
m & \rightarrow s \\
m & \\
s & \quad \text{reducing the argument further to}
\begin{align*}
s & \rightarrow \sim b \\
\sim b & \rightarrow \sim s \\
b & \rightarrow \sim s \\
Q &
\end{align*}
\end{align*}
\]

In this last argument, replacing \( \sim b \rightarrow \sim s \) by its equivalent \( s \rightarrow b \) (using the substitution principle from page 10) and switching the order of the premises, we see that the argument reduces to modus tollens:

\[
\begin{align*}
s & \rightarrow b \\
s & \\
b & \quad \text{reducing the argument further to}
\begin{align*}
s & \\
\sim b & \rightarrow \sim s \\
b & \rightarrow \sim s \\
Q &
\end{align*}
\end{align*}
\]
Thus the original argument is valid because it can be reduced to a valid argument.

Practice Problem 2
Give a second solution of Example 3 by first replacing $P_2$ with $j \rightarrow m$ and $P_3$ with $s \rightarrow b$ (justify these substitutions!), and then showing that the resulting argument is valid. ➤ Solution on page 38

Deducing Valid Conclusions
Besides verifying that an argument is valid, symbolic logic also enables us to deduce consequences from lists of assumptions, as shown in the following example.

EXAMPLE FINDING A VALID CONCLUSION
Supply a valid conclusion for the following premises.

If it is Saturday, then the stock exchange is closed. Premise $P_1$
The stock exchange is open or my loan payment is due. Premise $P_2$
My loan payment isn’t due. Premise $P_3$

Solution
We first assign names to each statement used in these premises:

$a$ is “it is Saturday”
$b$ is “the stock exchange is open”
$c$ is “my loan payment is due”

so $\sim a$ is “it is not Saturday”
so $\sim b$ is “the stock exchange is closed”
so $\sim c$ is “my loan payment isn’t due”

Then these premises take the symbolic form

$$a \rightarrow \sim b \quad P_1$$
$$b \lor c \quad P_2$$
$$\sim c \quad P_3$$
By the disjunctive syllogism, \( b \lor c \) and \( \sim c \) imply \( b \), and then \( a \rightarrow \sim b \) and \( b \) imply \( \sim a \) by modus tollens.

It is not Saturday.

is a valid conclusion from the given premises.

**Proof That the Hypothetical Syllogism Is Valid**

The following symbolic calculation establishes the validity of the hypothetical syllogism.

\[
[(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r)
\]

Hypothetical syllogism

\[= \sim [(p \rightarrow q) \land (q \rightarrow r)] \lor (p \rightarrow r)\]

Conditional equivalence

\[= [\sim (p \rightarrow q) \lor \sim (q \rightarrow r)] \lor (p \rightarrow r)\]

De Morgan’s laws

\[= (p \land \sim q) \lor (q \land \sim r) \lor (\sim p \lor r)\]

Negation of conditionals

\[= [(p \land \sim q) \lor \sim p] \lor [(q \land \sim r) \lor r]\]

Commutative and associative laws

\[= [\sim p \lor \sim q \lor q \lor r]\]

Distributive and commutative laws

\[= (\sim p \lor r) \lor (\sim q \lor q)\]

Identity and associative laws

\[= (p \rightarrow r) \lor t = t\]

Identity law 4

Since the hypothetical syllogism is logically equivalent to a tautology, it is a valid argument. Notice that the calculation removes the tautologies \( \sim p \lor p \) and \( \sim r \lor r \) to reduce it to the disjunction of the conclusion \( p \rightarrow r \) and the tautology \( \sim q \lor q \) of the linking statement \( q \).

**Section Summary**

The argument that the premises \( P_1, P_2, \ldots, P_n \) imply the conclusion \( Q \) is valid if and only if \( P_1 \land P_2 \land \ldots \land P_n \rightarrow Q \) is a tautology. The following arguments are valid:
Valid Arguments

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extension</td>
<td>( p \rightarrow (p \lor q) )</td>
</tr>
<tr>
<td>Simplification</td>
<td>((p \land q) \rightarrow p)</td>
</tr>
<tr>
<td>Modus ponens</td>
<td>[(p \rightarrow q) \land p \rightarrow q]</td>
</tr>
<tr>
<td>Modus tollens</td>
<td>[(p \rightarrow q) \land \sim q \rightarrow \sim p]</td>
</tr>
<tr>
<td>Disjunctive syllogism</td>
<td>[(p \lor q) \land \sim q \rightarrow q]</td>
</tr>
<tr>
<td>Hypothetical syllogism</td>
<td>[(p \rightarrow q) \land (q \rightarrow r) \rightarrow (p \rightarrow r)]</td>
</tr>
</tbody>
</table>

The following arguments are not valid:

<table>
<thead>
<tr>
<th>Name</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fallacy of the converse</td>
<td>((p \rightarrow q) \land q \rightarrow p)</td>
</tr>
<tr>
<td>Fallacy of the inverse</td>
<td>[(p \rightarrow q) \land \sim q \rightarrow \sim q]</td>
</tr>
</tbody>
</table>

A longer argument can often be simplified by replacing several premises with their conclusion, and a valid conclusion can similarly be found from a list of premises.

Solutions to Practice Problems

1. Since \((p \rightarrow q) \land \sim p = (\sim p \lor q) \land \sim p = \sim p\) by the absorption laws, this syllogism is equivalent to \(~ p \rightarrow \sim q\), which is false when \(p\) is false (so that \(\sim p\) is true) and \(q\) is true (so that \(\sim q\) is false).

2. Since \(\sim j \lor m = j \rightarrow m\) and \(\sim b \rightarrow \sim s = s \rightarrow b\), the argument becomes

\[\begin{align*}
   m & \rightarrow s & \text{ } P_1 \\
   j & \rightarrow m & \text{ } P_2 \\
   s & \rightarrow b & \text{ } P_3 \\
   j & & \text{ } P_4 \\
   b & & \text{ } Q
\end{align*}\]

Changing the order, this is the same as

\[ (j \rightarrow m) \land (m \rightarrow s) \land (s \rightarrow b) \land j \rightarrow b \]

Implies \((j \rightarrow b\) by the hypothetical syllogism

Since \([i \rightarrow b] \land j] \rightarrow b\) is valid by modus ponens, the argument of Example 3 is valid.
L.4 Exercises

Identify the type of each syllogism and whether it is "valid" or a "fallacy."

1. If you like chocolate, you’ll love this cake.
   You like chocolate.
   You’ll love this cake.

2. Fred plays third base if Bob pitches.
   Fred plays shortstop.
   Bob doesn’t pitch.

3. He’s either studying or watching TV.
   He’s not watching TV.
   He’s studying.

4. The dog barks if there is a knock at the door.
   The dog barks.
   There is a knock at the door.

5. You’ll burn your tongue if the soup is hot.
   If you burn your tongue, you’ll be in a bad mood.
   You’ll be in a bad mood if the soup is hot.

6. If the book is thick, it must be difficult.
   I can’t understand it if it is difficult.
   If the book is thick, I can’t understand it.

7. If it’s hot, we’ll go to the beach.
   We go to the beach.
   It’s hot.

8. The mystery will be solved if the detective is clever.
   The detective is not clever.
   The mystery will not be solved.

9. If it is snowing, the train will be late.
   The train is not late or dinner is ruined.
   It is snowing.
   Dinner is ruined.

10. If it is raining, the road is muddy.
    The road is not muddy or the car is stuck.
    The car is stuck.
    It is raining.

11. If many are sick, the hospitals will be crowded.
    If the disease is contagious, many will be sick.
    The hospitals are not crowded.
    The disease is not contagious.

12. If I can’t drive my car, I can’t get to work.
    I can’t drive my car if gas prices go up.
    I can get to work.
    Gas prices did not go up.

13. I’ll get my term paper done if I stay up all night and the computer doesn’t break.
    My term paper isn’t done.
    I didn’t stay up all night and the computer broke.

14. If the batter concentrates, it will be a triple play or a home run.
    It wasn’t a triple play and it wasn’t a home run.
    The batter didn’t concentrate.

15. I can graduate and get a job if I work hard.
    I can’t graduate or I am a success or I can’t get a job.
    I am not a success.
    I don’t work hard.

16. If the stock market is up, then the investors are confident.
    The investors are not confident or the investors buy new stock offerings.
    The investors are taking risks if they buy new stock offerings.
    The stock market is up.
    The investors are taking risks.

Supply a valid conclusion for each list of premises.

17. If I have chicken for dinner, then I have ice cream for dessert.
    I don’t have ice cream for dessert or I go to bed early.

18. The flower is yellow or the leaf is green.
    The spider is poisonous if the leaf is green.
    The flower is red.
19. I go swimming and I have fun if I go to the beach.
   I do not go swimming or I do not have fun or I am sunburned.
   I go to the beach.

20. If I go to the mountains, I hike and canoe.
   I do not hike or I do not canoe or I camp.
   I do not camp.

Explorations and Excursions

The following problems extend and augment the material presented in the text.

Truth Tables and Valid Arguments

Although symbolic manipulations give simple and direct proofs of the validity of the arguments presented in this section, it is also a worthwhile undertaking to establish the same results by truth tables.

Construct a complete truth table for each argument and verify that it is a tautology and thus is valid.

21. \( p \to (p \lor q) \) (extension)

22. \( (p \land q) \to p \) (simplification)

23. \( [(p \to q) \land p] \to q \) (modus ponens)

24. \( [(p \to q) \land \neg q] \to \neg p \) (modus tollens)

25. \( [(p \lor q) \land \neg p] \to q \) (disjunctive syllogism)

26. \( [(p \to q) \land (q \to r)] \to (p \to r) \) (hypothetical syllogism)

27. \( [(p \to \neg q) \land (q \lor r) \land (\neg r)] \to \neg p \) (See Example 4 on pages 36–37.)

28. \( [(p \to q) \land (\neg r \lor p) \land (\neg s \to \neg q) \land r] \to s \) (See Example 3 on pages 34–36.)

Construct a complete truth table for each argument and verify that it is not a tautology and thus is a fallacy.

29. \( [(p \to q) \land q] \to p \) (fallacy of the converse)

30. \( [(p \to q) \land \neg p] \to \neg q \) (fallacy of the inverse)

Application Preview

The Dog Walking Ordinance*

The following transcript of a Borough Council meeting in England illustrates the difficulties of expressing a simple idea in precise and unambiguous language.

From the Minutes of a Borough Council Meeting:

Councillor Trafford took exception to the proposed notice at the entrance of South Park: “No dogs must be brought to this Park except on a lead.” He pointed out that this order would not prevent an

owner from releasing his pets, or pet, from a lead when once safely inside the Park.

The Chairman (Colonel Vine): What alternative wording would you propose, Councillor?

Councillor Trafford: “Dogs are not allowed in this Park without leads.”

Councillor Hogg: Mr. Chairman, I object. The order should be addressed to the owners, not to the dogs.

Councillor Trafford: That is a nice point. Very well then: “Owners of dogs are not allowed in this Park unless they keep them on leads.”

Councillor Hogg: Mr. Chairman, I object. Strictly speaking, this would prevent me as a dog-owner from leaving my dog in the back-garden at home and walking with Mrs. Hogg across the Park.

Councillor Trafford: Mr. Chairman, I suggest that our legalistic friend be asked to redraft the notice himself.

Councillor Hogg: Mr. Chairman, since Councillor Trafford finds it so difficult to improve on my original wording, I accept. “Nobody without his dog on a lead is allowed in this Park.”

Councillor Trafford: Mr. Chairman, I object. Strictly speaking, this notice would prevent me, as a citizen, who owns no dog, from walking in the Park without first acquiring one.

Councillor Hogg (with some warmth): Very simply, then: “Dogs must be led in this Park.”

Councillor Trafford: Mr. Chairman, I object: this reads as if it were a general injunction to the Borough to lead their dogs into the Park.

Councillor Hogg interposed a remark for which he was called to order; upon his withdrawing it, it was directed to be expunged from the Minutes.

The Chairman: Councillor Trafford, Councillor Hogg has had three tries; you have had only two . . .

Councillor Trafford: “All dogs must be kept on leads in this Park.”

The Chairman: I see Councillor Hogg rising quite rightly to raise another objection. May I anticipate him with another amendment: “All dogs in this Park must be kept on the lead.”

This draft was put to the vote and carried unanimously, with two abstentions.

Introduction

If you get into a conversation with someone, you might hear statements such as “Rich people are snobs,” “Midwesterners are
friendly,” and “There’s someone out there who’s right for you.” In this section we will consider statements that say that everyone in a certain class has a certain property (universal statements) or that there is at least one person in a class with a certain property (existential statements). We begin by considering statements whose truth or falsity depends on the individual to whom they are applied.

**Open Statements**

Consider the sentence “x is a registered Democrat,” where x may be any citizen of the United States. We denote the sentence as \( p(x) \), which is read “\( p \) of \( x \),” since its truth or falsity depends on the particular citizen \( x \) to whom it is applied. When \( x \) is replaced by a particular citizen, then \( p(x) \) will be a statement, since it will then be true or false (depending on the individual chosen).

In general, an open statement \( p(x) \) is a statement about \( x \), where \( x \) may be any member of a specified universal set \( U \) of allowable objects. An open statement \( p(x) \) is not a statement, since its truth value cannot be determined until \( x \) is specified.

**EXAMPLE FINDING TRUTH AND FALSETY USING AN OPEN STATEMENT**

For the universal set

\[
U = \{\text{table, lamp, taxi, cat, horse, crow}\}
\]

and the open statement

\[
p(x) = \text{“the } x \text{ has four legs”}
\]

find the truth or falsity of

\[
p(\text{table}) \quad \text{and} \quad p(\text{crow}).
\]

**Solution**

Although \( p(x) \) is neither true nor false (since \( x \) is not specified), we have that

\[
p(\text{table}) \quad \text{is true} \quad \text{“The table has four legs” is true}
\]

and

\[
p(\text{crow}) \quad \text{is false} \quad \text{“The crow has four legs” is false}
\]
Truth Sets

The truth set of an open statement $p(x)$ is the set of all $x$ in $U$ such that $p(x)$ is true. We denote the truth set of $p(x)$ by $P$. For the $U$ and $p(x)$ in Example 1, the truth set is

$$P = \{\text{table, cat, horse}\} \quad \text{Set of } x \text{ in } U \text{ with four legs}$$

In general, if $p(x)$ is a tautology, then $P$ is the universal set, $P = U$, while if $p(x)$ is a contradiction, then $P$ is the null set, $P = \emptyset$. The truth set of the negation $\sim p(x)$ is the complement $P^c$ of the truth set of $p(x)$. For the preceding example, the truth set of $\sim p(x)$ is

$$P^c = \{\text{lamp, taxi, crow}\} \quad \text{Set of } x \text{ in } U \text{ not with four legs}$$

Practice Problem 1

Find (a) the truth set $Q$ of the open statement $q(x) = \text{“the } x \text{ can give you a ride”}$, where $U = \{\text{table, lamp, taxi, cat, horse, crow}\}$, and (b) the truth set of $\sim q(x)$.  

Quantifiers

We will use the symbol $\forall$ to mean “for all” and the usual set notation symbol $\in$ to mean “is an element of.” Again letting $p(x) = \text{“the } x \text{ has four legs,”}$ the statement that all elements of the set $S = \{\text{table, horse}\}$ have four legs may be written as

$$\forall x \in S, p(x) \quad \text{For all } x \text{ in } \{\text{table, horse}\}, \text{ the } x \text{ has four legs}$$

Universal Quantifier

Let $S$ be a subset of $U$, and let $p(x)$ be an open statement for $x$ in $U$. Then the statement “for all $x \in S, p(x)$” is written

$$\forall x \in S, p(x) \quad \text{For all } x \text{ in } S, p \text{ of } x$$

This statement is true if and only if $p(x)$ is true for all $x$ in $S$.

Other phrases for the universal quantifier $\forall$ are “for each” and “for every.” Clearly, the truth set $P$ of $p(x)$ has the property that

$$\forall x \in P, p(x) \quad p(x) \text{ holds for all } x \text{ in its truth set}$$

Since the truth set $P$ is the largest set of $x$’s in $U$ for which $p(x)$ is true, the statement $\forall x \in S, p(x)$” is true if and only if $S$ is a subset of $P$.

Be careful! “$\forall x \in S, p(x)$” is a statement about the truth of $p(x)$ for each $x$ in $S$, and is not a claim that $S$ contains every possible $x$ for which $p(x)$ is true.
We use the symbol $\exists$ to mean “there exists.”

**Existential Quantifier**

Let $S$ be a subset of $U$, and let $p(x)$ be an open statement for $x$ in $U$. Then the statement “there exists an $x \in S$ such that $p(x)$” is written

$$\exists x \in S, p(x)$$

This statement is true if and only if $p(x)$ is true for at least one $x$ in $S$.

Other phrases for the existential quantifier $\exists$ are “there is an,” “for at least one,” and “for some.” Since the truth set $P$ of $p(x)$ contains every $x$ in $U$ for which $p(x)$ is true, the statement “$\exists x \in S, p(x)$” is true if and only if $P$ and $S$ have at least one element in common (thus the intersection of $P$ and $S$ is nonempty: $P \cap S \neq \emptyset$).

**EXAMPLE USING QUANTIFIERS**

Let $U$ be the set of colors

$$U = \{\text{violet, blue, green, yellow, orange, pink, red}\}$$

with subsets

$$A = \{\text{violet, yellow}\} \quad \text{and} \quad B = \{\text{blue, orange, red}\}.$$

Let $p(x)$ and $q(x)$ be the open statements

$$p(x) = \text{“the word } x \text{ has more than five letters”}$$

$$q(x) = \text{“the word } x \text{ contains the letter } n.$$"

Determine the truth values of

a. $\forall x \in A, p(x)$

b. $\forall x \in B, p(x)$

c. $\exists x \in A, q(x)$

d. $\exists x \in B, q(x)$

**Solution**

a. “$\forall x \in A, p(x)$” means that all words in $A = \{\text{violet, yellow}\}$ have more than five letters, which is clearly true.

b. “$\forall x \in B, p(x)$” means that all words in $B = \{\text{blue, orange, red}\}$ have more than five letters, which is clearly false.

c. “$\exists x \in A, q(x)$” means that there exists a word in $A = \{\text{violet, yellow}\}$ that contains the letter $n$, which is clearly false.

d. “$\exists x \in B, q(x)$” means that there exists a word in $B = \{\text{blue, orange, red}\}$ that contains the letter $n$, which is clearly true.
We could also have found these answers using truth sets. For instance, the truth set for \( p(x) \) is \( P = \{ \text{violet, yellow, orange} \} \), and since \( A = \{ \text{violet, yellow} \} \) is a subset of \( P \), the statement “\( \forall x \in A, p(x) \)” in part (a) is true, just as we found above.

Let \( r(x) \) be the open statement “the word \( x \) contains the letter \( o \)” with the same sets \( U, A, \) and \( B \) used in Example 2. Determine the truth values of: 

\( a. \ \forall x \in A, r(x) \quad b. \ \exists x \in B, r(x) \).

**Solution on page 48**

**De Morgan’s Laws**

The statement “\( \forall x \in S, p(x) \)” is not true if we can identify even one single \( x \) in \( S \) for which \( p(x) \) is false. Similarly, “\( \exists x \in S, p(x) \)” is not true if it happens that \( p(x) \) is false for every \( x \) in \( S \). These observations, known as De Morgan’s laws, are generalizations of the identically named laws on page 14.

\begin{align*}
1. \quad & \sim [\forall x \in S, p(x)] = \exists x \in S, \sim p(x) \quad \text{To negate an } \forall, \text{ change it to an } \exists \text{ and negate the inside statement} \\
2. \quad & \sim [\exists x \in S, p(x)] = \forall x \in S, \sim p(x) \quad \text{To negate an } \exists, \text{ change it to an } \forall \text{ and negate the inside statement}
\end{align*}

That is, the negation of a universally quantified statement is the existence of its negation, and the negation of an existentially quantified statement is the universality of its negation.

**NEGATING QUANTIFIED STATEMENTS**

Let \( U = \{ \text{penny, nickel, dime, quarter} \} \) with subsets \( A = \{ \text{penny, nickel, dime} \} \) and \( B = \{ \text{dime, quarter} \} \). Let \( p(x) \) be the open statement “\( x \) is worth less than fifteen cents.” Express each statement in sentence form, use De Morgan’s laws to state the negation in symbolic and sentence form, and determine the truth value of each.

\( a. \ \forall x \in A, p(x) \quad b. \ \exists x \in B, \sim p(x) \)

**Solution**

\( a. \ \forall x \in A, p(x) \)” means “every coin in the set \( \{ \text{penny, nickel, dime} \} \) is worth less than fifteen cents,” which is clearly true.
By De Morgan’s law 1, the negation of “∀x ∈ A, p(x)” is “∃x ∈ A, ¬ p(x),” which means “there is a coin in {penny, nickel, dime} that is worth fifteen or more cents.” This statement is clearly false, as we would expect for the negation of a true statement.

b. “∃x ∈ B, ¬ p(x)” means “there is a coin in the set {dime, quarter} that is worth fifteen or more cents,” which is clearly true.

By De Morgan’s law 2, the negation of “∃x ∈ B, ¬ p(x)” is “∀x ∈ B, p(x)” [using the double negation ¬ (¬ p(x)) = p(x)]. This means “every coin in the set {dime, quarter} is worth less than fifteen cents,” which is clearly false, as we would expect for the negation of a true statement.

Practice Problem 3
For the same sets U and A used in Example 3, express “∃x ∈ A, ¬ p(x)” in sentence form, use De Morgan’s laws to state its negation in symbolic and sentence form, and determine the truth value of each.

Solution on page 48

Quantified Syllogisms and Euler Diagrams
A syllogism involving “all,” “some,” or “none” is called a quantified syllogism, and the validity of such an argument can often be established visually by using Euler diagrams,* as shown in the following examples. We begin with a syllogism due to Aristotle (384–322 B.C.), who was the first to systematically investigate logic and its relationship to mathematics.

Example 4
Is the following syllogism valid?

All men are mortal.
Socrates is a man.

Therefore: Socrates is mortal.

Solution
Since the first premise states that all men are mortal, we represent “all men” as a region within a larger region representing “all mortals.”

* Leonhard Euler (1707–1783), a prolific Swiss mathematician, first used his diagrams more than a century before the formal beginnings of modern set theory and Venn diagrams (see Section 1 of Chapter 5).
If we represent Socrates by a dot •, where should this dot be placed? By the second premise, this dot must be inside the region representing “all men.” But then the dot representing Socrates is clearly inside the region representing “all mortals,” showing that “Socrates is mortal,” and thus the syllogism is valid.

When drawing Euler diagrams, we must draw each region to reflect all possible positions, rather than just those favorable to the conclusion.

**EXAMPLE MORE ABOUT EULER DIAGRAMS**

Is the following syllogism valid?

- Some home owners are in debt.
- Charles is in debt.

Therefore: Charles is a home owner.

**Solution**

The regions for “home owners” and those who are “in debt” overlap by the first premise, but we are not told that one lies completely within the other. So we must begin by drawing

If we represent Charles by a dot •, we must place this dot within the “in debt” region. But where?

Since it is possible to place the dot within the “in debt” region but not within the “home owners” region, this syllogism is not valid.
Section Summary

An open statement \( p(x) \) is a statement about \( x \), where \( x \) may be any member of a specified universal set \( U \) of allowable objects. The truth set \( P(p(x)) \) of \( p(x) \) is the set of all \( x \) in \( U \) such that \( p(x) \) is true.

Universal quantifier \( \forall \quad \forall x \in S, p(x) \) \( p(x) \) is true for all \( x \) in \( S \)

Existential quantifier \( \exists \quad \exists x \in S, p(x) \) \( p(x) \) is true for at least one \( x \) in \( S \)

\[
\sim [\forall x \in S, p(x)] = \exists x \in S, \sim p(x)
\]
\[
\sim [\exists x \in S, p(x)] = \forall x \in S, \sim p(x)
\]

De Morgan’s laws

The validity of a quantified syllogism can often be demonstrated by an Euler diagram showing the relationship between the truth sets of quantified statements.

Solutions to Practice Problems

1. a. \( Q = \{\text{taxi, horse}\} \), since \( q(x) \) is true only when \( x \) is replaced by “taxi” or “horse.”
   
   b. \( Q^c = \{\text{table, lamp, cat, crow}\} \), since the truth set of \( \sim q(x) \) is the complement of the truth set of \( q(x) \).

2. a. True, since the letter \( o \) appears in each element of \( A = \{\text{violet, yellow}\} \).
   
   b. True, since the letter \( o \) appears in at least one element “orange,” of \( B = \{\text{blue, orange, red}\} \).

3. \( \exists x \in A, \sim p(x) \) means “there is a coin in the set \{penny, nickel, dime\} that is worth fifteen or more cents,” which is clearly false.

   By De Morgan’s law 2, the negation is “\( \forall x \in A, p(x) \)” [using the double negation \( \sim (\sim p(x)) = p(x) \)], which means “every coin in the set \{penny, nickel, dime\} is worth less than fifteen cents.” This statement is true, as we would expect for the negation of a false statement.

Exercises

Let \( U = \{\text{apple, broccoli, grape, pear, potato}\} \) with subsets \( A = \{\text{apple, pear}\} \) and \( B = \{\text{broccoli, grape, potato}\} \). Let \( p(x) \) be the open statement “\( x \) is a fruit.” Express each symbolic statement as a sentence and identify it as “true” or “false.”

1. \( p(\text{apple}) \)  
2. \( \forall x \in A, p(x) \)  
3. \( \forall x \in B, p(x) \)  
4. \( \exists x \in A, \sim p(x) \)  
5. \( \exists x \in B, \sim p(x) \)
Let \( U = \{\text{coffee, tea, milk, cookies, cake}\} \) with subsets \( A = \{\text{milk, cookies}\} \) and \( B = \{\text{coffee, tea, milk}\} \). Let \( p(x) \) be the open statement “\( x \) is a beverage.” Express each symbolic statement as a sentence and identify it as “true” or “false.”

6. \( \sim p(\text{tea}) \)
7. \( \exists x \in A, p(x) \)
8. \( \forall x \in B, p(x) \)
9. \( \forall x \in A, p(x) \)
10. \( \exists x \in B, \sim p(x) \)

Negate each statement using De Morgan’s laws.

31. \( \exists x \in A, \sim p(x) \)
32. \( \forall x \in B, \sim q(x) \)
33. \( \forall x \in A, p(x) \land q(x) \)
34. \( \exists x \in B, p(x) \lor q(x) \)
35. \( \exists x \in A, p(x) \rightarrow q(x) \)
36. \( \forall x \in B, \sim p(x) \land \sim q(x) \)
37. Every rich person is a snob.
38. All midwesterners are friendly.
39. There is a quadrilateral that is not a square.
40. There is a positive integer that is neither odd nor even.

Use Euler diagrams to identify each argument as “valid” or a “fallacy.”

41. All pears are ripe. 42. All pears are ripe.
Try this pear. This is ripe. It is ripe. This is a pear.

43. Some apples are rotten.
Try this apple.
It is rotten.

44. Some apples are rotten.
All the apples are green.
This apple is rotten and green.

45. All hurricanes are dangerous.
Floyd is a hurricane.
Floyd is dangerous.

46. Some storms are hurricanes.
All hurricanes are dangerous.
This storm is dangerous.

47. All business majors take mathematics.
Jim takes mathematics.
Jim is a business major.

48. All business majors take mathematics.
Jim is a business major.
Jim takes mathematics.

49. All sports cars are fast.
All fast cars are expensive.
All sports cars are expensive.
50. Some cold days are snowy.
   Today is not snowy.
   Today is not cold.

**Explorations and Excursions**

The following problems extend and augment the material presented in the text.

**More About Sets and Logic**

Let \( U \) be a universal set for the open statements \( p(x) \) and \( q(x) \), let \( P \) be the truth set for \( p(x) \), and let \( Q \) be the truth set for \( q(x) \). Justify each of the following facts relating set theory to logic.

51. The truth set of \( p(x) \land q(x) \) is \( P \cap Q \).

52. The truth set of \( p(x) \lor q(x) \) is \( P \cup Q \).

53. The truth set of a tautology is \( U \).

54. The truth set of a contradiction is \( \emptyset \) (the null set).

55. The truth set of \( \sim p(x) \) is \( P^c \).

56. \( P \subseteq Q \leftrightarrow \forall x, p(x) \rightarrow q(x) \) is true.

57. \( P = Q \leftrightarrow \forall x, p(x) \leftrightarrow q(x) \) is true.

58. The truth set of \( \sim p(x) \land q(x) \) is \( P^c \cap Q \).

59. The truth set of \( p(x) \land \sim q(x) \) is \( P \cup Q^c \).

60. Show that De Morgan’s laws for symbolic logic (see page 14) imply De Morgan’s laws for sets (see Exercises 43 and 44 at the end of Section 1 of Chapter 5).

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**Chapter Summary with Hints and Suggestions**

Reading the text and doing the exercises in this chapter have helped you to master the following skills, which are listed by section (in case you need to review them) and are keyed to particular Review Exercises. Answers for all Review Exercises are given at the end of this chapter.

**L.1 Statements and Connectives**

- Express a symbolic statement as a sentence.  
  (Review Exercises 1–4.)
  
  \(~ p ~) \quad \text{not} \quad p

  \pmb{\begin{align*}
  p \land q & \quad \text{and} \quad p \land q \\
  p \lor q & \quad \text{or} \quad p \lor q \\
  p \rightarrow q & \quad \text{if} \quad p, \ \text{then} \quad q
  \end{align*}}

- Express a sentence in symbolic form.  
  (Review Exercises 5–10.)

**L.2 Truth Tables**

- Find the truth value of a compound statement using the correct order of operations.  
  (Review Exercises 11–12.)
  \[(\), \sim , \land , \lor , \text{ and then } \rightarrow\]

- Construct a truth table for a compound statement.  (Review Exercises 13–14.)

- Simplify a compound statement by symbolic manipulation using the laws of symbolic logic.  
  (Review Exercises 15–18.)

  \(~ (~ p) = p \)

  \(p \lor q = q \lor p \quad p \land q = q \land p \)

  \(p \lor (q \land r) = (p \lor q) \land r \quad p \land (q \land r) = (p \land q) \lor r \)

  \(p \lor (q \land r) = (p \lor q) \land (p \land r) \quad p \land (q \lor r) = (p \land q) \lor (p \land r) \)

  \(p \lor (p \land q) = p \quad p \land (p \lor q) = p \)

- Use De Morgan’s laws to express the negation of a conjunction as a disjunction or the negation of a disjunction as a conjunction.  
  (Review Exercises 19–20.)

  \(~ (p \land q) = \sim p \lor \sim q \quad \sim (p \lor q) = \sim p \land \sim q \)
L.3 Implications
- State the converse, inverse, contrapositive, and negation of a conditional statement.
  (Review Exercises 21–22.)
  - Direct statement
  - Converse
  - Inverse
  - Contrapositive
  - Negation

- Construct a truth table for a compound statement involving the conditional or biconditional connective.
  (Review Exercises 23–26.)
- Simplify a compound statement involving the conditional connective by using the laws of symbolic logic.
  (Review Exercises 27–30.)

L.4 Valid Arguments
- Identify the type of a syllogism and whether it is “valid” or a “fallacy.”
  (Review Exercises 31–34.)
  - Modus ponens
  - Modus tollens
  - Disjunctive syllogism
  - Hypothetical syllogism
  - Fallacy of the converse
  - Fallacy of the inverse

- Analyze an argument and identify it as “valid” or a “fallacy.”
  (Review Exercises 35–38.)
- Supply a valid conclusion for a list of premises.
  (Review Exercises 39–40.)

L.5 Quantifiers and Euler Diagrams
- Find the truth value of an open statement $p(x)$ for a particular choice of $x \in U$.
  (Review Exercises 41–42.)
- Find the truth set of an open statement.
  (Review Exercises 43–44.)
  \[ P = \{ x \in U \text{ such that } p(x) \text{ is true} \} \]
- Find the truth value of a quantified statement.
  (Review Exercises 45–46.)
- Use De Morgan’s laws to negate a quantified statement.
  (Review Exercises 47–48.)
  \[ \sim [\forall x \in S, p(x)] = \exists x \in S, \sim p(x) \]
  \[ \sim [\exists x \in S, p(x)] = \forall x \in S, \sim p(x) \]
- Determine the validity of a quantified syllogism using an Euler diagram.
  (Review Exercises 49–50.)

Hints and Suggestions
- (Overview) Although useful conclusions require correct premises, the validity of logical arguments does not depend on the truth of the premises. Truth values lead to truth table definitions of the logical connectives, and logical equivalences and tautologies lead to valid arguments.
- Truth tables are useful for small calculations, but involved compound statements can be analyzed more easily using the laws of symbolic logic.
- The four variations of the implication $p \to q$ are not all equivalent: The converse and the inverse are equivalent, as are the direct statement and the contrapositive.
- Logically equivalent statements are interchangeable, but it is usually best to replace a complicated statement by a simpler version when analyzing a statement or argument.
- The implication $p \to q$ is equivalent to $\sim p \lor q$, so the negation $\sim (p \to q)$ is equivalent to $p \land \sim q$ by De Morgan’s laws.
**L.1 Statements and Connectives**

Let \( p \) represent the statement “roses are red,” and let \( q \) represent “violets are blue.” Express each symbolic statement as a sentence.

1. \( p \land q \)
2. \( p \lor q \)
3. \( \sim q \)
4. \( \sim p \rightarrow q \)

Let \( p \) represent the statement “it is summer,” let \( q \) represent “the leaves are green,” and let \( r \) represent “it is snowing.” Express each sentence in symbolic form.

5. It is summer and the leaves are green.
6. It is summer and it is not snowing.
7. The leaves are green or it is not summer.
8. It is not snowing or the leaves are green and it is summer.
9. If it is snowing, then it is not summer.
10. If it is summer and the leaves are green, then it is not snowing.

**L.2 Truth Tables**

For each compound statement:

a. Find its truth value given that \( p \) is \( T \), \( q \) is \( T \), and \( r \) is \( F \).

11. \( p \land (\sim q \lor r) \)
12. \( (p \lor \sim q) \land r \)

b. Check your answer using a graphing calculator.

13. \( (p \lor \sim q) \land \sim p \)
14. \( (p \land r) \lor (\sim q \lor r) \)

For each compound statement:

a. Construct a truth table. Identify any tautologies and contradictions.

15. \( (p \lor \sim q) \land (p \lor q) \)
16. \( (\sim p \lor q) \land (p \land \sim q) \)
17. \( p \land (p \lor q) \)
18. \( \sim (p \land q) \lor p \)

b. Check your answer using a graphing calculator.

Use De Morgan’s laws to negate each statement.

19. The glasses are broken and the watch won’t run.
20. His ankle is fine or his knee is bruised.

**L.3 Implications**

Write in sentence form the (a) converse, (b) inverse, (c) contrapositive, and (d) negation of the conditional statement. \([\text{Hint: It may be helpful to rewrite}]\)
the statement in “if . . . , then . . .” form to identify
the antecedent and the consequent.
21. The frame is ornate if the painting is expensive.
22. The freezer works properly only if the ice cream is hard.

Construct a truth table for each statement. Identify any tautologies and contradictions.
23. \( \neg p \to \neg q \)
24. \([(p \to q) \land p] \to q\)
25. \((p \lor q) \to (q \land r)\)
26. \(\neg p \leftrightarrow q\)

Simplify each statement using the laws of symbolic logic. Identify any tautologies and contradictions.
27. \(p \lor q \to p\)
28. \((p \land q) \to p\)
29. \((p \to q) \to \neg p\)
30. \(p \to \neg p\)

L.4 Valid Arguments

Identify the type of each syllogism and whether it is “valid” or a “fallacy.”
31. Freshmen must take English composition.
   Tim is a freshman.
   Tim takes English composition.
32. The band sounds great when Justin plays bass.
   The band was awful.
   Justin didn’t play bass.
33. She’s talking on the phone or visiting her sister.
   She’s not visiting her sister.
   She’s talking on the phone.
34. If it’s cloudy, we’ll go fishing.
   It’s clear.
   We’re not going fishing.

Analyze each argument and identify it as “valid” or a “fallacy.”
35. If the virus is virulent, then many will die.
   Not many will die or it will be an economic disaster.
   The virus is virulent.
   It will be an economic disaster.

36. Many old people will freeze this winter if home heating oil is too expensive.
   If oil prices go up, home heating oil will be too expensive.
   Oil prices have not gone up.

   Not many old people will freeze this winter.
37. If I win the lottery or get a job, then I can afford a car.
   I can’t afford a car or I won’t take the bus.
   I ride the bus.

   I didn’t win the lottery and I didn’t get a job.
38. If interest rates are down, industry can invest to increase productivity.
   Industry cannot invest to increase productivity or unemployment will decrease.
   If unemployment decreases, then housing starts will increase.
   Housing starts have decreased.

   Interest rates are up.

Supply a valid conclusion for each list of premises.
39. If I have soup for lunch, then I have cookies for tea.
   I don’t have cookies for tea or I stay late at work.

   I ride the bus.
40. The book is heavy if it is thick.
   The book is light or I won’t buy it.
   I bought the book.

L.5 Quantifiers and Euler Diagrams

Let \( U = \{\text{Brahms, Cézanne, Debussy, Dürer, Pachelbel, Vermeer}\} \) with subsets \( A = \{\text{Debussy, Dürer}\} \) and \( B = \{\text{Brahms, Cézanne}\}. \) Let \( p(x) \) be the open statement “\( x \) is a famous composer,” and let \( q(x) \) be the open statement “\( x \) is a famous Frenchman.”

Find the truth value of each statement.
41. \( p(\text{Brahms})\)
42. \( q(\text{Vermeer})\)
43. Find the truth set \( P \) of \( p(x) \).
44. Find the truth set \( Q \) of \( q(x) \).
Determine the truth value of each statement.

45. \( \forall x \in A, p(x) \)

46. \( \exists x \in B, q(x) \)

Negate each statement using De Morgan’s laws.

47. All great composers have names beginning with the letter B.

48. There is a logic problem that I cannot solve.

49. All snowy days are cold.
   Today is not cold.
   Today is not snowy.

50. Some sports cars are expensive.
   All sports cars are fast.
   All fast cars are expensive.

**Projects and Essays**

The following projects and essays are based on this chapter. There are no right or wrong answers—the results depend only on your imagination and resourcefulness.

1. Locate and read the article on “Symbolic Logic” by John E. Pfeiffer in the December 1960 issue of *Scientific American*, and write a one-page report on either (a) one of his examples or (b) how computer advances since 1960 have (or haven’t) fulfilled some of his predictions.

2. Besides Alice’s Adventures in Wonderland and Through the Looking Glass, written under his pen name “Lewis Carroll,” Charles Dodgson (1832–1898) also wrote The Game of Logic. Locate a reprint of this classic or some excerpts included in a logic book, and write a one-page analysis of one of his examples.

3. Although our discussion of the conditional explicitly rejected any cause-and-effect linkage between the antecedent and the consequent, such an approach can be challenged on the basis that there is a legitimate need for “meaning” in logic statements and analysis. Investigate the differences between “implication in formal meaning” and “implication in material meaning,” perhaps by starting with the article “Symbolic Logic” by Alfred Tarski as reprinted in *The World of Mathematics*, vol. 3, by James R. Newman (New York: Simon and Schuster, 1956), pp. 1901–1931, and write a one-page report on your findings.

4. The problem of separating logically valid facts from conclusions that we might subconsciously draw from the content of the statements has been addressed by the philosopher H. P. Grice in his theory of “conversational implicature.” Find out more about Grice and his work, and write a one-page report on some of his examples.

5. Closely read a credit card or car loan agreement and find examples of the legal phrase “and/or.” Write a report on how this language clarifies possible confusions between the inclusive and exclusive meanings of the disjunction.