Chapter 2: Derivatives and Their Uses

EXERCISES 2.1

1. \( x \) | \( 5x - 7 \) | \( x \) | \( 5x - 7 \)
   \[\begin{array}{ccc}
   1.9 & 2.500 & 2.1 & 3.500 \\
   1.99 & 2.950 & 2.01 & 3.050 \\
   1.999 & 2.995 & 2.001 & 3.005 \\
   \end{array}\]
   \( \lim_{x \to 2} (5x - 7) = 3 \)

5. \( x \) | \((1 + 2x)^{1/x}\) | \( x \) | \((1 + 2x)^{1/x}\)
   \[\begin{array}{ccc}
   -0.1 & 9.313 & 0.1 & 6.192 \\
   -0.01 & 7.540 & 0.01 & 7.245 \\
   -0.001 & 7.404 & 0.001 & 7.374 \\
   \end{array}\]
   \( \lim_{x \to 0} (1 + 2x)^{1/x} \approx 7.4 \)

9. \( \lim_{x \to 1} x^{\frac{-1}{x}} = 1 \)

13. \( \lim_{x \to 2} 4x^2 - 10x + 2 = 4(3)^2 - 10(3) + 2 = 8 \)

15. \( \lim_{x \to 5} \frac{3x^2 - 5x}{7x - 10} = \frac{3(5)^2 - 5(5)}{7(5) - 10} = 2 \)

17. \( \lim_{x \to 3} \sqrt{2} = \sqrt{2} \) because the limit of a constant is just the constant.

21. \( \lim_{h \to 0} (5x^3 + 2x^2h - xh^2) = 5x^3 + 2x^2 \cdot 0 - x(0)^2 = 5x^3 \)

25. \( \lim_{x \to 3} \frac{x+3}{x^2 + 8x + 15} = \lim_{x \to 3} \frac{x+3}{(x+3)(x+5)} = \frac{1}{-3 + 5} = \frac{1}{2} \)

29. \( \lim_{h \to 0} \frac{2xh - 3h^2}{h} = \lim_{h \to 0} (2x - 3h) = 2x - 3(0) = 2x \)

31. \( \lim_{h \to 0} \frac{4x^2h + xh^2 - h^3}{h} = \lim_{h \to 0} (4x^2 + xh - h^2) \\
   = 4x^2 + x(0) - (0)^2 = 4x^2 \)
33. a. \( \lim_{{x \to 2^-}} f(x) = 1 \)
   b. \( \lim_{{x \to 2^+}} f(x) = 3 \)
   c. \( \lim_{{x \to 2}} f(x) \) does not exist.

37. a. \( \lim_{{x \to 4^+}} f(x) = \lim_{{x \to 4^+}} (3 - x) = 3 - 4 = -1 \)
   b. \( \lim_{{x \to 4^-}} f(x) = \lim_{{x \to 4^-}} (10 - 2x) = 10 - 2(4) = 2 \)
   c. \( \lim_{{x \to 4}} f(x) \) does not exist.

39. a. \( \lim_{{x \to 4^+}} f(x) = \lim_{{x \to 4^+}} (2 - x) = 2 - 4 = -2 \)
   b. \( \lim_{{x \to 4^+}} f(x) = \lim_{{x \to 4^+}} (x - 6) = 4 - 6 = -2 \)
   c. \( \lim_{{x \to 4}} f(x) = -2 \)

41. a. \( \lim_{{x \to 0^+}} f(x) = \lim_{{x \to 0^+}} (-x) = -0 = 0 \)
   b. \( \lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^-}} (x) = 0 \)
   c. \( \lim_{{x \to 0}} f(x) = 0 \)

43. a. \[
\begin{array}{c|c}
  x & \frac{\left| x \right|}{x} \\
  \hline
  -0.1 & -1 \\
  -0.01 & -1 \\
  -0.001 & -1 \\
  \lim_{{x \to 0}} f(x) & -1 \\
\end{array}
\]
   b. \[
\begin{array}{c|c}
  x & \frac{\left| x \right|}{x} \\
  \hline
  0.1 & 1 \\
  0.01 & 1 \\
  0.001 & 1 \\
  \lim_{{x \to 0}} f(x) & 1 \\
\end{array}
\]
   c. \( \lim_{{x \to 0}} f(x) \) does not exist.

45. \( \lim_{{x \to 3^-}} f(x) = \infty \)
   \( \lim_{{x \to 3^+}} f(x) = -\infty \)
   \( \lim_{{x \to 3}} f(x) \) does not exist.

47. \( \lim_{{x \to 0^-}} f(x) = \infty \)
   \( \lim_{{x \to 0^+}} f(x) = \infty \)
   \( \lim_{{x \to 0}} f(x) = \infty \)

49. on \([-3, -1]\) by \([-50, 50]\)
   \( \lim_{{x \to -2^-}} f(x) = -\infty ; \lim_{{x \to -2^+}} f(x) = \infty \)
   \( \lim_{{x \to -2}} f(x) \) does not exist.

51. on \([2, 4]\) by \([-50, 50]\)
   \( \lim_{{x \to 3^-}} f(x) = \infty ; \lim_{{x \to 3^+}} f(x) = \infty \)
   \( \lim_{{x \to 3}} f(x) = \infty \)

53. Continuous

55. Discontinuous at \( c \) because \( \lim_{{x \to c^-}} f(x) \neq f(c) \).

57. Discontinuous at \( c \) because \( f(c) \) is not defined.

59. Discontinuous at \( c \) because \( \lim_{{x \to c}} f(x) \) does not exist.
61. **a.**

![Graph](image)

b. \( \lim_{x \to 3^-} f(x) = 3; \lim_{x \to 3^+} f(x) = 3 \)

c. Continuous

63. **a.**

![Graph](image)

b. \( \lim_{x \to 3^-} f(x) = 3; \lim_{x \to 3^+} f(x) = 4 \)

c. Discontinuous because \( \lim_{x \to 3} f(x) \) does not exist.

65. Continuous

67. Discontinuous at \( x = 1 \)

69. \( f(x) = \frac{-12}{5x^3 - 5x} \) is discontinuous at values of \( x \) for which the denominator is zero. Thus, consider

\[
5x^3 - 5x = 0
\]

\[
5x(x^2 - 1) = 0
\]

\( 5x \) equals zero at \( x = 0 \) and \( x^2 - 1 \) equals zero at \( x = \pm 1. \)

Thus, the function is discontinuous at \( x = 0, x = -1, \) and \( x = 1. \)

73. From Exercise 37, we know
\( \lim_{x \to 4} f(x) = -2 = f(4). \) Therefore, the function is continuous.

75. From Exercise 39, we know \( \lim_{x \to 0} f(x) = 0 = f(0). \) Therefore, the function is continuous.

77. From the graph, we can see that \( \lim_{x \to 6} f(x) \) does not exist because the left and right limits do not agree. \( f(x) \) is discontinuous at \( x = 6. \)

81. \[
\begin{array}{c|c|c}
 x & \left(1 + \frac{x}{10}\right)^{1/x} & x \\
-0.1 & 1.11 & 0.1 \\
-0.01 & 1.11 & 0.01 \\
-0.001 & 1.11 & 0.001 \\
-0.0001 & 1.11 & 0.0001 \\
\end{array}
\]

\[ \lim_{x \to 0} \left(1 + \frac{x}{10}\right)^{1/x} \approx 1.11 \]

83. \[
\lim_{x \to 1^+.001x^2} = \frac{100}{1 + .001(0)^2} = \frac{100}{1} = 100
\]

85. As \( x \) approaches \( c \), the function is approaching \( \lim f(x) \) even if the value of the function at \( c \) is different, so the limit is the function is "going".

87. False: The value of the function at 2 has nothing to do with the limit as \( x \) approaches 2.
89. False: Both one-sided limits would have to exist and agree to guarantee that the limit exists.

91. False: On the left side of the limit exists and equals 2 (as we saw in Example 4 on page 75, but on the right side of the denominator of the fraction is zero. Therefore one side of the equation is defined and the other is not.

93. True: The third requirement for continuity at $x = 2$ is that the limit and the value at 2 must agree, so if one is 7 the other must be 7.

EXERCISES 2.2

1. The slope is positive at $P_1$.
The slope is negative at $P_2$.
The slope is zero at $P_3$.

5. The tangent line at $P_1$ contains the points (0, 2) and (1, 5). The slope of this line is
$$m = \frac{5 - 2}{1 - 0} = 3.$$ The slope of the curve at $P_1$ is 3.
The tangent line at $P_2$ contains the points (3, 5) and (5, 4). The slope of this line is
$$m = \frac{4 - 5}{5 - 3} = -\frac{1}{2}.$$ The slope of the curve at $P_2$ is $-\frac{1}{2}$.

7. Your graph should look roughly like the following:

11. a. $\frac{4 - 8}{2} = \frac{34 - 8}{2} = 13$
b. $\frac{5 - 8}{1} = \frac{19 - 8}{1} = 11$
c. $\frac{5 - 8}{.5} = \frac{13 - 8}{.5} = 10$
d. $\frac{2.2 - 2}{.1} = 2.236 - 2 = 9.2$
e. $\frac{2.002 - 2}{.01} = 8.0092 - 8 = 9.02$
f. Answers seem to be approaching 9.

13. a. $\frac{26 - 16}{2} = \frac{10 - 16}{2} = 5$
b. $\frac{21 - 16}{1} = \frac{5 - 16}{1} = 5$
c. $\frac{18.5 - 16}{.5} = 5$
d. $\frac{5 - 16}{.1} = 5$
e. $\frac{16.05 - 16}{.01} = 5$
f. Answers seem to be approaching 5.

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17. \[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \]
   \[ = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \]
   \[ = \lim_{h \to 0} \frac{2xh + h^2}{h} \]
   Evaluating at \( h = 0 \) gives \( 2(2) + 1 = 3 \), which matches the answer from Exercise 9.

19. \[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5(x+h) + 1 - (5x + 1)}{h} \]
   \[ = \lim_{h \to 0} \frac{5x + 5h + 1 - 5x - 1}{h} \]
   \[ = \lim_{h \to 0} \frac{5h}{h} \]
   \[ = \lim_{h \to 0} 5 = 5 \]
   Evaluating at \( x = 3 \) gives 5, which matches the answer from Exercise 13.

23. \[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \]
   \[ = \lim_{h \to 0} \left( \frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left( \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \]
   \[ = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \]
   \[ = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \frac{1}{\sqrt{x+h} + \sqrt{x}} \]
   \[ = \frac{1}{2\sqrt{x}} \]
   The slope of the tangent line at \( x = 4 \) is \( \frac{1}{2\sqrt{4}} = 0.25 \), which matches the answer from Exercise 15.

25. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 5 - (x^2 - 3x + 5)}{h} \]
   \[ = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 5 - x^2 + 3x - 5}{h} \]
   \[ = \lim_{h \to 0} \frac{2x + h - 3}{h} \]
   \[ = \lim_{h \to 0} (2x + h - 3) \]
   \[ = 2x - 3 \]

27. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
   \[ = \lim_{h \to 0} \frac{1 - (x+h)^2 - (1 - x^2)}{h} \]
   \[ = \lim_{h \to 0} \frac{1 - x^2 - 2xh - h^2 + 2x^2 - x^2}{h} \]
   \[ = \lim_{h \to 0} \frac{2x^2 - 2xh - h^2 + 2x^2 - 2xh}{h} \]
   \[ = \lim_{h \to 0} -2x - h \]
   \[ = \lim_{h \to 0} -2x \]
   \[ = -2x \]

31. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
   \[ = \lim_{h \to 0} \frac{x + h - x}{h} \]
   \[ = \lim_{h \to 0} \frac{2}{h} \]
   \[ = \lim_{h \to 0} \frac{h}{2h} \]
   \[ = \lim_{h \to 0} \frac{1}{2} \]

35. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
   \[ = \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \]
   \[ = \lim_{h \to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h} \]
   \[ = \lim_{h \to 0} \frac{(2ax + ah + b) = 2ax + b}{h} \]

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37. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{(x+h)^5 - x^5}{h} \]
\[ = \lim_{h \to 0} \frac{x^5 + 5x^4h + 10x^3h^2 + 5x^2h^3 + xh^4 - x^5}{h} \]
\[ = \lim_{h \to 0} 5x^4 + 10x^3h + 5x^2h^2 + xh^3 = 5x^4 \]

39. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{2x - 2}{x(x+h)} \]
\[ = \lim_{h \to 0} \frac{2}{x(x+h)} \cdot \frac{1}{h} \]
\[ = \lim_{h \to 0} \frac{2}{x(x+h)} \]
\[ = \frac{-2}{x^2} \]

41. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \]
\[ = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \]
\[ = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \]
\[ = \lim_{h \to 0} \frac{1}{h(\sqrt{x+h} + \sqrt{x})} \]
\[ = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \]

43. \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{(x+h)^3 + (x+h)^2 - (x^3 + x^2)}{h} \]
\[ = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + x^2 + 2xh + h^2 - x^3 - x^2}{h} \]
\[ = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 2xh + h^2}{h} \]
\[ = \lim_{h \to 0} 3x^2 + 3xh + h^2 + 2x + h \]
\[ = 3x^2 + 2x \]

45. a. The slope of the tangent line at \( x = 2 \) is \( f'(2) = 2(2) - 3 = 1 \). To find the point of the curve at \( x = 2 \), we calculate \( y = f(2) = 2^2 - 3(2) + 5 = 3 \). Using the point-slope form with the point \( (2, 3) \), we have \( y - 3 = 1(x - 2) \)
\[ y - 3 = x - 2 \]
\[ y = x + 1 \]

b. [Image of graph]

on viewing window \([-10, 10] \) by \([-10, 10] \)
47. a. 

\[ f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x + h) - 4 - (3x - 4)}{h} = \lim_{h \to 0} \frac{3x + 3h - 4 - 3x + 4}{h} = \lim_{h \to 0} 3 = 3 \]

b. The graph of \( f(x) = 3x - 4 \) is a straight line with slope 3.

49. a. \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4 - (3x - 4)}{h} = \lim_{h \to 0} 3x + 3h - 4 - 3x + 4 = \lim_{h \to 0} 3 = 3 \)

b. The graph of \( f(x) = 3x - 4 \) is a straight line with slope 3.

51. a. \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{5 - 5}{h} = \lim_{h \to 0} 0 = 0 \)

b. The graph of \( f(x) = 5 \) is a straight line with slope 0.

53. a. \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{m(x + h) + b - (mx + b)}{h} = \lim_{h \to 0} \frac{mx + mh + b - mx - b}{h} = \lim_{h \to 0} \frac{mh}{h} \)

b. The graph of \( f(x) = mx + b \) is a straight line with slope \( m \).

55. a. \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - 8(x + h) + 110 - (x^2 - 8x + 110)}{h} \)

\[ = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - 8x - 8h + 110 - x^2 + 8x - 110}{h} \]

\[ = \lim_{h \to 0} \frac{2x + h - 8}{h} \]

b. \( f'(2) = 2(2) - 8 = -4 \). The temperature is decreasing at a rate of 4 degrees per minute after 2 minutes.

c. \( f'(5) = 2(5) - 8 = 2 \). The temperature is increasing at a rate of 2 degrees per minute after 5 minutes.

57. a. \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x + h)^2 - (x + h) - (2x^2 - x)}{h} \)

\[ = \lim_{h \to 0} \frac{2x^2 + 4hx + 2h^2 - x - h - 2x^2 + x}{h} \]

\[ = \lim_{h \to 0} \frac{2x^2 + 4hx + 2h^2 - x - h - 2x^2 + x}{h} = \lim_{h \to 0} (4x + 2h - 1) = 4x - 1 \]

b. \( f'(5) = 4(5) - 1 = 19 \). When 5 words have been memorized, the memorization time is increasing at a rate of 19 seconds per word.

59. a. \( T'(x) = \lim_{h \to 0} \frac{T(x + h) - T(x)}{h} = \lim_{h \to 0} \frac{-(x + h)^2 + 5(x + h) + 100 - (-x^2 + 5x + 100)}{h} \)

\[ = \lim_{h \to 0} \frac{-x^2 - 2hx - h^2 + 5x + 5h + 100 + x^2 - 5x - 100}{h} \]

\[ = \lim_{h \to 0} \frac{-2x - h + 5}{h} = \lim_{h \to 0} \frac{-2x + 5}{h} \]

b. \( T'(2) = -2(2) + 5 = 1 \). The patient’s temperature is increasing at a rate of 1 degree per day.

c. \( T'(3) = -2(3) + 5 = -1 \). The patient’s temperature is decreasing at a rate of 1 degree per day.

d. On day 2, the patient’s health is worsening. On day 3, the patient’s health is improving.

61. The average rate of change requires two \( x \)-values and is the change in the function - values divided by the change in the \( x \)-values. The instantaneous rate of change is at a single \( x \)-value, and is found from the formula \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \).

63. Substitution \( h = 0 \) would make the denominator zero, and we can’t divide by zero. That’s why we need to do some algebra on the difference quotient, to cancel out the terms that are zero so that afterwards we can evaluate by direct substitution.
65. The units of $x$ are blargs and the units of $f$ are prendles because the derivative $f'(x)$ is equivalent to the slope of $f(x)$, which is the change in $f$ over the change in $x$.

67. The patient’s health is deteriorating during the first day (temperature is rising above normal). The patient’s health is improving during the second day (temperature is falling).

**EXERCISES 2.3**

1. \[ f'(x) = \frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3 \]

5. \[ f'(x) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2} \]

9. \[ g'(w) = \frac{d}{dw}(6w^{1/3}) = 6 \cdot \frac{1}{3} w^{1/3-1} = 2w^{-2/3} \]

13. \[ f'(x) = \frac{d}{dx}(4x^2 - 3x + 2) = 4 \cdot 2x - 3 \cdot 1 + 0 = 8x - 3 \]

17. \[ f'(x) = \frac{d}{dx}\left(\frac{6}{\sqrt[3]{x}}\right) = \frac{d}{dx}(6x^{-1/3}) = 6 \left(-\frac{1}{3}\right)x^{-4/3} = -\frac{2}{x^{4/3}} \]

21. \[ f'(x) = \frac{d}{dx}\left(\frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1\right) = \frac{1}{6}(3x^2) + \frac{1}{2}(2x) + 1 = \frac{1}{2}x^2 + x + 1 \]

25. \[ h'(x) = \frac{d}{dx}(6x^{2/3} - 12x^{-1/3}) = 6 \cdot \frac{2}{3} x^{2/3-1} - 12 \left(-\frac{1}{3}\right)x^{-1/3-1} = 4x^{-1/3} + 4x^{-4/3} \]

27. \[ f'(x) = \frac{d}{dx}\left(10x^{-1/2} - 9x^{5/3} + 17\right) = 10 \left(-\frac{1}{2}\right)x^{-1/2-1} - 9 \left(\frac{5}{3}\right)x^{5/3-1} = -5x^{-3/2} - 15x^{2/3} \]

29. \[ f'(x) = \frac{d}{dx}\left(\frac{x^2 + x^3}{x}\right) = \frac{d}{dx}(x + x^2) = 1 + 2x \]

31. \[ f'(x) = \frac{d}{dx}(x^5) = 5x^4; \quad f'(-2) = 5(-2)^4 = 80 \]

33. \[ f'(x) = \frac{d}{dx}\left(6x^{2/3} - 48x^{-1/3}\right) = 6 \left(\frac{2}{3}\right)x^{-1/3} - 48 \left(-\frac{1}{3}\right)x^{-4/3} = 4x^{-1/3} + 16x^{-4/3} \]

\[
\begin{align*}
\frac{df}{dx}(8) & = 4(8)^{-1/3} + 16(8)^{-4/3} = \frac{4}{\sqrt[3]{8}} + \frac{16}{\sqrt[3]{8^4}} = \frac{4}{2} + \frac{16}{8} \\
& = 2 + \frac{16}{8} = 3 \\
\end{align*}
\]

35. \[ \frac{df}{dx} = \frac{d}{dx}(x^3) = 3x^2 \quad \left. \frac{df}{dx}\right|_{x=-3} = 3(-3)^2 = 27 \]
37. \[
\frac{df}{dx} = \frac{d}{dx} (16x^{-1/2} + 8x^{1/2}) = 16\left(-\frac{1}{2}\right)x^{-3/2} + 8\left(\frac{1}{2}\right)x^{-1/2} = 8x^{-3/2} + 4x^{-1/2}
\]
\[
\frac{df}{dx}_{x=4} = -8(4)^{-3/2} + 4(4)^{-1/2} = -\frac{8}{4\sqrt{3}} + \frac{4}{4} = -\frac{8}{8} + \frac{4}{2} = 1
\]

39. a. \[f'(x) = 2 \cdot x^1 - 2 + 0 = 2x + 2\]

b. 

\[\text{The slope of the tangent line is } 4.\]
\[y - 5 = 4(x - 3)\]
\[\Rightarrow y = 4x - 7\]

The equation for the tangent line is 
\[y = 4x - 7.\]

41. a. \[f'(x) = 3 \cdot x^2 - 3(2)x^1 + 2 + 0 = 3x^2 - 6x + 2\]

b. 

\[\text{The slope of the tangent line is } 2.\]
\[y + 2 = 2(x - 2)\]
\[\Rightarrow y = 2x - 6\]

The equation for the tangent line is 
\[y = 2x - 6.\]

43. For \(y_1 = 5\) and viewing rectangle \([-10, 10]\) by \([-10, 10]\), your graph should look roughly like the following:

45. a. The marginal profit function is the derivative of the profit function \(P(x) = 0.02x^{3/2} - 3000\).

\[MP(x) = 0.02\left(\frac{3}{2}\right)x^{1/2} - 0 = 0.03x^{1/2}\]

b. The marginal profit when 10,000 units have been sold is found by evaluating the marginal profit function at \(x = 10,000\).

\[MP(10,000) = 0.03(10,000)^{1/2} = 0.03\sqrt{10,000} = 0.03(100) = 3\]

When 10,000 units have been sold, the profit on each additional unit is about $3.

47. \[P(10,001) - P(10,000) = 0.02(10,001)^{1/2} - 3000 - [0.02(10,000)^{1/2} - 3000] \approx 3.00007\]

The answer is close to $3.

49. a. The rate of change of the teenage population in \(x\) years is the derivative of the population function \(P(x) = 12,000,000 - 12,000x + 600x^2 + 100x^3\).

\[P'(x) = -12,000 + 2 \cdot 600x + 3 \cdot 100x^2 = -12,000 + 1200x + 300x^2\]

b. To find the rate of change of the teenage population 1 year from now, evaluate \(P'(x)\) for \(x = 1\).

\[P'(1) = -12,000 + 1200(1) + 300(1)^2 = -12,000 + 1500 = -10,500\]

One year from now, the teenage population will be decreasing at a rate of 10,500 per year.

c. To find the rate of change of the teenage population 10 years from now, evaluate \(P'(x)\) for \(x = 10\).

\[P'(10) = -12,000 + 1200(10) + 300(10)^2 = -12,000 + 12,000 + 30,000 = 30,000\]

Ten years from now, the teenage population will be increasing at a rate of 30,000 per year.
51. The rate of change of the pool of potential customers is the derivative of the function
\[ N(x) = 400,000 - \frac{200,000}{x}. \]
\[ N'(x) = \frac{d}{dx}(400,000 - 200,000x^{-1}) \]
\[ N'(x) = 0 - (-1)200,000x^{-2} = \frac{200,000}{x^2} \]
To find the rate of change of the pool of potential customers when the ad has run for 5 days, evaluate \( N'(x) \) for \( x = 5 \).
\[ N'(5) = \frac{200,000}{5^2} = 8000 \]
The pool of potential customers is increasing by about 8000 people per additional day.

53. \( A(t) = 0.01t^2 \quad 1 \leq t \leq 5 \)
The instantaneous rate of change of the cross-sectional area \( t \) hours after administration of nitroglycerin is given by
\[ A'(t) = 2(0.01)t^{2-1} = 0.02t \]
\[ A'(4) = 0.02(4) = 0.08 \]
After 4 hours the cross-sectional area is increasing by about 0.08 cm\(^2\) per hour.

55. The instantaneous rate of change of the number of phrases students can memorize is the derivative of the function \( p(t) = 24\sqrt{t} \).
\[ p'(t) = \frac{d}{dt}(24t^{1/2}) \]
\[ = \frac{1}{2}(24t^{-1/2}) = 12t^{-1/2} \]
\[ p'(4) = 12(4)^{-1/2} = \frac{12}{\sqrt{4}} = 6 \]
The number of phrases students can memorize after 4 hours is increasing by 6.

57. a. \( U(x) = 100\sqrt{x} = 100x^{1/2} \)
\[ MU(x) = U'(x) = \frac{1}{2}(100)x^{1/2-1} = 50x^{-1/2} \]
b. \( MU(1) = U'(1) = 50(1)^{-1/2} = 50 \)
The marginal utility of the first dollar is 50.
c. \( MU(1,000,000) = U'(1,000,000) = 50(10^6)^{-1/2} = 50(10)^{-3} = \frac{50}{1000} = 0.05 \)
The marginal utility of the millionth dollar is 0.05.

59. a. \( f(12) = 0.831(12)^2 - 18.1(12) + 137.3 = 39.764 \)
A smoker who is a high school graduate has a 39.8% chance of quitting.
\[ f'(12) = 0.831(2)12 - 18.1 = 1.662 \cdot 12 - 18.1 \]
When a smoker has a high school diploma, the chance of quitting is increasing at the rate of 1.8% per year of education.

b. \( f(16) = 0.831(16)^2 - 18.1(16) + 137.3 = 60.436 \)
A smoker who is a college graduate has a 60.4% chance of quitting.
\[ f'(16) = 1.662(16) - 18.1 = 8.492 \]
When a smoker has a college degree, the chance of quitting is increasing at the rate of 8.5% per year of education.
61. a. [Graph of a function with the point (0,0) and the point (5,0) marked.] on [0,50] by [0,20000]

b. \( y_1 = 14.435x^2 + 36.67x + 1479.7 \) on [0,50] by [0,20000]

c. Tuition will be about $26,043 in the year 2010.

e. Tuition will be increasing at a rate of about $1,191 per year.

63. \( f(x) = 2 \) will have a graph that is a horizontal line at height 2, and the slope (the derivative) of a horizontal line is zero. A function that stays constant will have a rate of change of zero, so its derivative (instantaneous rate of change) will be zero.

65. If \( f \) has a particular rate of change, then the rate of change of \( 2 \cdot f(x) \) will be twice as large, and the rate of change of \( c \cdot f(x) \) will be \( c \cdot f'(x) \), which is just the constant multiple rule.

67. Since \( -f \) slopes down by the same amount that \( f \) slopes up, the slope of \( -f \) should be the negative of the slope of \( f \). The constant multiple rule with \( c = -1 \) also says that the slope of \( -f \) will be the negative of the slope of \( f \).

69. Evaluating first would give a constant and the derivative of a constant is zero, so evaluating and the differentiating would always give zero, regardless of the function and number. This supports the idea that we should always differentiate and then evaluate to obtain anything meaningful.

EXERCISES 2.4

1. a. Using the product rule:
\[
\frac{d}{dx}(x^4 \cdot x^6) = 4x^3 \cdot x^6 + x^4 \cdot 6x^5 = 4x^9 + 6x^9 = 10x^9
\]
b. Using the power rule:
\[
\frac{d}{dx}(x^4 \cdot x^6) = \frac{d}{dx}(x^{10}) = 10x^9
\]

3. a. Using the product rule:
\[
\frac{d}{dx}[x^4(x^5 + 1)] = 4x^3(x^5 + 1) + x^4(5x^4) = 4x^8 + 4x^8 + 5x^8 = 9x^8 + 4x^3
\]
b. Using the power rule:
\[
\frac{d}{dx}[x^4(x^5 + 1)] = \frac{d}{dx}(x^9 + x^4) = 9x^8 + 4x^3
\]

5. \( f'(x) = 2x(x^3 + 1) + x^2(3x^2) = 2x^4 + 2x + 3x^4 = 5x^4 + 2x \)

7. \( f'(x) = 1(5x^2 - 1) + x(10x) = 5x^2 - 1 + 10x^2 = 15x^2 - 1 \)

9. \( f'(x) = (2x)(x^2 - 1) + (x^2 + 1)(2x) = 2x^3 - 2x + 2x^3 + 2x = 4x^3 \)
11. \[ f'(x) = (2x + 1)(3x + 1) + (x^2 + x)(3) = 6x^2 + 5x + 1 + 3x + 3 = 9x^2 + 8x + 1 \]

13. \[ f'(x) = 2x(2x + 3x - 1) + x^2(2x + 3) = 2x^3 + 6x^2 - 2x + 2x^3 + 3x^2 = 4x^3 + 9x^2 - 2x \]

15. \[ f'(x) = (4x)(1 - x) + (2x^2 + 1)(1) = 4x - 4x^2 - 2x^2 - 1 = -6x^2 + 4x - 1 \]

17. \[ f'(x) = \left(\frac{1}{2}x^{-1/2}\right)x^{1/2} + 1 + \left(x^{1/2} - 1\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2} + \frac{1}{2}x^{-1/2} + \frac{1}{2} - \frac{1}{2}x^{-1/2} = 1 \]

19. \[ f'(t) = 8t^{1/3}(3t^{2/3} + 1) + 6t^{4/3}(2t^{-1/3}) = 24t + 8t^{1/3} + 12t = 36t + 8t^{1/3} \]

21. \[ f'(z) = (4z^3 + 2z)(z^3 - z) + (z^4 + z^2 + 1)(3z^2 - 1) = 4z^6 - 2z^4 - 2z^2 + 3z^6 + 3z^4 + 3z^2 - z^4 - z^2 - 1 = 7z^6 - 1 \]

23. a. Using the quotient rule:
\[ \frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{x^3(8x^7) - 2x(x^8)}{(x^2)^2} = \frac{8x^9 - 2x^9}{x^4} = 6x^5 \]

b. Using the power rule:
\[ \frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x^6) = 6x^5 \]

25. a. Using the quotient rule:
\[ \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{x^3(0) - 3x^2(1)}{(x^3)^2} = -\frac{3x^2}{x^6} = -\frac{3}{x^4} \]

b. Using the power rule:
\[ \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-4} = -\frac{3}{x^4} \]

27. \[ f'(x) = \frac{3(4x^3) - 3x^2(x^4 + 1)}{(x^3)^2} = \frac{4x^6 - 3x^6 - 3x^2}{x^6} = \frac{x^6 - 3x^2}{x^6} = 1 - \frac{3}{x^4} \]

29. \[ f'(x) = \frac{(x - 1)(1) - (1)(x + 1)}{(x - 1)^2} = \frac{x - 1 - x - 1}{(x - 1)^2} = -\frac{2}{(x - 1)^2} \]

31. \[ f'(x) = \frac{(2 + x)(3) - (1)(3x + 1)}{(2 + x)^2} = \frac{6 + 3x - 3x - 1}{(2 + x)^2} = \frac{5}{(2 + x)^2} \]

33. \[ f'(t) = \frac{(t^2 + 1)(2t) - (2t)(t^2 - 1)}{(t^2 + 1)^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2} = \frac{4t}{(t^2 + 1)^2} \]

35. \[ f'(s) = \frac{(s + 1)(3s^2) - (1)(s^3 - 1)}{(s + 1)^2} = \frac{3s^3 + 3s^2 - s^3 + 1}{(s + 1)^2} = \frac{2s^3 + 3s^2 + 1}{(s + 1)^2} \]

37. \[ f'(x) = \frac{(x + 1)(2x - 2) - (1)(x^2 - 2x + 3)}{(x + 1)^2} = \frac{2x^2 - 2 - x^2 + 2x - 3}{(x + 1)^2} = \frac{x^2 + 2x - 5}{(x + 1)^2} \]
39. \[ f'(x) = \frac{(x^2 + 1)(4x^3 + 2x) - (2x)(x^4 + x^2 + 1)}{(x^2 + 1)^2} = \frac{4x^3(x^2 + 1) + 2x(x^2 + 1) - 2x(x^4) - 2x(x^2 + 1)}{(x^2 + 1)^2} \]

\[ = \frac{4x^5 + 4x^3 - 2x^5}{(x^2 + 1)^2} = \frac{2x^5 + 4x^3}{(x^2 + 1)^2} \]

41. Rewrite \( y = 3x^{-1} \) \( \frac{dy}{dx} = -3x^{-2} \) Rewritten \( \frac{dy}{dx} = -\frac{3}{x^2} \)

45. Rewrite \( y = \frac{1}{3}x^2 - \frac{5}{3}x \) \( \frac{dy}{dx} = \frac{2}{3}x - \frac{5}{3} \) Rewritten \( \frac{dy}{dx} = \frac{2x - 5}{3} \)

47. \[ \frac{d}{dx} \left[ (x^3 + 2) \frac{x^2 + 1}{x + 1} \right] = \frac{d}{dx} (x^3 + 2) \frac{x^2 + 1}{x + 1} + (x^3 + 2) \frac{d}{dx} \left( \frac{x^2 + 1}{x + 1} \right) \]

\[ = 3x^2 \left( \frac{x^2 + 1}{x + 1} \right) + (x^3 + 2) \frac{(x + 1)(2x) - (x^2 + 1)}{(x + 1)^2} \]

\[ = 3x^2 \left( \frac{x^2 + 1}{x + 1} \right) + (x^3 + 2) \frac{2x^2 + 2x - 1}{(x + 1)^2} \]

49. \[ \frac{d}{dx} \left( \frac{x^2 + 3)(x^3 + 1)}{x^2 + 2} \right) = \frac{d}{dx} \left( \frac{x^2 + 3)(x^3 + 1)}{x^2 + 2} \right) \]

\[ = (x^2 + 2) \left\{ \frac{d}{dx} (x^2 + 3) (x^3 + 1) + (x^2 + 3) \frac{d}{dx} (x^3 + 1) \right\} - 2x \left( \frac{(x^2 + 3)(x^3 + 1)}{(x^2 + 2)^2} \right) \]

\[ = (x^2 + 2) \left( 2x(x^3 + 1) + (x^2 + 3)(3x^2) \right) - 2x \left( \frac{(x^2 + 3)(x^3 + 1)}{(x^2 + 2)^2} \right) \]

\[ = (x^2 + 2) \left( 2x^3 + 2x + 3x^4 + 9x^2 \right) - 2x \left( \frac{(x^2 + 3)(x^3 + 1)}{(x^2 + 2)^2} \right) \]

\[ = \frac{2x^6 + 2x^3 + 3x^6 + 9x^4 + 4x^4 + 4x^4 + 18x^2 - 2x^6 - 6x^4 - 2x^3 - 6x}{(x^2 + 2)^2} \]

\[ = \frac{3x^6 + 13x^4 + 18x^2 - 2x}{(x^2 + 2)^2} \]

51. \[ \frac{d}{dx} \left( \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right) = \frac{d}{dx} \left( \frac{x^{1/2} - 1}{x^{1/2} + 1} \right) = \frac{(x^{1/2} + 1)(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-1/2})(x^{1/2} - 1)}{(x^{1/2} + 1)^2} \]

\[ = \frac{\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-1/2}}{(x^{1/2} + 1)^2} = \frac{x^{-1/2}}{(x^{1/2} + 1)^2} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2} \]

53. \[ \frac{d}{dx} \left[ \frac{R(x)}{x^2} \right] = \frac{x \cdot R'(x) - 1 \cdot R(x)}{x^2} = \frac{xR'(x) - R(x)}{x^2} \]
55. a. The instantaneous rate of change of cost with respect to purity is the derivative of the cost function \( C(x) = \frac{100}{100-x} \) on \( 50 \leq x < 100 \).
\[
C'(x) = \frac{(100-x) \cdot 0 - (-1)(100)}{(100-x)^2} = \frac{100}{(100-x)^2} \text{ on } 50 \leq x < 100
\]

b. To find the rate of change for a purity of 95%, evaluate \( C'(x) \) at \( x = 95 \).
\[
C'(95) = \frac{100}{(100-95)^2} = \frac{100}{5^2} = 4
\]
The cost is increasing by 4 cents per additional percent of purity.

c. To find the rate of change for a purity of 98%, evaluate \( C'(x) \) at \( x = 98 \).
\[
C'(98) = \frac{100}{(100-98)^2} = \frac{100}{2^2} = 25
\]
The cost is increasing by 25 cents per additional percent of purity.

57. a.\[\text{Graph}\]
on \([50, 100]\) by \([0, 20]\)

b. Rate of change of cost is 4 for \( x = 95 \); rate of change of cost is 25 for \( x = 98 \).

59. a. \[
AP(x) = \frac{P(x)}{x} = \frac{12x-1800}{x}
\]

b. The marginal average profit function \( MAP(x) \) is the derivative of the average profit function \( AP(x) \).
\[
MAP(x) = \frac{d}{dx} \left( \frac{12x-1800}{x} \right) = \frac{x(12)-1(12x-1800)}{x^2} = \frac{12x-12x+1800}{x^2} = \frac{1800}{x^2}
\]
c. \[
MAP(300) = \frac{1800}{(300)^2} = \frac{1800}{90,000} = \frac{2}{100}
\]
The average profit is increasing at the rate of 2 cents per additional unit after 300 units.

61. To find the rate of change of temperature, find \( T'(x) \).
\[
T(x) = 3x^2(4-x^2) + x^3(-2x) = 12x^2 - 3x^4 - 2x^4 = 12x^2 - 5x^4
\]
For \( x = 1 \), \( T'(1) = 12(1)^2 - 5(1)^4 = 12 - 5 = 7 \).
After 1 hour, the person’s temperature is increasing by 7 degrees per hour.

63. a.\[\text{Graph}\]
on \([0, 2]\) by \([90, 110]\)

b. The rate of change at \( x = 1 \) is 7.

c. The maximum temperature is about 104.5 degrees.
65. a-b.

\[ y = 5002.9x^2 - 43,623.6x + 221,931 \]

don \([2, 25]\) by \([0, 5,000,000]\)

d.

on \([0, 40]\) by \([0, 45,000]\)

f.

on \([0, 40]\) by \([0, 2000]\)

\[ y(40) \approx 33,325 \]

so in the year 2010, per capita national debt should be \$33,325.

\[ y(40) \approx 1,219 \]

so in the year 2010, per capita national debt should be growing by \$1,219 per year.

67. a.

\[ \frac{d}{dx} \left( \frac{1}{x^2} \right) = \frac{d}{dx} (x^{-2}) = -2x^{-3} = -\frac{2}{x^3} \]

\[ \left. \frac{d}{dx} \right|_{x=0} = -\frac{2}{0^3} \quad \text{Undefined} \]

b. Answers will vary.

69. False: the product rule gives the correct right-hand side.

71. True:

\[ \frac{d}{dx} (x \cdot f) = \frac{d}{dx} x \cdot f = f + x \cdot f' \]

73. \[
\frac{d}{dx} (f \cdot g) = \frac{(f \cdot g)'}{f} + \frac{(f \cdot g)}{g} = g \cdot f' + f \cdot g'
\]

The right-hand side multiplies out to \(g \cdot f' + f \cdot g'\) which agrees with the product rule.

75. False: This would be the same as saying that the derivative (instantaneous rate of change) of a product is a product of the derivatives. The product rule gives the correct way of finding the derivative of a product.

77. \[
\frac{d}{dx} (f \cdot g \cdot h) = \frac{d}{dx} [f \cdot (g \cdot h)] = \frac{df}{dx} (g \cdot h) + f \cdot \frac{d}{dx} (g \cdot h)
\]

\[ = \frac{df}{dx} (g \cdot h) + f \left( \frac{dg}{dx} h + g \cdot \frac{dh}{dx} \right) \]

\[ = \frac{df}{dx} g \cdot h + f \cdot \frac{dg}{dx} h + f \cdot g \cdot \frac{dh}{dx} \]

79. \[
\frac{d}{dx} [f(x)]^2 = \frac{d}{dx} [f(x) \cdot f(x)]
\]

\[ = \left[ \frac{d}{dx} f(x) \right] f(x) + f(x) \left[ \frac{d}{dx} f(x) \right]
\]

\[ = f'(x) \cdot f(x) + f(x) \cdot f'(x) = 2f(x) \cdot f'(x) \]