

8.8 Chaos and Fractals



Iterations

Have you read the book *Jurassic Park* by Michael Crichton? In the book, a mathematician named Ian Malcolm had several quotes referring to the first, second, and third iterations of a *fractal* curve. Figure 1 shows the curves and his comments. Can you see that the fractals are repetitions of the first drawing?

The point being illustrated is that even though you start with very controlled and predictable patterns, the end result may be completely unexpected; that is, chaos may occur. What is chaos theory? Let us quote Ian Malcolm in *Jurassic Park*.*

Chaos theory grew out of attempts to make models of weather in the 1960s. If I have a weather system that I start up with a certain temperature and a certain humidity and if I then repeat it with almost the same temperature, wind, and humidity the second system will not behave almost the same. It'll wander off and rapidly will become very different from the first. Thunderstorms instead of sunshine. The shorthand is the “butterfly effect.” A butterfly flaps its wings in Peking and weather in New York is different.

In this section we shall draw fractals and see how chaos theory works.

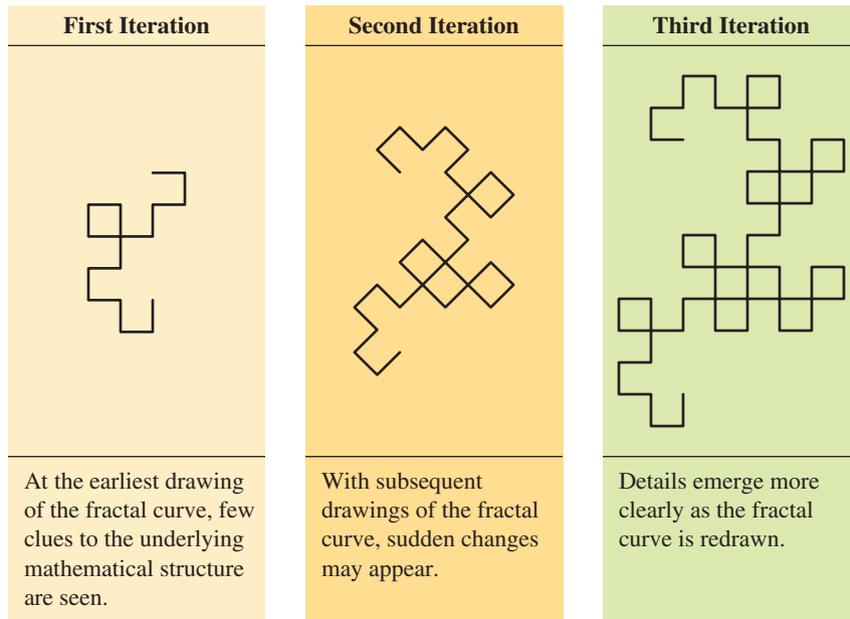


FIGURE 1

A. Chaos

There are many things in the universe around us that are not completely understood even by the most advanced scientists. It has been asked by some whether our universe is not mostly chaos. The shape of the seashore, the arrangement of branches on a tree, and the way tornadoes hit inhabited areas are just some of these seemingly random things that have not yielded to scientific analysis.

* Excerpt from *Jurassic Park* by Michael Crichton, copyright © 1993 by Alfred A. Knopf.

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It was not until the 1960s that mathematicians began a serious study of what seemed to be random processes. A problem of this sort is as follows: Consider the equation $y = 2.5x(1 - x)$. Take a value of x between 0 and 1, say 0.5. Calculate the corresponding value of $y = (2.5)(0.5)(1 - 0.5) = 0.625$. Now let $x = 0.625$, and find the next corresponding value of y . A calculator gives the value 0.586 to three decimal places. Next let $x = 0.586$ and find the corresponding value of y . It is 0.607 to three decimal places. Now let $x = 0.607$, and so on. The following table shows the results of this procedure. The arithmetic can easily be done with a scientific calculator, but *be sure to round off each calculation to three decimal places before doing the next calculation.*

x	0.5	0.586	0.596	0.599	0.600	0.600
y	0.625	0.607	0.602	0.600	0.600	0.600

Note that after seven steps the resulting number seems to have settled down to 0.600. This number is called an *attractor*.


GRAPH IT

A graphing calculator is an attractive way to do attractors! Recall that x is entered as $\boxed{X, \tau, \theta, \eta}$ and that we want 3 decimals, so press $\boxed{\text{MODE}}$, go down one line and right four spaces and press $\boxed{\text{ENTER}}$ to specify 3 decimals in the answer. Go to the home screen. Press 0.498 $\boxed{\text{STO}} \rightarrow \boxed{x}$ $\boxed{\text{ENTER}}$ to use the value 0.498 for \boxed{x} . Next enter $3.5x(1 - x)$ $\boxed{\text{STO}} \rightarrow \boxed{x}$ to let x (the y in Example 2) be $3.5x(1 - x)$. Finally, simply press $\boxed{\text{ENTER}}$ and you get .875. Press $\boxed{\text{ENTER}}$ again and you get .383. You can continue as many steps as you want! A few of them are shown.

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0.498 → X          .498
3.5X(1-X) → X     .875
                  .383
                  .827
                  .501

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EXAMPLE 1 ▶ Finding Attractors Given an Equation

Use the equation $y = 3x(1 - x)$ starting with $x = 0.2$ and follow the procedure discussed above. Use a calculator and round your answers to three decimal places.

Solution

It takes about 43 steps for the procedure to settle down. Here are the last of these steps with three decimal place answers.

x	0.702	0.701	0.700	0.699	0.699
y	0.628	0.629	0.630	0.631	0.631

There are two attractors, 0.631 and 0.699. ■

EXAMPLE 2 ▶ Finding More Attractors Given an Equation

Start with $x = 0.4$ and follow the same procedure with the equation $y = 3.5x(1 - x)$.

Solution

It takes about 11 steps for the procedure to settle down to four attractors. Here are the steps starting with the tenth calculation, which gives $x = 0.498$ to three decimal places.

x	0.498	0.383	0.501	0.383	0.501
y	0.875	0.827	0.875	0.827	0.875

The four attractors are 0.383, 0.501, 0.827, and 0.875. ■

If slightly larger values of k are used in $y = kx(1 - x)$, it turns out that the sequence can end up by doubling the number of attractors repeatedly. However, for values of k close to 4, the sequences go on and on being totally random without any attractors. Mathematicians and scientists have called this kind of result **chaos**.

B. Fractal Geometry

In our study of geometry we have come across one-, two-, and three-dimensional figures. There are many objects, however, that can hardly be classified as one-, two-, or three-dimensional. Examples of such shapes are coastlines, mountain ranges, and paths of lightning strokes. Until quite recently it was assumed that it was impossible to make realistic geometric models of these types of figures. The discovery and development of **fractal geometry** now makes this possible. This important branch of geometry has been one of the most interesting topics in the mathematics of the last twenty years.

Benoit Mandelbrot, a mathematician, applied the word *fractal* (which means “broken up, fragmented”) to certain shapes such as that of the Koch snowflake, which you can construct as follows:

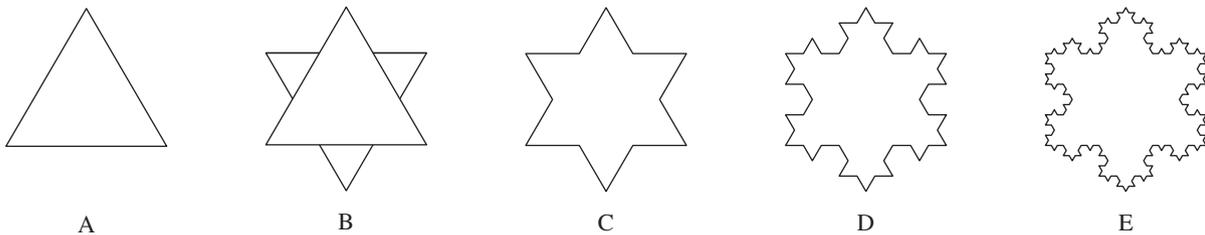


FIGURE 2

Step 1 Start with an equilateral triangle, as in Figure 2, panel A.

Step 2 On the middle part of each side, draw an equilateral triangle with side one-third the length of the side of the original triangle, as in panel B.

Step 3 Erase the side of each smaller triangle that is on a side of the original triangle, as in panel C.

Step 4 Repeat steps 1, 2, and 3 for each of the smaller triangles, as shown in panel D.

Step 5 Continue the same kind of construction on each little straight line segment of the figure. You will end up with the “snowflake” shown in panel E.

The snowflake in Figure 2, panel E, was named for the Swedish mathematician Helga von Koch, who was the first to discover its very remarkable properties. For example, note that the curve consists of an infinite number of small pieces of the form $\text{—}\wedge\text{—}$. It can be shown that the perimeter is infinite in length, but that the area is just 1.6 times the area of the original starting triangle.

EXAMPLE 3 ► Finding Lengths of Lines in the Koch Snowflake

Suppose that the side of the starting triangle in the construction of the Koch snowflake is 1 in. long. In the first step of the construction, how long is the broken line that replaces one side of the triangle?

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Solution

Note that the middle third of the original side is replaced by two sides of an equilateral triangle, the length of whose side is this middle third. Thus, the broken line consists of four pieces, each of length $\frac{1}{3}$ in., so that the length of the broken line is $\frac{4}{3}$ in. ■

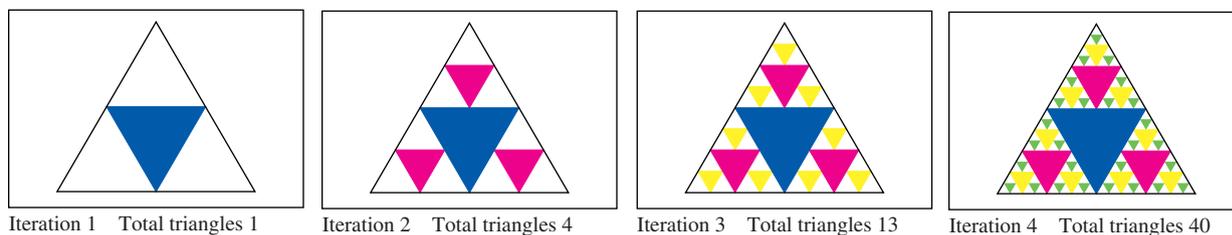
The result of Example 3 shows that the perimeter of the entire figure obtained in the first step is 4 in. when the perimeter of the starting triangle is 3 in.

EXAMPLE 4 ▶ **Constructing Sierpinski's Triangle**

We are now ready to construct the beautiful Sierpinski triangle fractal (Figure 3). Inside the white equilateral triangle, we draw a second equilateral triangle pointing down (blue). Now the original triangle is divided into 4 identical equilateral triangles, 3 white and 1 blue. We repeat the process with each new equilateral triangle drawn red. If we do it two more times with yellow (iteration 3) and green (iteration 4), we obtain a total of 40 triangles.

(a) How many triangles will the fifth iteration have?

(b) What about the n th iteration?

**FIGURE 3**

The Sierpinski triangle.

Solution

(a) To find the fifth iteration, we have to find the pattern at each step.

The number of triangles in going from iteration 1 to iteration 2 increased by 3.

The number of triangles in going from iteration 2 to iteration 3 increased by 3^2 .

The number of triangles in going from iteration 3 to iteration 4 increased by 3^3 .

Thus, to go from iteration 4 to iteration 5, we should add 3^4 triangles, obtaining a total of $40 + 3^4 = 40 + 81 = 121$ triangles.

(b) Let us again look at the pattern for the number of triangles.

Iteration 1	1	= 1
Iteration 2	$1 + 3$	= 4
Iteration 3	$1 + 3 + 3^2$	= 13
Iteration 4	$1 + 3 + 3^2 + 3^3$	= 40
Iteration 5	$1 + 3 + 3^2 + 3^3 + 3^4$	= 121
⋮	⋮	⋮
Iteration n	$1 + 3 + 3^2 + 3^3 + 3^4 + \cdots + 3^{n-1}$	■



To further explore the Sierpinski triangle, access link 8.8.1 on this textbook's Online Study Center.

EXERCISES 8.8

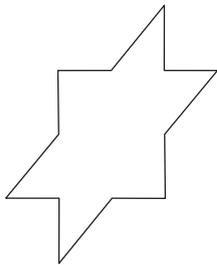
A Chaos

In problems 1–4, follow the procedure of Example 1 to find the possible attractors.

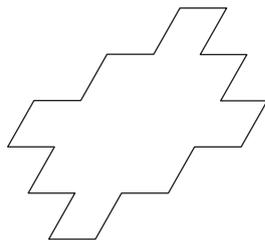
1. In the equation $y = kx(1 - x)$, take $k = 3.5$ and a starting value of $x = 0.2$
2. For the same k as in problem 1, take a starting value of $x = 0.7$.
3. In the equation $y = kx(1 - x)$, take $k = 3$ and a starting value of $x = 0.4$.
4. In the same equation, take $k = 2$ and a starting value of $x = 0.8$.

B Fractal Geometry

5. In the construction of the snowflake fractal, what is the perimeter after
 - a. the third step?
 - b. the fourth step?
6. Instead of an equilateral triangle, use a 3-4-5 right triangle and construct a fractal by replacing one-third of each side by a right triangle with sides respectively parallel to the other two sides of the original triangle. The first step results in the following figure:



7. Instead of a triangle, use an equilateral parallelogram and construct a fractal by replacing one-third of each side by a parallelogram with its sides parallel to the sides of the original parallelogram. The first step results in the figure shown below.

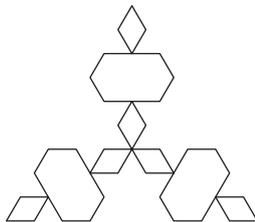


8. Consider a square of side 1 in. Note that the smallest number of complete squares that you can divide the starting square into is four. (Just draw two lines, each joining the midpoints of a pair of opposite sides of the original square.) Can you continue this procedure to show that the area of the original square can be considered as the sum of the areas of an ever-increasing number of small squares?
9. In the construction of problem 8, the area of one of the first four squares is $\frac{1}{4}$ in.². In the next step, there will be 16 small squares, each of area $\frac{1}{16}$ in.². If you go through five steps of this construction, what will be the area of one of the smallest squares?
10.
 - a. Start with a square and develop a fractal by replacing each side, \square , by a \square . Repeat this procedure.
 - b. If this procedure is continued indefinitely, will the perimeter of the fractal be finite? Explain why or why not.
 - c. Will the area of the fractal be finite? Explain why or why not.
11. Start with a square and develop a fractal by replacing each side as in problem 10 but with the small square drawn *inside* the larger square. Then if the common segments are erased in each step, the areas of the small squares are subtracted from the area of the larger square.
12. With the procedure of problem 11, what area remains after the first step? Assume the original square to be of side s .
13. Referring to problem 12, what area remains after the second step?
14. As the procedure of problem 11 is continued, what happens to the length of the perimeter of the fractal? Does it get larger and larger—that is, does it have infinite length?
15. Referring to problems 11–13, what happens to the area enclosed by the fractal? Does it become smaller and smaller; that is, will it have zero area?



Using Your Knowledge

16. You can construct an “antisnowflake” fractal by starting with an equilateral triangle as for the Koch snowflake but drawing the new triangles inside the preceding ones. This results in the area of each new triangle’s being subtracted from the area enclosed by the preceding step. Carry this out for two steps. You should get a figure like the one shown below.



17. Use the fact that the area of an equilateral triangle of side s is $\frac{\sqrt{3}}{4}s^2$ to find the area remaining after the first step of problem 16.
18. Refer to problem 17 and find the area remaining after the second step.



In Other Words

19. Refer to problem 16 and decide whether the perimeter of the fractal remains finite as the procedure is carried on indefinitely. Explain.
20. Refer to problem 16 and decide whether the area enclosed by the fractal tends to zero. Explain.



Web It Exercises

You can construct many of the fractals in this section (Koch, Sierpinski, and Jurassic Park) if your computer can run Java applets. Form several groups and go to link 8.8.2 on this textbook’s Online Study Center.

Team 1

1. Does the Plus Fractal have the self-similarity property?
2. Can you find the number of line ends for each iteration?
3. Can you find the area of the rhombus?
4. Can you find its total length?

Hints are given on the Web!

Team 2

1. The length of the lines in the Koch fractal get smaller with each iteration. What about the rate of growth of the total length of the perimeter?
2. What will be the length of the 50th iteration?
3. What will happen to the length if the number of iterations increases indefinitely? (See Example 3.)

Team 3

1. Can you find the number of lines for each iteration in the Jurassic Park fractal?
2. Can you find the formula to predict the number of lines after the n th iteration in the Jurassic Park fractal?